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AREA OF THE COMPLEMENT OF THE FAST ESCAPING SETS OF A FAMILY OF ENTIRE FUNCTIONS

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Abstract

Let f be an entire function with the form $f(z) = P(e^z)/e^z$, where P is a polynomial with $\deg(P) \geq 2$ and $P(0) \neq 0$. We prove that the area of the complement of the fast escaping set (hence the Fatou set) of f in a horizontal strip of width 2π is finite. In particular, the corresponding result can be applied to the sine family $\alpha \sin(z + \beta)$, where $\alpha \neq 0$ and $\beta \in \mathbf{C}$.

1. Introduction

Let $f : \mathbf{C} \to \mathbf{C}$ be a transcendental entire function. Denote by $f^{\circ n}$ the n-th iterate of f. The *Fatou set* $F(f)$ of f is defined as the maximal open set in which the family of iterates $\{f^{\circ n} : n \in \mathbf{N}\}$ is normal in the sense of Montel. The complement of $F(f)$ is called the *Julia set* of f, which is denoted by $J(f)$. It is well known that $J(f)$ is a perfect completely invariant set which is either nowhere dense or coincides with $\mathbf{C}$. For more details about these sets, one can refer [3], [6] and [17] for rational maps, and [4] and [11] for meromorphic functions.

Already in 1920s, Fatou considered the iteration of transcendental entire functions [13] and one of his study object was $f(z) = \alpha \sin(z) + \beta$, where $0 < \alpha < 1$ and $\beta \in \mathbf{R}$. After Misiurewicz showed that the Fatou set of $f(z) = e^z$ is empty in 1981 [18], the dynamics of exponential maps and trigonometric functions attracted many interests from then on. See [8], [9] and [7] for example. In particular, in 1987 McMullen [16] proved a remarkable result which states that the Julia set of $\sin(\alpha z + \beta)$, $\alpha \neq 0$ always has positive Lebesgue area and the Hausdorff dimension of the Julia set of λe^z, $\lambda \neq 0$ is always 2. From then on a series of papers considered the area and the Hausdorff dimension of the dynamical objects of the transcendental entire functions, not only for the Julia sets in dynamical planes (see [29], [14], [15], [30], [23], [2], [21], [1], [20], [27] and the references therein for example), but also the bifurcation loci in the parameter spaces (see [19] and [31]).

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Unlike the polynomials, the Julia set of a transcendental entire function f is always unbounded. Since the Fatou set of f is dense in the complex plane (if $F(f) \neq \emptyset$), it is interesting to ask when the Fatou set of f has finite area. For the sine function $f(z) = \sin z$, Milnor conjectured that the area of the Fatou set of f is finite in a vertical strip of width 2π. By applying the tools in [16], Schubert proved this conjecture in 2008 [24].

For a transcendental entire function f, the *escaping set* $I(f)$ was studied firstly by Eremenko in [10]. A subset of the escaping set, called the *fast escaping set* $A(f)$, was introduced by Bergweiler and Hinkkanen in [5]. These sets have received quite a lot of attention recently. Especially for the fast escaping set, see [25], [22], [26], [28], [12] and the references therein. In this paper, we consider the area of the complement of the fast escaping sets of a family of entire functions and try to extend the result of Schubert to this class. Our main result is the following.

Theorem 1.1. *Let P be a polynomial with $\deg(P) \geq 2$ and $P(0) \neq 0$. Then the area of the complement of the fast escaping set of any function with the form $f(z) = P(e^z)/e^z$ is finite in any horizontal strip of width 2π.*

The method in this paper is strongly inspired by the work of McMullen and Schubert ([16] and [24]). It is worth to mention that we give also a specific formula of the upper bound of $\text{Area}(S \cap A(f)^c)$ in terms of the coefficients of the polynomial P (see Theorem 3.1), where S is any horizontal strip of width 2π and $A(f)^c$ is the complement of the fast escaping set of f. In fact, we believe that our method can be adopted also to the type of entire functions with the form

$$f(z) = \frac{P(w)}{w^m} \circ \exp(z)$$

completely similarly, where $m \geq 1$ is a positive integer, P is a polynomial with degree $\deg(P) \geq m+1$ and $P(0) \neq 0$.

As a consequence of Theorem 1.1 and Theorem 3.1, we have the following result on the area of the complement of the fast escaping set of the sine family.

Theorem 1.2. *Let S be any vertical strip of width 2π. Then the area of the complement of the fast escaping set of $f(z) = \alpha \sin(z + \beta)$ with $\alpha \neq 0$ satisfies*

$$\text{Area}(S \cap A(f)^c) \leq (4\pi + 4r)\left(x^* + r + 8ce^{4 - x^*/2}\frac{r}{1 - e^{-r/2}}\right),$$

where

$$r = \frac{1}{8}, \quad c = \frac{536\sqrt{2}}{|\alpha|} + \frac{1}{|\alpha|^2}$$

and

$$x^* = \max\left\{\log\left(1 + \frac{18K}{|\alpha|}\right), \log\left(\frac{8(K+1)}{|\alpha|}\right), 6\log 2, 12 + 2\log c\right\}$$

with $K = \max\{|\alpha|/2, |\beta|\}$. *In particular, if* $f(z) = \sin z$ *or* $\cos z$, *then*

$$\mathrm{Area}(S \cap A(f)^c) < 361.$$

Since the fast escaping set of $f(z) = P(e^z)/e^z$ is contained in the Julia set (see Corollary 2.11), it means that the complement of the fast escaping set contains the Fatou set and hence Theorem 1.2 is a generalization of Schubert's result. In [24] Schubert proved that $\mathrm{Area}(S \cap F(f)) < 574$ for $f(z) = \sin z$, where S is a vertical strip with width 2π. See Figure 1.

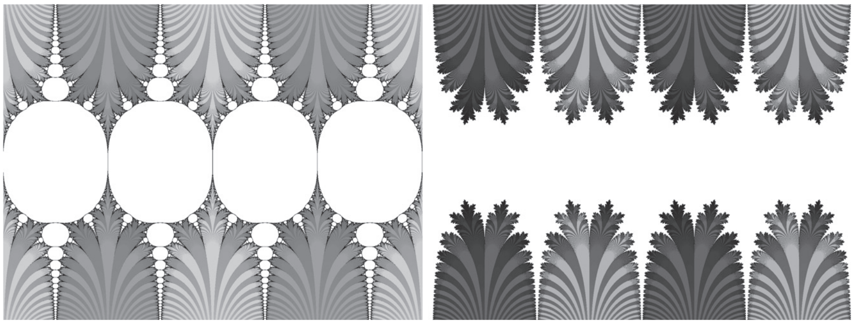

Figure 1. The Fatou sets (white regions) of $f(z) = \sin z$ and $f(z) = \cos z$. Both of these functions have period 2π. It is shown in Theorem 1.2 that the area of the complement of the fast escaping set (hence the Fatou set) of f in a vertical strip with width of 2π is bounded above by 361.

We collect some notations which will be used throughout of this paper. Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$, respectively, be the set of natural numbers, integers, real numbers and complex numbers. For any $x \geq 0$, we use $[x]$ to denote the integer part of x. Hence $x - 1 < [x] \leq x$. For a subset X of $\mathbf{C}$, we use X^c to denote the complement of X in $\mathbf{C}$. All the distance and diameter in this paper are measured in the Euclidean metric and the area is regarded as the two-dimensional planar Lebesgue area. We use $\mathbf{D}(a, r) := \{z \in \mathbf{C} : |z - a| < r\}$ to denote the round disk with center $a \in \mathbf{C}$ and radius $r > 0$.

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2. Distortion lemmas and some basic settings

2.1. Distortion quantities. As in [16] and [24], we introduce some quantities of distortion in this subsection. Let D be a bounded set in the complex plane $\mathbf{C}$ and let f be a holomorphic function defined in a neighbourhood of D. We say that f has *bounded distortion* on D if there are positive constants c and C, such that for all distinct x and y in D, one has

$$c < \frac{|f(x) - f(y)|}{|x - y|} < C. \tag{1}$$

The quantity

$$L(f|_D) := \inf\{C/c : c \text{ and } C \text{ satisfy (1)}\}$$

is the *distortion* of f on D. By (1) we have

$$\sup_{z \in D} |f'(z)| \le C \quad \text{and} \quad \inf_{z \in D} |f'(z)| \ge c.$$

Therefore, $L(f|_D)$ has a lower bound satisfying

$$L(f|_D) \ge \frac{\sup_{z \in D}|f'(z)|}{\inf_{z \in D}|f'(z)|}. \tag{2}$$

The equality holds in this inequality if D is a convex domain.

