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ON PEREZ DEL POZO'S LOWER BOUND OF WEIERSTRASS WEIGHT

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Abstract

Let V be a smooth projective curve over the complex number field with genus $g \geq 2$, and let σ be an automorphism on V such that the quotient curve $V/\langle\sigma\rangle$ has genus 0. We write d (resp., b) for the order of σ (resp., the number of fixed points of σ). When d and b are fixed, the lower bound of the (Weierstrass) weights of fixed points of σ was obtained by Perez del Pozo [7]. We obtain necessary and sufficient conditions for when the lower bound is attained.

1. Introduction

Let V be a smooth projective curve over the complex number field with genus $g \geq 2$, let σ be a nontrivial automorphism on V, and let d (resp., b) be the order of σ (resp., the number of fixed points of σ). If $(n_1, \dots, n_g)$ is the gap sequence of a point P on V, the *weight* of P, denoted by $w(P)$, is defined by $w(P) = \sum_{i=1}^{g}(n_i - i)$. Take a fixed point P of σ. According to A. L. Perez Del Pozo [7], if $b \geq 2$, then we have $w(P) \geq \underline{w}$, where

$$\underline{w} = \begin{cases} \dfrac{(d-1)(b-2)(b-4)}{8} & \text{if } b \geq 2 \text{ is even;} \\ \dfrac{(d-1)(b-3)^2}{8} & \text{if } b \geq 3 \text{ is odd.} \end{cases} \tag{1.1}$$

In this paper, we consider the case where the genus of the quotient curve $V/\langle\sigma\rangle$ is 0. In §3, we provide

(1) necessary and sufficient conditions (Proposition 5, Theorems 8, 10, 11, and 15) for whether such curve V has a fixed point P of σ with $w(P) = \underline{w}$.

We will see that when $b \geq 3$, if σ has a fixed point P such that $w(P) = \underline{w}$, then the d-cyclic covering $\pi : V \to V/\langle\sigma\rangle \cong \mathbf{P}^1$ must satisfy the following condition:

(∗) all the ramification points of π are total ramifications,

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i.e., around every ramification point of π, there is a local coordinate z such that π is expressed by $z \mapsto z^d$. As an intermediate result, in §2 we provide

(2) an algorithm (Theorem 3) to compute the gap sequences of the ramification points of π under the condition $(*)$.

Finally, in §4, we observe the case where $b = 1$, and provide

(3) the classification (Proposition 17) of those curves V such that the unique fixed point of σ has weight 1, 2, or 3.

2. Gap sequences at the fixed points

Let V be a curve of genus $g \geq 2$ with an automorphism σ of order d such that $V/\langle\sigma\rangle$ has genus 0. Let b be the number of fixed points of σ, and π be the d-cyclic covering $V \to V/\langle\sigma\rangle$. In this and next section, we assume that $b \geq 2$, i.e., σ fixes at least two points of V.

As we will see, for most part of our work, it suffices to consider the curve V under the assumption $(*)$. We write $N(d) = \{i \in \mathbf{N} : 1 \leq i \leq d-1, \gcd(d, i) = 1\}$. It is well-known that the curve V satisfying the condition $(*)$ has the following plane model:

$$\Gamma_1: \quad y^d = \prod_{j \in N(d)} \left(\prod_{k=1}^{s_j} (x - \lambda_{j,k})^j \right), \tag{2.1}$$

where the $\lambda_{j,k}$'s are mutually distinct, $s_j \geq 0$, and $\sum_{j \in N(d)} j s_j$ is divisible by d. Thus, we can write $sd = \sum_{j \in N(d)} j s_j$ for some positive integer s. Under the (x, y)-coordinate of Γ_1, the automorphism σ is given by $(x, y) \mapsto (x, \varepsilon y)$, and the d-cyclic covering $\pi : V \to \mathbf{P}^1$ is given by $(x, y) \mapsto x$, where $\varepsilon = \exp(2\pi\sqrt{-1}/d)$.