Let $\mathrm{Area}(E)$ be the Lebesgue area of the measurable set $E \subset \mathbf{C}$. If X and D are two measurable subsets of the complex plane with $\mathrm{Area}(D) > 0$, we use

$$\mathrm{density}(X, D) := \frac{\mathrm{Area}(X \cap D)}{\mathrm{Area}(D)}$$

to denote the *density* of X in D. If c and C satisfy (1), then $c^2 \,\mathrm{Area}(X) \le \mathrm{Area}(f(X)) \le C^2 \,\mathrm{Area}(X)$. This means that

$$\mathrm{density}(f(X), f(D)) \le L(f|_D)^2 \ \mathrm{density}(X, D). \tag{3}$$

The *nonlinearity* of f on D is defined as

$$N(f|_D) := \sup\left\{\frac{|f''(z)|}{|f'(z)|} : z \in D\right\} \cdot \mathrm{diam}(D), \tag{4}$$

provided the right-hand side is finite. In the following by *square* we mean a closed square whose sides are parallel to the coordinate axes. We will use the following relation between the distortion and nonlinearity on squares.

Lemma 2.1. *Let Q be a compact and convex domain in $\mathbf{C}$ (in particular if Q is a square) and let f be a conformal map defined in a neighbourhood of Q with $N(f|_Q) < 1$. Then*

$$L(f|_Q) \le 1 + 2N(f|_Q).$$

Proof. Since f is conformal, let z_0 be a point in Q such that

$$|f'(z_0)| = \sup_{z \in Q} |f'(z)| > 0.$$

Since Q is convex, for any $z \in Q$ we have

$$\frac{|f'(z) - f'(z_0)|}{|f'(z_0)|} = \frac{|\int_{z_0}^{z} f''(\zeta)\, d\zeta|}{|f'(z_0)|} \le \frac{\sup_{z \in Q}|f''(z)|}{|f'(z_0)|} \cdot |z - z_0|$$
$$\le \sup_{z \in Q}\left\{\frac{|f''(z)|}{|f'(z)|}\right\} \cdot \operatorname{diam}(Q) = N(f|_Q) < 1.$$

Therefore, the image of Q under $f'(z)$ is contained in the disk $\mathbf{D}(f'(z_0), |f'(z_0)|)$ and hence $\log f'(z)$ is well-defined on Q.

Since Q is compact, let $z_1 \in Q$ such that

$$|f'(z_1)| = \inf_{z \in Q} |f'(z)| > 0.$$

Since Q is convex and $\log f'(z)$ is well-defined, we have

$$\log L(f|_Q) = \log \frac{|f'(z_0)|}{|f'(z_1)|} \le |\log f'(z_1) - \log f'(z_0)|$$
$$= \left|\int_{z_0}^{z_1} (\log f'(z))'\, dz\right| = \left|\int_{z_0}^{z_1} \frac{f''(z)}{f'(z)}\, dz\right|$$
$$\le \sup_{z \in Q}\left\{\frac{|f''(z)|}{|f'(z)|}\right\} \cdot \operatorname{diam}(Q) = N(f|_Q).$$

Since $e^x \le 1 + 2x$ for $x \in [0, 1)$, we have

$$L(f|_Q) \le \exp(N(f|_Q)) \le 1 + 2N(f|_Q). \qquad \square$$

Remark. McMullen notes in [16] that $L(f|_Q)$ is bounded above by $1 + O(N(f|_Q))$ if $N(f|_Q)$ is small. After that Schubert states in [24] that $L(f|_Q) \le 1 + 8N(f|_Q)$ if $N(f|_Q) < 1/4$ but without a proof.

Let n be a positive integer. For each $1 \le i \le n$, let $D_i \subset \mathbf{C}$ be an open set and $f_i : D_i \to \mathbf{C}$ a conformal map. Let σ and $M > 0$ be two constants satisfying

$$|f_i'(z)| > \sigma > 1 \quad \text{and} \quad \frac{|f_i''(z)|}{|f_i'(z)|} < M, \quad \text{where } z \in D_i \text{ and } 1 \le i \le n.$$

Furthermore, let $Q_i \subset D_i$, $1 \le i \le n$ be squares with sides of length $r > 0$ satisfying $Q_{i+1} \subset f_i(Q_i)$ for all $1 \le i \le n-1$. Define $V := f_n(Q_n)$ and

$$F := (f_n \circ \cdots \circ f_1)^{-1} : V \to Q_1.$$

Then F is a conformal map. McMullen proved that the distortion of F on V is bounded above by a constant depending only on σ, M and r, but not on f_i and n ([16]). Actually, this upper bound can be formulated in the following lemma.

LEMMA 2.2. *If the sides of length r of Q_i is chosen such that $r \le 1/(4M)$ for all $1 \le i \le n$, then the distortion of F on V satisfies*

$$L(F|_V) \le \exp\left(\frac{\sigma}{\sigma - 1}\right).$$

Proof. Let g_i be the inverse of f_i which maps $f_i(Q_i)$ to Q_i for $1 \le i \le n$. Recall that $V = f_n(Q_n)$. Define $V_i := g_i \circ \cdots \circ g_n(V)$, where $1 \le i \le n$. In particular, $V_n = g_n(V) = Q_n$. Since $|f_i'(z)| > \sigma > 1$ for all $1 \le i \le n$, we have

$$\operatorname{diam}(V_i) \le \frac{\sqrt{2}r}{\sigma^{n-i}}, \quad \text{for all } 1 \le i \le n.$$

Note that $V_i \subset Q_i \subset D_i$ for $1 \le i \le n$ since $Q_{i+1} \subset f_i(Q_i)$ for all $1 \le i \le n-1$. This means that there exists a square $Q_i' \subset Q_i$ such that $V_i \subset Q_i'$ and the length of the sides of Q_i' is at most $\sqrt{2}r/\sigma^{n-i}$. Hence by (4), the nonlinearity of f_i on Q_i' satisfies

$$N(f_i|_{Q_i'}) = \left(\sup_{z \in Q_i'} \frac{|f_i''(z)|}{|f_i'(z)|}\right) \cdot \operatorname{diam}(Q_i') \le \frac{2Mr}{\sigma^{n-i}} \le \frac{1}{2}.$$

By Lemma 2.1, we have

$$L(f_i|_{Q_i'}) \le 1 + \frac{4Mr}{\sigma^{n-i}}, \quad \text{for all } 1 \le i \le n.$$

For any holomorphic functions f and g, it is straightforward to verify that the distortion of f and g satisfies[1]

$$L(f|_V) = L(f^{-1}|_{f(V)}) \quad \text{and} \quad L((g \circ f)|_V) \le L(f|_V)L(g|_{f(V)}).$$

Hence, we have

$$\begin{aligned} L(F|_V) &= L((f_n \circ \cdots \circ f_1)|_{V_1}) \\ &\le L(f_1|_{V_1})L(f_2|_{V_2}) \cdots L(f_n|_{V_n}) \le L(f_1|_{Q_1'})L(f_2|_{Q_2'}) \cdots L(f_n|_{Q_n'}) \\ &\le \prod_{i=0}^{n-1}\left(1 + \frac{4Mr}{\sigma^i}\right) \le \prod_{i=0}^{n-1}\left(1 + \frac{1}{\sigma^i}\right). \end{aligned}$$

Since $\log(1 + x) \le x$ for all $x > 0$, we have

$$L(F|_V) \le \exp\left(\sum_{i=0}^{n-1} \frac{1}{\sigma^i}\right) < \exp\left(\sum_{i=0}^{\infty} \frac{1}{\sigma^i}\right) = \exp\left(\frac{\sigma}{\sigma - 1}\right). \qquad \square$$

2.2. Nesting conditions, density and area. In his proof of the existence of Julia sets of entire functions having positive area, McMullen introduced a system

[1] We suppose that the inverse of f exists in the first equality.

of compact sets which satisfies the nesting conditions [16]. We now recall the precise definition.

Definition (Nesting conditions). For $k \in \mathbf{N}$, let $\mathscr{E}_k$ be a finite collection of measurable subsets of $\mathbf{C}$, i.e. $\mathscr{E}_k := \{E_{k,i} : 1 \le i \le d_k\}$, where each $E_{k,i}$ is a measurable subset of $\mathbf{C}$ and $d_k := \#\mathscr{E}_k < +\infty$. We say that $\{\mathscr{E}_k\}_{k=0}^{\infty}$ satisfies the *nesting conditions* if $\mathscr{E}_0 = \{E_{0,1}\}$, where $E_{0,1}$ is a compact connected measurable set and for all $k \in \mathbf{N}$,

(a) every $E_{k+1,i} \in \mathscr{E}_{k+1}$ is contained in a $E_{k,j} \in \mathscr{E}_k$, where $1 \le i \le d_{k+1}$ and $1 \le j \le d_k$;
(b) every $E_{k,i} \in \mathscr{E}_k$ contains a $E_{k+1,j} \in \mathscr{E}_{k+1}$, where $1 \le i \le d_k$ and $1 \le j \le d_{k+1}$;
(c) $\mathrm{Area}(E_{k,i} \cap E_{k,j}) = 0$ for all $1 \le i, j \le d_k$ with $i \ne j$; and
(d) there is $\rho_k > 0$ such that for all $1 \le i \le d_k$ and $E_{k,i} \in \mathscr{E}_k$, we have[2]

$$\mathrm{density}(\mathscr{E}_{k+1}, E_{k,i}) := \mathrm{density}\left(\bigcup_{j=1}^{d_{k+1}} E_{k+1,j}, E_{k,i}\right) \ge \rho_k.$$

Let $\{\mathscr{E}_k\}_{k=0}^{\infty}$ be a sequence satisfying the nesting conditions. Define $E := \bigcap_{k=0}^{\infty} \mathscr{E}_k$. The following lemma was established in [16, Proposition 2.1].

Lemma 2.3. *The density of E in $E_{0,1}$ satisfies*

$$\mathrm{density}(E, E_{0,1}) \ge \prod_{k=0}^{\infty} \rho_k.$$

Now we give the definition of some regions which are needed in the following. For $x > 0$, we define

$$\Lambda(x) := \{z \in \mathbf{C} : |\mathrm{Re}\, z| > x\}. \tag{5}$$

For any given $m, n \in \mathbf{Z}$ and $r > 0$, we define the closed square by

$$Q_r^{m,n} := \{z \in \mathbf{C} : mr \le \mathrm{Re}\, z \le (m+1)r \text{ and } nr \le \mathrm{Im}\, z \le (n+1)r\}.$$

Let

$$\mathscr{Q}_r := \{Q_r^{m,n} : m, n \in \mathbf{Z}\} \tag{6}$$

be a partition of $\mathbf{C}$ by the grids with sides of length $r > 0$. Sometimes we write $Q_r^{m,n} \in \mathscr{Q}_r$ as Q_r if we don't want to emphasize the superscript of $Q_r^{m,n}$.

[2] Note that $\mathscr{E}_k$ is a collection of measurable sets for $k \in \mathbf{N}$. For simplicity, sometimes we will not distinguish $\mathscr{E}_k$ and the union of its elements $\bigcup_{i=1}^{d_k} E_{k,i}$.

Lemma 2.4. *Let $Q \subset \mathbf{C}$ be a square with sides of length $r > 0$ and suppose that f is conformal in a neighbourhood of Q with distortion $L(f|_Q) < \infty$. For any $x > 0$ and $z_0 \in Q$, we have*

$$\mathrm{Area}\Big(\bigcup\{Q_r \in \mathcal{Q}_r : Q_r \cap (\partial f(Q) \cup (\partial\Lambda(x) \cap f(Q))) \neq \emptyset\}\Big) \leq cr^2,$$

where $c = 16 + 12\sqrt{2}L(f|_Q)|f'(z_0)|$.

This lemma was established in [24, Lemma 2.3] with a different coefficient c. For completeness we include a proof here and the argument is slightly different.

Proof. If $\gamma \subset \mathbf{C}$ is a vertical line with length $l_1 > 0$, it is clear that

$$\#\{Q_r \in \mathcal{Q}_r : Q_r \cap \gamma \neq \emptyset\} \leq 4 + \frac{2l_1}{r}. \tag{7}$$

Let $\gamma \subset \mathbf{C}$ be a continuous curve with length $l_2 = 2\sqrt{2}kr > 0$, where k is a positive integer. We claim that

$$k' := \#\{Q_r \in \mathcal{Q}_r : Q_r \cap \gamma \neq \emptyset\} \leq 4 + 8k. \tag{8}$$

Indeed, if $k = 1$, then it is easy to see $k' \leq 12$. Assume that $k = n$ and in this case $k' \leq 4 + 8n$. If $k = n + 1$, let $\gamma(t) : [0, 1] \to \mathbf{C}$ be a parameterization of γ such that the length of $\gamma([0, t_0])$ is $2\sqrt{2}nr$ while the length of $\gamma([t_0, 1])$ is $2\sqrt{2}r$, where $0 < t_0 < 1$. Since $\gamma([t_0, 1])$ can intersect at most 8 squares while $\gamma([0, t_0])$ can intersect at most $4 + 8n$ by the assumption, it follows that $k' \leq 4 + 8(n + 1)$ if $k = n + 1$. Hence the claim (8) is proved.

For the general case, we assume that $\gamma \subset \mathbf{C}$ is a continuous curve with length $l_3 > 0$. Let $[x]$ be the integer part of $x > 0$. By (8), we have

$$\#\{Q_r \in \mathcal{Q}_r : Q_r \cap \gamma \neq \emptyset\} \leq 4 + 8\left[\frac{l_3}{2\sqrt{2}r}\right] + 8 \leq 12 + \frac{2\sqrt{2}l_3}{r}. \tag{9}$$

Since f is a conformal map in a neighbourhood of Q, we conclude that $\partial f(Q) = f(\partial Q)$. From (2), the length of $\partial f(Q)$ satisfies

$$\begin{aligned} l_4 := \int_{\partial f(Q)} |d\xi| = \int_{\partial Q} |f'(z)|\,|dz| \leq \sup_{z \in Q} |f'(z)| \cdot 4r \\ \leq 4L(f|_Q)|f'(z_0)|r. \end{aligned} \tag{10}$$

Similarly, the length of $\partial\Lambda(x) \cap f(Q)$ satisfies

$$\begin{aligned} l_5 \leq 2 \operatorname{diam} f(Q) \leq 2 \sup_{z \in Q} |f'(z)| \cdot \operatorname{diam}(Q) \\ \leq 2\sqrt{2}L(f|_Q)|f'(z_0)|r. \end{aligned} \tag{11}$$

By (7), (9), (10) and (11), we have

$$\begin{aligned}\#\{Q_r \in \mathcal{Q}_r : Q_r \cap (\partial f(Q) \cup (\partial\Lambda(x) \cap f(Q))) \neq \emptyset\} \\ \leq \left(4 + \frac{2l_5}{r}\right) + \left(12 + \frac{2\sqrt{2}l_4}{r}\right) = 16 + \frac{2l_5 + 2\sqrt{2}l_4}{r} \\ \leq 16 + 12\sqrt{2}L(f|_Q)|f'(z_0)|.\end{aligned}$$

The proof is finished if we notice that the area of each Q_r is r^2. □

2.3. Basic properties of the polynomial and entire function. For $N \geq 2$, let P be a polynomial with degree at least 2 which has the form

$$P(z) = a_0 + a_1 z + \cdots + a_N z^N,$$

where $a_i \in \mathbf{C}$ for $0 \leq i \leq N$ and $a_0 a_N \neq 0$. In the rest of this article, the polynomial P will be fixed. We denote

$$K := \max\{|a_0|, |a_1|, \ldots, |a_N|\} > 0. \tag{12}$$

LEMMA 2.5. *Let* $\varepsilon > 0$ *be any given constant. The following statements hold*:

(a) *If* $|z| \geq 1 + \dfrac{K}{\varepsilon|a_N|} > 1$, *then*

$$|P(z) - a_N z^N| \leq \varepsilon|a_N|\,|z|^N;$$

(b) *If* $|z| \leq \dfrac{\varepsilon|a_0|}{K + \varepsilon|a_0|} < 1$, *then*

$$|P(z) - a_0| \leq \varepsilon|a_0|.$$

Proof. By the definition of K in (12), if $|z| \geq 1 + \dfrac{K}{\varepsilon|a_N|} > 1$, then

$$|P(z) - a_N z^N| \leq K(1 + |z| + \cdots + |z|^{N-1}) < K\frac{|z|^N}{|z| - 1} \leq \varepsilon|a_N|\,|z|^N.$$