Note that for the curve given by (2.1), the number b of fixed points of σ is equal to the sum of the s_j's. Thus, by the Riemann-Hurwitz formula, we have

$$g = \frac{(b-2)(d-1)}{2}. \tag{2.2}$$

Since $g \geq 2$, we always have $b \geq 3$. When $b = 3$, we have $d \geq 5$; and when $b = 4$ or 5, we have $d \geq 3$.

For further discussions, we introduce some notations: for two integers i and j, let $q_{i,j}$ and $r_{i,j}$ denote the quotient and remainder for the division of ij by d. Namely, we have $ij = dq_{i,j} + r_{i,j}$ and $0 \leq r_{i,j} \leq d-1$. The following well-known fact will be used frequently in this paper:

Lemma 1 (see [6], Main Theorem for instance). *Let ρ be an arbitrary projective transformation of $\mathbf{P}^1$, let $\mu_{j,k} = \rho(\lambda_{j,k})$ for $j \in N(d)$ and $1 \leq k \leq s_j$, and let e be an arbitrary integer relatively prime to d. Then there exists a birational map γ*

from Γ_1 *to the following curve* Γ_e

$$\Gamma_e: \quad y^d = \prod_{j \in N(d)} \left(\prod_{k=1}^{s_j} (x - \mu_{j,k})^{r_{j,e}} \right),$$

such that $\gamma(\lambda_{j,k}) = \mu_{j,k}$ *for any* j *and* k.

For $j \in N(d)$ and $k \in \{1, \dots, s_j\}$, we write $P_{j,k}$ for the point on V corresponding to the $\lambda_{j,k}$ in the equation (2.1), write P_j for the divisor $\sum_{k=1}^{s_j} P_{j,k}$, and write $Q = Q_1 + \cdots + Q_d$ for the divisor on V over the point at infinite in $\mathbf{P}^1$, i.e., $Q = \pi^{-1}(\infty)$. Moreover, we write f_j for the rational function $(x - \lambda_{j,1}) \cdots (x - \lambda_{j,s_j})$ on V. Then we can compute the divisors of the following meromorphic functions and differential 1-form on V

$$\begin{aligned} &(x - \lambda_{j,k}) = dP_{j,k} - Q, \quad (f_j) = dP_j - s_j Q, \\ &(y) = \sum_{j \in N(d)} jP_j - sQ, \quad (dx) = \sum_{j \in N(d)} (d-1)P_j - 2Q. \end{aligned} \tag{2.3}$$

In what follows, we will construct a family of holomorphic 1-forms on V, and show that they form a basis of $H^0(V, K)$, where K is a canonical divisor on V. But we still need some materials: for $i \in \{1, \dots, d-1\}$, we define

$$\eta_i = \frac{dx}{y^i} \quad \text{and} \quad \omega_i = \eta_i \prod_{j \in N(d)} f_j^{q_{i,j}}.$$

Note that $\eta_1 = \omega_1$. Using (2.3), we can compute the divisors of η_i and ω_i:

$$(\eta_i) = \sum_{j \in N(d)} (d - ij - 1)P_j + (si - 2)Q,$$

$$(\omega_i) = \sum_{j \in N(d)} ((q_{i,j} + 1)d - 1 - ij)P_j + \left(si - 2 - \sum_{j \in N(d)} q_{i,j} s_j \right) Q.$$

We observe the coefficients of P_j's and Q in (ω_i):

$$\begin{aligned} \text{cof. of } P_j \text{ in } (\omega_i) &= (q_{i,j} + 1)d - 1 - (q_{i,j} d + r_{i,j}) \\ &= d - r_{i,j} - 1 \geq 0, \\ \text{cof. of } Q \text{ in } (\omega_i) &= -2 + \sum_{j \in N(d)} \left(\frac{ij}{d} - q_{i,j} \right) s_j \\ &= -2 + \sum_{j \in N(d)} \frac{r_{i,j} s_j}{d}. \end{aligned} \tag{2.4}$$

Now we write

$$l_i = 1 + \text{cof. of } Q \text{ in } (\omega_i) = -1 + \sum_{j \in N(d)} \frac{r_{i,j} s_j}{d}, \quad 1 \le i \le d-1.$$

By definition, all the l_i's are integers. Indeed, this can be checked directly:

$$\sum_{j \in N(d)} r_{i,j} s_j \equiv \sum_{j \in N(d)} ij s_j \equiv i \left(\sum_{j \in N(d)} j s_j \right) \equiv isd \equiv 0 \pmod{d}.$$

Moreover, the s_j's are not all zero, so all of the l_i's are nonnegative.