On the other hand, if $|z| \leq \dfrac{\varepsilon|a_0|}{K + \varepsilon|a_0|} < 1$, then

$$|P(z) - a_0| \leq K(|z| + \cdots + |z|^N) < K\frac{|z|}{1 - |z|} \leq \varepsilon|a_0|.$$ □

Note that

$$P(z)/z = a_0 z^{-1} + a_1 + \cdots + a_N z^{N-1}$$

is a rational function. Let $\mathbf{D}(a, r) := \{z \in \mathbf{C} : |z - a| < r\}$ be the open disk centered at $a \in \mathbf{C}$ with radius $r > 0$. For each $R > 0$ and $\theta, \xi \in [0, 2\pi)$, we denote a closed domain

$$\mathbf{U}(R,\theta,\xi) := \left\{ z \in \mathbf{C} : |z| \geq R \text{ and } \theta - \frac{\xi}{2} \leq \arg(z) \leq \theta + \frac{\xi}{2} \right\}.$$

LEMMA 2.6. *For every $\theta \in [0, 2\pi)$, the rational function $P(z)/z$ is univalent in a neighborhood of* $\mathbf{U}\left(2R_1, \theta, \frac{\pi}{N-1}\right)$ *and* $\overline{\mathbf{D}}(0, R_2/2)$, *where*

$$R_1 = 1 + \frac{4K}{|a_N|} \quad \text{and} \quad R_2 = \frac{|a_0|}{4K + |a_0|}.$$

Proof. (a) If $|z| \geq R_1$, by Lemma 2.5(a) we have

$$\left| \frac{P(z)}{z} - a_N z^{N-1} \right| \leq \frac{1}{4} |a_N|\, |z|^{N-1}.$$

Then one can write $P(z)/z$ as

$$P_1(z) = \frac{P(z)}{z} = a_N z^{N-1}(1 + \varphi(z)), \tag{13}$$

where $\varphi(z)$ is holomorphic in $\mathbf{C}\backslash\{0\}$ and $|\varphi(z)| \leq 1/4$ if $|z| \geq R_1$.

Let $w_0 \in \mathbf{C}\backslash\{0\}$. For any $w \in \partial\mathbf{U}(|w_0|/2, \arg(w_0), \pi)$, we have

$$|w - w_0| > \frac{1}{4}(|w| + |w_0|). \tag{14}$$

Let $g(z) := z^{N-1}$. For each $z_0 \in \mathbf{C}$ such that $|z_0| \geq 2R_1$, we define $w_0 := g(z_0) = z_0^{N-1}$. Note that $g^{-1}(\mathbf{U}(|w_0|/2, \arg(w_0), \pi))$ consists of $N-1$ disjoint closed domains:

$$D_k := \mathbf{U}\left(2^{-1/(N-1)}|z_0|, \arg(z_0) + \frac{2k\pi}{N-1}, \frac{\pi}{N-1}\right),$$

where $0 \leq k \leq N-2$. Then for $0 \leq k \leq N-2$, $z_k := z_0 e^{2k\pi \mathrm{i}/(N-1)}$ is contained in the interior of D_k.

For any $z \in \partial D_k$ with $0 \leq k \leq N-2$, we have $z^{N-1} \in \partial\mathbf{U}(|w_0|/2, \arg(w_0), \pi)$. Combining (13) and (14), we have

$$|z^{N-1} - z_0^{N-1}| > \frac{1}{4}(|z|^{N-1} + |z_0|^{N-1}) \geq |z^{N-1}\varphi(z) - z_0^{N-1}\varphi(z_0)|.$$

Define $\varphi_1(z) := a_N(z^{N-1} - z_0^{N-1})$ and $\varphi_2(z) := P_1(z) - P_1(z_0) = a_N z^{N-1}(1 + \varphi(z)) - a_N z_0^{N-1}(1 + \varphi(z_0))$. By Rouché's theorem, $\varphi_1(z) = 0$ and $\varphi_2(z) = 0$ have the same number of roots in each D_k, where $0 \leq k \leq N-2$. Since $\varphi_1(z) = 0$ has exactly one root z_k in each D_k, this means that $\varphi_2(z) = 0$ has exactly one root in each D_k, where $0 \leq k \leq N-2$.

On the other hand, (14) holds also for $w \in \partial\mathbf{U}(|w_0|/2, -\arg(w_0), \pi)$. By Rouché's theorem again, $\varphi_2(z) = 0$ has no root in each $-D_k$, where $0 \le k \le N-2$. By the arbitrariness of z_0, it means that $P_1(z) = P(z)/z$ is univalent in a neighborhood of $\mathbf{U}\left(2R_1, \theta, \dfrac{\pi}{N-1}\right)$, where $\theta \in [0, 2\pi)$.

(b) Similarly, by Lemma 2.5(b) one can write $P(z)/z$ as

$$P_1(z) = \frac{P(z)}{z} = \frac{a_0}{z}(1 + \psi(z)),$$

where $\psi(z)$ is holomorphic in $\mathbf{C}$ and $|\psi(z)| \le 1/4$ if $|z| \le R_2$. For each $z_0 \in \overline{\mathbf{D}}(0, R_2/2)\setminus\{0\}$ and $z \in \partial\mathbf{D}(0, R_2)$, we have

$$|z - z_0| > \frac{1}{4}(|z| + |z_0|).$$

Hence

$$\left|\frac{1}{z} - \frac{1}{z_0}\right| > \frac{1}{4}\frac{|z| + |z_0|}{|zz_0|} \ge \left|\frac{\psi(z)}{z} - \frac{\psi(z_0)}{z_0}\right|.$$

Define $\psi_1(z) := a_0(1/z - 1/z_0)$ and $\psi_2(z) := P_1(z) - P_1(z_0) = \dfrac{a_0}{z}(1 + \psi(z)) - \dfrac{a_0}{z_0}(1 + \psi(z_0))$. By Rouché's theorem, $\psi_1(z) = 0$ and $\psi_2(z) = 0$ have the same number of roots in $\mathbf{D}(0, R_2)$. Since $\psi_1(z) = 0$ has exactly one root z_0 in $\mathbf{D}(0, R_2)$, this means that $\psi_2(z) = 0$ has exactly one root in $\mathbf{D}(0, R_2)$. By the arbitrariness of z_0, it means that $P_1(z) = P(z)/z$ is univalent in a neighborhood of $\overline{\mathbf{D}}(0, R_2/2)$. □

Since P is a polynomial, it is easy to see that $P(e^z)/e^z$ is a transcendental entire function. We now give some quantitative estimations on the mapping properties of $f(z) = P(e^z)/e^z$ by applying some properties of $P(z)/z$ obtained above. Recall that $\Lambda(x) = \{z \in \mathbf{C} : |\mathrm{Re}\, z| > x\}$ for $x > 0$. We denote

$$K_0 := \min\{|a_0|, |a_N|\} > 0. \tag{15}$$

Corollary 2.7. *Let*

$$r_0 := \frac{\pi}{N-1} \quad \text{and} \quad R_3 := \log\left(2 + \frac{8K}{K_0}\right). \tag{16}$$

Then for any square $Q \subset \Lambda(R_3)$ with sides of length $r \le r_0$, the restriction of $f(z) = P(e^z)/e^z$ on a neighbourhood of Q is a conformal map.

Proof. We have $|e^z| \ge 2R_1$ if $\mathrm{Re}\, z \ge \log(2R_1)$ and $|e^z| \le R_2/2$ if $\mathrm{Re}\, z \le \log(R_2/2)$. Let $Q \subset \Lambda(R_3)$ be a square with sides of length $\pi/(N-1)$. It is easy to see that exp is injective in a neighbourhood of Q and $\exp(Q)$ is contained in $\overline{\mathbf{D}}(0, R_2/2)$ or $\mathbf{U}\left(2R_1, \theta, \dfrac{\pi}{N-1}\right)$ for some $\theta \in [0, 2\pi)$. This means that $f(z) = P(e^z)/e^z$ is conformal in a neighborhood of Q by Lemma 2.6. □

We will use the following lemma to estimate $|f'(z)|$ and $|f''(z)/f'(z)|$ for $f(z) = P(e^z)/e^z$.