LEMMA 2. *Under the assumption* $(*)$, *for any fixed* $j \in N(d)$ *and any fixed* $k \in \{1, \dots, s_j\}$, *the following differential* 1*-forms give a basis of* $H^0(V, K)$:

$$\omega_i (x - \lambda_{j,k})^m, \quad 1 \le i \le d-1, \, 0 \le m \le l_i - 1.$$

Here, if $l_{i_0} = 0$ *for some* i_0, *then the condition for* m *becomes* $0 \le m \le -1$. *In this case, we ignore the corresponding* $\omega_{i_0}(x - \lambda_{j,k})^m$*'s, since there does not exist such* m.

Proof. The divisor of $(x - \lambda_{j,k})^m$ is equal to $mdP_{j,k} - mQ$. We have seen in (2.4) that all the coefficients of P_j's are nonnegative in ω_i, so they remain nonnegative in $\omega_i(x - \lambda_{j,k})^m$. On the other hand, since all the i's with $l_i = 0$ are ignored, the coefficient of Q is also nonnegative in ω_i. It remains nonnegative in $\omega_i(x - \lambda_{j,k})^m$ under the condition $0 \le m \le l_i - 1$. Therefore, the differential forms $\omega_i(x - \lambda_{j,k})^m$'s are all holomorphic. It is easy to see that they are **C**-linearly independent. Hence, it suffices to show that the number of these differential forms is equal to the genus of V, i.e., to show $\sum_{i=1}^{d-1} l_i = g$. For any integer k relatively prime to d, the remainders of $k, 2k, \dots, (d-1)k$ divided by d form a permutation of $1, 2, \dots, d-1$. Thus, we have $\sum_{i=1}^{d-1} r_{i,k} = \sum_{i=1}^{d-1} i = d(d-1)/2$. Therefore, from the definition of the l_i's and (2.2), we deduce that

$$\begin{aligned} \sum_{i=1}^{d-1} l_i &= -(d-1) + \sum_{i=1}^{d-1} \sum_{k \in N(d)} \frac{r_{i,k} s_k}{d} = -(d-1) + \sum_{k \in N(d)} \left(\sum_{i=1}^{d-1} r_{i,k} \right) \frac{s_k}{d} \\ &= -(d-1) + \sum_{k \in N(d)} \frac{d(d-1)}{2} \frac{s_k}{d} = (d-1) \left(-1 + \sum_{k \in N(d)} s_k/2 \right) = g, \end{aligned}$$

which completes the proof. □

THEOREM 3. *Under the assumption* $(*)$, *for any* $j \in N(d)$ *and any* $k \in \{1, \dots, s_j\}$, *the gap sequence at* $P_{j,k}$ *consists of the following integers*

$$(m+1)d - r_{i,j}, \quad 1 \le i \le d-1, \, 0 \le m \le l_i - 1.$$

That is, the gap sequence at $P_{j,k}$ *has the following form*

$$\begin{aligned}\{&d - r_{1,j},\ 2d - r_{1,j},\ \ldots,\ l_1 d - r_{1,j},\\ &d - r_{2,j},\ 2d - r_{2,j},\ \ldots,\ l_2 d - r_{2,j},\\ &\qquad\vdots\\ &d - r_{d-1,j},\ 2d - r_{d-1,j},\ \ldots,\ l_{d-1} d - r_{d-1,j}\}.\end{aligned}$$