LEMMA 2.8. *Suppose that* $|z| \geq R_4$ *or* $|z| \leq R_5$, *where*

$$R_4 = 1 + \max\left\{\frac{2K+4}{|a_N|}, \frac{K}{|a_N|}\left(\frac{2N^2}{N-1} + 1\right)\right\} \quad \textit{and}$$

$$R_5 = \min\left\{\frac{|a_0|}{2(KN+2)}, \frac{1}{2N}\sqrt{\frac{|a_0|}{K}}\right\}.$$

Then

$$\left|P'(z) - \frac{P(z)}{z}\right| > 2 \quad \textit{and} \quad \left|\frac{z^2 P''(z)}{zP'(z) - P(z)} - 1\right| < N.$$

Proof. A direct calculation shows that

$$P'(z) = \sum_{k=1}^{N} k a_k z^{k-1} \quad \text{and} \quad P''(z) = \sum_{k=2}^{N} k(k-1) a_k z^{k-2}.$$

This means that

$$P'(z) - \frac{P(z)}{z} = \sum_{k=1}^{N} k a_k z^{k-1} - \sum_{k=0}^{N} a_k z^{k-1} = \sum_{k=0}^{N} (k-1) a_k z^{k-1} \tag{17}$$

and

$$\frac{z^2 P''(z)}{zP'(z) - P(z)} - 1 = \frac{\sum_{k=0}^{N} k(k-1) a_k z^k}{\sum_{k=0}^{N} (k-1) a_k z^k} - 1 = \frac{\sum_{k=0}^{N} (k-1)^2 a_k z^k}{\sum_{k=0}^{N} (k-1) a_k z^k}. \tag{18}$$

If $|z| \geq 1 + \dfrac{2K+4}{|a_N|} > 3$, by (17) we have

$$\begin{aligned} \left|P'(z) - \frac{P(z)}{z}\right| &\geq |a_N|(N-1)|z|^{N-1} - K(N-1)(|z|^{N-2} + \cdots + |z| + 1) \\ &\geq (N-1)|z|^{N-1}\left(|a_N| - \frac{K}{|z|-1}\right) \\ &\geq \frac{|a_N|}{2}|z|^{N-1} \geq \frac{|a_N|}{2}|z| > 2. \end{aligned} \tag{19}$$

If $|z| \leq \dfrac{|a_0|}{2(KN+2)} < \dfrac{1}{2}$, we have

$$\left|P'(z)-\frac{P(z)}{z}\right| \ge \frac{|a_0|}{|z|}-K(N-1)(|z|+\cdots+|z|^{N-1}) \tag{20}$$
$$\ge \frac{|a_0|}{|z|}-K(N-1) > \frac{|a_0|}{2|z|} \ge KN+2>2.$$

For the second inequality, if $|z| \ge 1+\frac{K}{|a_N|}\left(\frac{2N^2}{N-1}+1\right) > 8$, by (18) we have

$$\left|\frac{z^2P''(z)}{zP'(z)-P(z)}-1\right| \le N-1+\left|\frac{\sum_{k=0}^{N-1}(k-1)(N-k)a_kz^k}{\sum_{k=0}^{N}(k-1)a_kz^k}\right|$$
$$\le N-1+\frac{KN^2}{N-1}\cdot\frac{|z|+\cdots+|z|^{N-1}}{|a_N|\,|z|^N-K(|z|+\cdots+|z|^{N-1})}$$
$$\le N-1+\frac{KN^2}{N-1}\cdot\frac{1}{|a_N|(|z|-1)-K} \le N-\frac{1}{2}<N.$$

If $|z| \le \frac{1}{2N}\sqrt{|a_0|/K} < \frac{1}{2}$, by (18) we have

$$\left|\frac{z^2P''(z)}{zP'(z)-P(z)}-1\right| \le 1+\left|\frac{\sum_{k=2}^{N}k(k-1)a_kz^k}{\sum_{k=0}^{N}(k-1)a_kz^k}\right|$$
$$\le 1+\frac{KN^2(|z|^2+\cdots+|z|^N)}{|a_0|-KN(|z|^2+\cdots+|z|^N)}$$
$$\le 1+\frac{2KN^2|z|^2}{|a_0|-2KN|z|^2} \le 1+\frac{N}{2N-1} \le \frac{5}{3}<N. \qquad \square$$

Corollary 2.9. *Let*

$$R_6 := \max\{\log R_4, -\log R_5\}. \tag{21}$$

Then for any $z \in \Lambda(R_6)$, *the function* $f(z)=P(e^z)/e^z$ *satisfies*

$$|f'(z)|>2 \quad \textit{and} \quad \frac{|f''(z)|}{|f'(z)|}<N.$$

Proof. Denote $P_1(w) := P(w)/w$. Therefore, $f(z)=P(e^z)/e^z=P_1\circ\exp(z)$. It is easy to check that

$$f'(z)=P_1'(e^z)e^z \quad \text{and} \quad f''(z)=P_1''(e^z)e^{2z}+P_1'(e^z)e^z.$$

Let $w=e^z$. By a straightforward computation, we have

$$f'(z)=P_1'(w)w=P'(w)-\frac{P(w)}{w} \tag{22}$$

and

$$\frac{f''(z)}{f'(z)} = \frac{P_1''(w)w^2 + P_1'(w)w}{P_1'(w)w} = \frac{w^2 P''(w)}{wP'(w) - P(w)} - 1.$$

Then the result follows from Lemma 2.8 immediately. □

2.4. Escaping and fast escaping sets. Let f be a transcendental entire function. A point $a \in \mathbf{C}$ is called an *asymptotic value* of f if there exists a continuous curve $\gamma(t) \subset \mathbf{C}$ with $0 < t < \infty$, such that $\gamma(t) \to \infty$ as $t \to \infty$ and $f(\gamma(t)) \to a$ as $t \to \infty$.

LEMMA 2.10. *The entire function $f(z) = P(e^z)/e^z$ does not have any finite asymptotic value.*

Proof. Assume that $a \in \mathbf{C}$ is a finite asymptotic value of $f(z)$. Then by definition, there exists a continuous curve $\gamma(t) \subset \mathbf{C}$ with $0 < t < \infty$, such that $\gamma(t) \to \infty$ as $t \to \infty$ and $f(\gamma(t)) \to a$ as $t \to \infty$. This means that

$$\lim_{t\to\infty} \frac{P(w)}{w} \circ e^{\gamma(t)} = a.$$

Denote $\gamma(t) = x(t) + \mathrm{i}y(t)$ and let $w_1, w_2, \dots, w_N$ be the N roots of the equation $P(w) = aw$. We define the set $Y := \{\arg w_i + 2k\pi : 1 \le i \le N, k \in \mathbf{Z}\}$. If $x(t)$ is unbounded as $t \to \infty$, then $f(\gamma(t))$ is also unbounded and this is a contradiction. Hence $|x(t)| \le A$ for some constant $A > 0$ for all t. Since $\gamma(t) \to \infty$ as $t \to \infty$, this implies that $y(t) \to \infty$ as $t \to \infty$. Therefore, for each $y_0 \in \mathbf{R}\backslash Y$, there exists a sequence $\{z_n\} \subset \gamma(t)$ such that $\mathrm{Im}\, z_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} e^{\mathrm{i}\,\mathrm{Im}\, z_n} = e^{\mathrm{i}y_0}$. Since $|x(t)| \le A$, it follows that $\lim_{t\to\infty} e^{x(t)} \ne 0$. This implies that $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} P(e^{z_n})/e^{z_n} \ne a$, which is a contradiction. □

Let f be a transcendental entire function. The set

$$I(f) := \{z \in \mathbf{C} : f^{\circ n}(z) \to \infty \text{ as } n \to \infty\} \tag{23}$$

is called the *escaping set* of f. We use $\mathrm{sing}(f^{-1})$ to denote the set of *singular values* of f which consists of all the critical values and asymptotic values of f and their accumulation points.

COROLLARY 2.11. *The escaping set $I(f)$ of $f(z) = P(e^z)/e^z$ is contained in the Julia set $J(f)$.*

Proof. It is clear that the set of the critical values of $f(z) = P(e^z)/e^z$ is finite. From Lemma 2.10, it follows that $\mathrm{sing}(f^{-1})$ is bounded. According to [11, Theorem 1], we have $I(f) \subset J(f)$. □

Actually, we will estimate the area of the complement of the fast escaping set in next section. Let f be a transcendental entire function. The *maximal modulus function* is defined by

$$M(r, f) := \max_{|z|=r} |f(z)|, \quad \text{where } r > 0.$$

We use $M^{\circ n}(r, f)$ to denote the n-th iterate of $M(r, f)$ with respect to the variable $r > 0$, where $n \in \mathbf{N}$. The notation $M(r, f)$ is written as $M(r)$ if the function f is known clearly. A subset of the escaping set, called the *fast escaping set* $A(f)$ was introduced in [5] and can be defined [22] by

$$A(f) := \{z : \text{there is } \ell \in \mathbf{N} \text{ such that } |f^{\circ(n+\ell)}(z)| \geq M^{\circ n}(R) \text{ for } n \in \mathbf{N}\}. \tag{24}$$

Here $R > 0$ is a constant such that $M^{\circ n}(R) \to \infty$ as $n \to \infty$. It is proved in [22, Theorem 2.2(b)] that $A(f)$ is independent of the choice of R such that $M^{\circ n}(R) \to \infty$ as $n \to \infty$.

Lemma 2.12. *Let $R > 0$ be a constant and define $u_0 := R$. For $n \geq 1$, define u_n inductively by $u_n := Re^{Ru_{n-1}}$. Let $v_0 \in \mathbf{R}$ and define v_n inductively by $v_n := 2e^{v_{n-1}/2}$ for $n \geq 1$. Then there is $\ell \in \mathbf{N}$ such that $v_{n+\ell} \geq 4Ru_n$ for all $n \in \mathbf{N}$.*

Proof. By the definition of v_n, we can obtain that there exists $l \in \mathbf{N}$ such that $v_l \geq 4Ru_0 = 4R^2$. Suppose that $n = k$ and in this case $v_{k+l} \geq 4Ru_k$. If $n = k + 1$, we hope to obtain that $v_{k+l+1} \geq 4Ru_{k+1}$. Note that $v_{k+l+1} = 2e^{v_{k+l}/2} \geq 2e^{2Ru_k} = 2R^{-1}e^{Ru_k}u_{k+1}$. It is sufficient to obtain $Ru_k \geq \log(2R^2)$. This is true since $u_k \geq R$ for all $k \in \mathbf{N}$ and $R^2 \geq \log(2R^2)$ for all $R > 0$. □

Corollary 2.13. *Let $z_0 \in \mathbf{C}$ and suppose that $z_n = f^{\circ n}(z_0)$ satisfies $|z_n| \geq \xi_n$ for all $n \in \mathbf{N}$, where $\xi_n > 0$ is defined inductively by*

$$\xi_n = 2\exp(\xi_{n-1}/2) \quad \text{with } \xi_0 > 0.$$

Then z_0 is contained in the fast escaping set of $f(z) = P(e^z)/e^z$.

Proof. Recall that $N \geq 2$ is the degree of the polynomial P and $K > 0$ is defined in (12). According to Lemma 2.5, there exists $\delta_0 \geq 1$ such that if $\delta \geq \delta_0$, then the maximal modulus function of f satisfies

$$M(\delta) = M(\delta, f) \leq 2Ke^{(N-1)\delta}.$$

On the other hand, there exists $\delta_1 > 0$ such that for all $\delta \geq \delta_1$, then $M^{\circ n}(\delta)$ is monotonically increasing as n increases. Since the Julia set of f is non-empty, this means that $M^{\circ n}(\delta) \to \infty$ as $n \to \infty$ if $\delta \geq \delta_1$.

Define

$$R := \max\{2K, (N-1)\delta_0, \delta_1\} \geq 1.$$

We denote $u_0 = R$ and for $n \geq 1$, define u_n inductively by $u_n = Re^{Ru_{n-1}}$. Then we have $M^{\circ n}(R) \leq u_n$ for all $n \in \mathbf{N}$. By the definition of ξ_n, we have $\xi_n =$

$2\exp^{\circ n}(\xi_0/2)$. According to Lemma 2.12, there exists $\ell \in \mathbf{N}$ such that for all $n \in \mathbf{N}$,

$$|f^{\circ(n+\ell)}(z_0)| = |z_{n+\ell}| \geq \xi_{n+\ell} \geq u_n \geq M^{\circ n}(R).$$

By the defintion of R, we have $M^{\circ n}(R) \to \infty$ as $n \to \infty$. This means that z_0 is contained in the fast escaping set of f. □

3. Proof of the theorems

3.1. Proof of Theorem 1.1. Recall that $N \geq 2$ is the degree of the polynomial P. Let $r > 0$ be fixed such that

$$r \leq \frac{1}{4N}. \tag{25}$$

We define

$$x' := \max\{R_3, R_6, 6\log 2\}, \tag{26}$$

where R_3 and R_6 are constants introduced in Corollary 2.7 and Corollary 2.9 respectively.

Recall that $\Lambda(x) = \{z \in \mathbf{C} : |\mathrm{Re}\, z| > x\}$ is the set defined in (5) for all $x > 0$. Let Q_0 be a square in $\Lambda(x)$ with sides of length r, where $x \geq x'$. Since $r < r_0 = \pi/(N-1)$, from Corollary 2.7 we know that f is conformal in a neighbourhood of Q_0. For $k \in \mathbf{N}$, define

$$x_k := 2\exp^{\circ k}(x/2). \tag{27}$$

In particular, $x_0 = x \geq x'$ and we have $x_{k+1} = 2\exp(x_k/2) > x_k \geq x'$ since $2e^{x/2} > x$ for all $x \in \mathbf{R}$. Recall that $\mathscr{Q}_r$ is a collection of grids with sides of length $r > 0$ defined in (6). For any subset E of Q_0 in $\Lambda(x_0)$ and $k \in \mathbf{N}$, define

$$\mathrm{pack}(f^{\circ k}(E)) := \{Q_r \in \mathscr{Q}_r : Q_r \subset f^{\circ k}(E) \cap \Lambda(x_k)\}.$$

We now define a sequence of families of measurable sets satisfying the nesting conditions based on the square Q_0. Let $\mathscr{E}_0 := \{Q_0\}$ and for $k \geq 1$, define inductively

$$\mathscr{E}_k := \{F_k \subset Q_0 : F_k \subset E_{k-1} \in \mathscr{E}_{k-1} \text{ and } f^{\circ k}(F_k) \in \mathrm{pack}(f^{\circ k}(E_{k-1}))\}.$$

It is clear that $\mathscr{E}_k$ is a finite collection of measurable subsets of $\mathbf{C}$ for all $k \in \mathbf{N}$. Denote the elements of $\mathscr{E}_k$ by $E_{k,i}$, where $1 \leq i \leq d_k$.

By definition, for all $k \in \mathbf{N}$, we have $f^{\circ(k+1)}(E_{k,i}) = f(Q_r^k)$, where[3] Q_r^k is a square with sides of length r and $Q_r^k \subset \Lambda(x_k)$. From (4), Corollary 2.9 and (25),

[3] Note that $Q_r^k \subset \Lambda(x_k)$ is a square depending also on the subscript 'i' of $E_{k,i}$, where $k \in \mathbf{N}$ and $1 \leq i \leq d_k$. We omit this index here for simplicity.

we have

$$N(f|_{Q_r^k}) < N\sqrt{2}r \le \frac{\sqrt{2}}{4}.$$

By Lemma 2.1, the distortion of f on Q_r^k satisfies

$$L(f|_{Q_r^k}) \le 1 + 2N(f|_{Q_r^k}) < 2. \tag{28}$$

For every $k \in \mathbf{N}$, let z_k be any point in $Q_r^k \subset \Lambda(x_k)$. From (2) and (28) we have

$$\begin{aligned}\mathrm{Area}(f(Q_r^k)) &= \int_{Q_r^k} |f'(z)|^2\, dxdy \ge \inf_{z\in Q_r^k} |f'(z)|^2 \cdot \mathrm{Area}(Q_r^k) \\ &\ge \frac{|f'(z_k)|^2}{(L(f|_{Q_r^k}))^2} \cdot r^2 > \frac{1}{4}|f'(z_k)|^2 r^2\end{aligned} \tag{29}$$

and

$$\begin{aligned}\mathrm{diam}(f(Q_r^k)) &\le \sup_{z\in Q_r^k} |f'(z)| \cdot \mathrm{diam}(Q_r^k) \\ &\le L(f|_{Q_r^k})|f'(z_k)| \cdot \sqrt{2}r < 2\sqrt{2}|f'(z_k)|r.\end{aligned} \tag{30}$$

Recall that $K_0 = \min\{|a_0|, |a_N|\} > 0$ is the constant defined in (15). By (19), (20) and (22), we have

$$|f'(z_k)| > \frac{1}{2}K_0 e^{|\mathrm{Re}\, z_k|} > \frac{1}{2}K_0 e^{x_k}. \tag{31}$$

For $k \in \mathbf{N}$ and $1 \le i \le d_k$, we denote

$$B_1 := \bigcup\{Q_r \in \mathcal{Q}_r : Q_r \subset f^{\circ(k+1)}(E_{k,i}) \cap (\mathbf{C}\backslash\Lambda(x_{k+1}))\}$$

and

$$B_2 := \bigcup\{Q_r \in \mathcal{Q}_r : Q_r \cap (\partial f^{\circ(k+1)}(E_{k,i}) \cup (\partial\Lambda(x_{k+1}) \cap f^{\circ(k+1)}(E_{k,i}))) \ne \emptyset\}.$$

Recall that $f^{\circ(k+1)}(E_{k,i}) = f(Q_r^k)$ for some square Q_r^k in $\Lambda(x_k)$ with sides of length r, where $k \in \mathbf{N}$ and $1 \le i \le d_k$. From (29), (30) and (31), we have

$$\begin{aligned}\frac{\mathrm{Area}(B_1)}{\mathrm{Area}(f^{\circ(k+1)}(E_{k,i}))} &\le \frac{2x_{k+1}\,\mathrm{diam}(f^{\circ(k+1)}(E_{k,i}))}{\mathrm{Area}(f^{\circ(k+1)}(E_{k,i}))} \\ &= \frac{2x_{k+1}\,\mathrm{diam}(f(Q_r^k))}{\mathrm{Area}(f(Q_r^k))} < \frac{16\sqrt{2}x_{k+1}}{|f'(z_k)|r} < \frac{32\sqrt{2}}{K_0 r}\cdot\frac{x_{k+1}}{e^{x_k}}.\end{aligned} \tag{32}$$

Note that $x_{k+1} \ge x_1 = 2e^{x/2}$ for all $k \in \mathbf{N}$ and $x \ge 6\log 2$ by (26). By Lemma 2.4, (28), (29) and (31), we have

$$\frac{\text{Area}(B_2)}{\text{Area}(f^{k+1}(E_{k,i}))} \le \frac{(16+12\sqrt{2}L(f|_{Q_r^k})|f'(z_k)|)r^2}{\text{Area}(f(Q_r^k))} \tag{33}$$
$$< \frac{32(2+3\sqrt{2}|f'(z_k)|)}{|f'(z_k)|^2} < \frac{256}{K_0^2e^{2x_k}} + \frac{192\sqrt{2}}{K_0e^{x_k}}$$
$$\le \left(\frac{128}{K_0^2}\cdot\frac{1}{e^{3x/2}} + \frac{96\sqrt{2}}{K_0}\cdot\frac{1}{e^{x/2}}\right)\cdot\frac{x_{k+1}}{e^{x_k}}$$
$$\le \left(\frac{1}{4K_0^2} + \frac{12\sqrt{2}}{K_0}\right)\cdot\frac{x_{k+1}}{e^{x_k}}.$$

For all $k \in \mathbf{N}$ and $1 \le i \le d_k$, by (32) and (33), we have

$$\text{density}\Big(\bigcup \text{pack}(f^{\circ(k+1)}(E_{k,i})), f^{\circ(k+1)}(E_{k,i})\Big) \tag{34}$$
$$\ge \frac{\text{Area}(\bigcup\{Q_r \in \mathcal{Q}_r : Q_r \cap f^{\circ(k+1)}(E_{k,i}) \neq \emptyset)}{\text{Area}(f^{\circ(k+1)}(E_{k,i}))} - \frac{\text{Area}(B_1)+\text{Area}(B_2)}{\text{Area}(f^{\circ(k+1)}(E_{k,i}))}$$
$$> 1 - c_0\frac{x_{k+1}}{e^{x_k}} \ge 1 - c_1\frac{x_{k+1}}{e^{x_k}},$$

where

$$c_1 \ge c_0 := \frac{32\sqrt{2}}{K_0 r} + \frac{1}{4K_0^2} + \frac{12\sqrt{2}}{K_0}. \tag{35}$$

Comparing (26), we assume that $x^* > 0$ is a fixed constant such that

$$x^* \ge \max\{R_3, R_6, 6\log 2, 12 + 2\log c_1\}. \tag{36}$$

Moreover, we suppose that the sequence $\{x_k\}_{k\in\mathbf{N}}$ in (27) is chosen such that the initial point satisfies $x_0 = x \ge x^*$. Then, all the statements above are still true since $x^* \ge x'$.

By a straightforward induction, one can show that for all $k \in \mathbf{N}$ and $x \in \mathbf{R}$,

$$\exp^{\circ(k+1)}(x) \ge \exp(k)\exp(x).$$

Since $x_{k+1} = 2e^{x_k/2}$, we have

$$\frac{x_{k+1}}{e^{x_k}} = \frac{2}{e^{x_k/2}} = \frac{2}{\exp^{\circ(k+1)}(x/2)} \le \frac{2}{e^k}\cdot\frac{1}{e^{x/2}}. \tag{37}$$

On the other hand, by (36), we have $e^{x/2} \ge c_1e^6 > 6c_1e^4$ since $x \ge x^*$. Therefore,

$$c_1e^4\frac{x_{k+1}}{e^{x_k}} \le c_1e^4\cdot\frac{2}{e^k}\cdot\frac{1}{e^{x/2}} \le c_1e^4\cdot\frac{2}{e^{x/2}} < \frac{1}{3}. \tag{38}$$

Define $V := f(Q_r^k)$ and let $G := f^{-(k+1)} : V \to Q_0$ be the inverse of $f^{\circ(k+1)}|_{E_{k,i}}$, where $k \in \mathbf{N}$ and $1 \le i \le d_k$. By Lemma 2.2, Corollary 2.9 and (25), the distortion of G on V satisfies

$$L(G|_V) < \exp\left(\frac{2}{2-1}\right) = e^2. \tag{39}$$

From (3) and (39), we have

$$\begin{aligned}
&\mathrm{density}(\mathscr{E}_{k+1}, E_{k,i}) \\
&\quad = 1 - \mathrm{density}(E_{k,i} \backslash \mathscr{E}_{k+1}, E_{k,i}) \\
&\quad = 1 - \mathrm{density}(G(f^{\circ(k+1)}(E_{k,i} \backslash \mathscr{E}_{k+1})), G(f^{\circ(k+1)}(E_{k,i}))) \\
&\quad \geq 1 - L(G|_V)^2 \,\mathrm{density}\Big(f^{\circ(k+1)}(E_{k,i}) \backslash \bigcup \mathrm{pack}(f^{\circ(k+1)}(E_{k,i}), f^{\circ(k+1)}(E_{k,i})\Big) \\
&\quad \geq 1 - e^4\Big(1 - \mathrm{density}\Big(\bigcup \mathrm{pack}(f^{\circ(k+1)}(E_{k,i})), f^{\circ(k+1)}(E_{k,i})\Big)\Big).
\end{aligned}$$

Therefore, by (34) and (38), we have

$$\mathrm{density}(\mathscr{E}_{k+1}, E_{k,i}) \geq 1 - c_1 e^4 \frac{x_{k+1}}{e^{x_k}} \geq \frac{2}{3}, \tag{40}$$

where $k \in \mathbf{N}$ and $1 \leq i \leq d_k$. For all $k \in \mathbf{N}$, by setting

$$\rho_k := 1 - c_1 e^4 \frac{x_{k+1}}{e^{x_k}}, \tag{41}$$

it is easy to see that $\{\mathscr{E}_k\}_{k=0}^{\infty}$ satisfies the nesting conditions.

Define $E = \bigcap_{k=0}^{\infty} \mathscr{E}_k$. Recall that $A(f)$ is the fast escaping set of f defined in (24). Since every point $z \in E_{k,i}$ satisfies $f^{\circ j}(z) \in \Lambda(x_j)$ for $0 \leq j \leq k$ and $x_k \to +\infty$ as $k \to \infty$, it means that E is contained in the fast escaping set $A(f)$ by (27) and Corollary 2.13. According to Lemma 2.3, we have

$$\mathrm{density}(A(f), Q_0) \geq \mathrm{density}(E, Q_0) \geq \prod_{k=0}^{\infty} \rho_k.$$

Note that $\log(1-t) > -2t$ for $t \in (0, 1/2)$. By (38) and (41) we have

$$\begin{aligned}
\log\left(\prod_{k=0}^{\infty} \rho_k\right) &= \sum_{k=0}^{\infty} \log\left(1 - c_1 e^4 \frac{x_{k+1}}{e^{x_k}}\right) \geq -2 \sum_{k=0}^{\infty} c_1 e^4 \frac{x_{k+1}}{e^{x_k}} \\
&\geq -\frac{4c_1 e^4}{e^{x/2}} \sum_{k=0}^{\infty} \frac{1}{e^k} > -\frac{8c_1 e^4}{e^{x/2}}.
\end{aligned}$$

Since $e^{-t} \geq 1 - t$ for all $t \in \mathbf{R}$, we have

$$\mathrm{density}(A(f), Q_0) > \exp\left(-\frac{8c_1 e^4}{e^{x/2}}\right) \geq 1 - \frac{8c_1 e^4}{e^{x/2}} \tag{42}$$

for all $x \geq x^*$ and all square $Q_0 \subset \Lambda(x)$ with sides of length r.

THEOREM 3.1. *Let S be any horizontal strip of width 2π. Then the area of the complement of the fast escaping set of $f(z) = P(e^z)/e^z$ satisfies*

$$\operatorname{Area}(S \cap A(f)^c) \le (4\pi + 4r)\left(x^* + r + 8c_1 e^{4-x^*/2}\frac{r}{1-e^{-r/2}}\right) < \infty, \tag{43}$$

where r, c_1 and x^* are any positive constants satisfying (25), (35) *and* (36) *respectively.*

Proof. Define the half strip S_+ by

$$S_+ := \{z \in \mathbf{C} : 0 \le \operatorname{Im} z \le 2\pi \text{ and } \operatorname{Re} z \ge 0\}.$$

We take

$$m_0 = [x^*/r] + 1 \quad \text{and} \quad n_0 = [2\pi/r] + 1, \tag{44}$$

where $[x]$ denotes the integer part of $x \ge 0$. Recall that $Q_r^{m,n}$ is defined as

$$Q_r^{m,n} := \{z \in \mathbf{C} : mr \le \operatorname{Re} z \le (m+1)r \text{ and } nr \le \operatorname{Im} z \le (n+1)r\},$$

where $m, n \in \mathbf{Z}$. Since $Q_r^{m,n} \subset \Lambda(x^*)$ for all $m \ge m_0$, we get

$$\operatorname{density}(A(f), Q_r^{m,n}) > 1 - \frac{8c_1 e^4}{\exp(mr/2)} \tag{45}$$

for all $m \ge m_0$ by (42). So

$$\begin{aligned}
\operatorname{Area}(S_+ \cap A(f)^c) &\le \operatorname{Area}\left(\left(\bigcup_{m=0}^{\infty}\bigcup_{n=0}^{n_0} Q_r^{m,n}\right)\Big\backslash A(f)\right)\\
&\le \sum_{m=0}^{\infty}\sum_{n=0}^{n_0}\operatorname{Area}(Q_r^{m,n}\backslash A(f))\\
&\le \sum_{m=0}^{\infty}\sum_{n=0}^{n_0}(1 - \operatorname{density}(A(f), Q_r^{m,n}))\cdot \operatorname{Area}(Q_r^{m,n}).
\end{aligned}$$

By (44) and (45), we obtain

$$\begin{aligned}
\operatorname{Area}(S_+ \cap A(f)^c) &\le r^2\left(\sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0} 1 + \sum_{m=m_0}^{\infty}\sum_{n=0}^{n_0}\frac{8c_1 e^4}{\exp(mr/2)}\right)\\
&\le (2\pi + 2r)\left(x^* + r + 8c_1 e^{4-x^*/2}\frac{r}{1-e^{-r/2}}\right).
\end{aligned}$$

This means that $\operatorname{Area}(S_+ \cap A(f)^c) < \infty$ for every fixed $r > 0$ satisfying (25). Similarly, one can obtain

$$\operatorname{Area}(S_- \cap A(f)^c) \le (2\pi + 2r)\left(x^* + r + 8c_1 e^{4-x^*/2}\frac{r}{1-e^{-r/2}}\right),$$

where $S_- = \{z \in \mathbf{C} : 0 \le \operatorname{Im} z \le 2\pi \text{ and } \operatorname{Re} z \le 0\}$. Since $f(z) = f(z + 2\pi \mathrm{i})$, for any horizontal strip S of width 2π, we have

$$\operatorname{Area}(S \cap A(f)^c) \le (4\pi + 4r)\left(x^* + r + 8c_1 e^{4 - x^*/2} \frac{r}{1 - e^{-r/2}}\right).$$

This completes the proof of Theorem 3.1 and hence Theorem 1.1. □

3.2. Proof of Theorem 1.2. Consider the quadratic polynomial

$$P(z) = \frac{\alpha}{2} z^2 + \mathrm{i}\beta z - \frac{\alpha}{2}, \quad \text{where } \alpha \ne 0 \text{ and } \beta \in \mathbf{C}.$$

We then have

$$f(z) := \frac{P(e^z)}{e^z} = \frac{\alpha}{2} e^z + \mathrm{i}\beta - \frac{\alpha}{2} e^{-z}.$$

Note that $\alpha \sin(z + \beta)$ is conjugated by $z \mapsto \mathrm{i}(z + \beta)$ to $f(z)$. In order to prove Theorem 1.2, it is sufficient to prove the corresponding statements on f.

Now we collect all the needing constants in the proof. Note that the degree of P is $\deg(P) = N = 2$. By (25) we fix the choice of $r > 0$ by setting

$$r = 1/8.$$

By (15), we have $K_0 = |\alpha|/2$. From (35), we fix

$$c_1 = c_0 = \frac{536\sqrt{2}}{|\alpha|} + \frac{1}{|\alpha|^2}.$$

By (16), we have

$$R_3 = \log\left(2 + \frac{16K}{|\alpha|}\right), \quad \text{where } K = \max\{|\alpha|/2, |\beta|\}.$$

According to Lemma 2.8, we have

$$R_4 = \max\left\{1 + \frac{4(K+2)}{|\alpha|}, 1 + \frac{18K}{|\alpha|}\right\} \quad \text{and} \quad R_5 = \min\left\{\frac{|\alpha|}{8(K+1)}, \frac{1}{4}\sqrt{\frac{|\alpha|}{2K}}\right\}.$$

Since $K \ge |\alpha|/2 > 0$, we have

$$\frac{8(K+1)}{|\alpha|} > \frac{8K}{|\alpha|} \ge 4\sqrt{\frac{2K}{|\alpha|}}, \quad \frac{8(K+1)}{|\alpha|} = \frac{4K}{|\alpha|} + \frac{4(K+2)}{|\alpha|} > 1 + \frac{4(K+2)}{|\alpha|}$$

and

$$1 + \frac{18K}{|\alpha|} = 1 + \frac{16K}{|\alpha|} + \frac{K}{|\alpha|/2} \ge 2 + \frac{16K}{|\alpha|}.$$

Hence by (36), we can fix

$$x^* = \max\left\{\log\left(1 + \frac{18K}{|\alpha|}\right), \log\left(\frac{8(K+1)}{|\alpha|}\right), 6\log 2, 12 + 2\log c_1\right\}.$$

By Theorem 3.1, the proof of Theorem 1.2 is finished module the statement on the sine and cosine functions.

Let S be a vertical strip with width 2π. If $\alpha = 1$ and $\beta = 0$, then $K = 1/2$ and

$$r = 1/8, \quad c_1 = 536\sqrt{2} + 1 \quad \text{and} \quad x^* = 12 + 2\log(536\sqrt{2} + 1). \tag{46}$$

From (43) we have

$$\begin{aligned} &\mathrm{Area}(S \cap A(\sin z)^c) \\ &\quad \le \left(4\pi + \frac{1}{2}\right)\left(\frac{97}{8} + 2\log(536\sqrt{2} + 1) + \frac{1}{e^2 - e^{31/16}}\right) < 361. \end{aligned}$$

If $\alpha = 1$ and $\beta = \pi/2$, then $K = \pi/2$ and we still have (46). Also from (43) we have

$$\mathrm{Area}(S \cap A(\cos z)^c) < 361.$$

This finishes the proof of Theorem 1.2. □

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