Note that the lengths l_i of the rows above do not necessarily match each other. If $l_{i_0} = 0$ for some $i_0 \in \{1, 2, \ldots, d-1\}$, then the length of the i_0th rows is 0, *i.e., that rows does not occur. Moreover, we have*

$$w(P_{j,k}) = \frac{1}{2}\left(g(d-g-1) + d\sum_{i=1}^{d-1} l_i^2 - 2\sum_{i=1}^{d-1} l_i r_{i,j}\right).$$

Proof. On a smooth curve, a positive integer n is a gap at a point P if and only if there exists a holomorphic 1-form ω such that ω has a zero of degree $n-1$ at P. By the computations in previous section, the holomorphic form $\omega_i(x - \lambda_{j,k})^m$ has a zero of order $(m+1)d - r_{i,j} - 1$ at $P_{j,k}$. Since $0 \le r_{i,j} \le d-1$ and since $\gcd(d, j) = 1$, for fixed $j \in N(d)$ and fixed $k \in \{1, \ldots, s_k\}$, when i runs over $\{1, 2, \ldots, d-1\}$ and m runs over $\{0, 1, \ldots, l_i - 1\}$, the orders $((m+1)d - r_{i,j} - 1)$'s are mutually distinct. In the proof of Lemma 2, we have shown that the number of these mutually distinct orders is equal to g. Hence, the $((m+1)d - r_{i,j} - 1)$'s form the gap sequence at $P_{j,k}$, and a direct computation gives the weight at $P_{j,k}$. □

Remark 4. By Theorem 3, the gap sequences at $P_{j,k}$'s are independent of the choice of the parameters $\lambda_{j,k}$'s in the equation (2.1). We see that two ramification points with the same exponent always have the same gap sequence.

3. Fixed point whose weight attains the lower bound

Now we begin to discuss what kind of curves can attain the Perez Del Pozo's lower bound $\underline{\omega}$. First of all, when $b = 2$, 3 or 4, we have $\underline{w} = 0$, i.e., we are just looking for the non-Weierstrass fixed points of σ. In fact, when $b = 2$, 3 or 4, we have the following results of K. Yoshida [10]:

Proposition 5. *Let V be a smooth curve of genus $g \ge 2$ with an automorphism σ of order d such that $V/\langle\sigma\rangle$ has genus* 0. *Assume that σ has $b = 2$, 3 or 4 fixed points. Then one of these fixed points is non-Weierstrass if and only if the curve V is given by the following equations* (*up to transformations in Lemma* 1):

(1) *when $b = 2$*:

$$y^d = (x^2 - 1)x^{d-2}, \quad d \ge 6 \text{ is even},$$

in this case, both fixed points $(\pm 1, 0)$ are non-Weierstrass;

(2) *when* $b = 3$:

$$y^d = (x^2 - 1)x^{d-2}, \quad d \geq 5 \text{ is odd},$$

in this case, the fixed points $(\pm 1, 0)$ *are non-Weierstrass*;

(3) *when* $b = 4$:

$$y^d = x(x-1)^v (x+1)^{d-v} (x-\lambda)^{d-1}, \quad d \geq 3 \text{ and } \gcd(d, v) = 1,$$

in this case, all fixed points $(0,0)$, $(\pm 1, 0)$, $(\lambda, 0)$ *are non-Weierstrass.*

Remark 6. In (2) of the theorem, the gap sequence at the third fixed point $(0,0)$ of σ is $(1, 3, 5, \ldots, d-2)$, i.e., a hyperelliptic gap sequence. In fact, all of the curves in (1) and (2) are hyperelliptic, i.e., have gonality 2, while the curves in (3) have various gonalities (see [8] and [9]). Note that the curves in (2) and (3) also satisfy the condition $(*)$.

Proof. The "only if" part follows from [10, Theorem 1] directly, so it remains to check the "if" part. When $b = 2$, the genus of the curve given above is $d/2 - 1$. The holomorphic 1-forms $x^{m-2}\,dx/y^m$ $(d/2 + 1 \leq m \leq d - 1)$ give a basis of $H^0(V, K)$, and their order at $(\pm 1, 0)$ are $0, 1, \ldots, g - 1$. When $b = 3$ or 4, the assertion can be checked by Theorem 3 directly. □

Now we consider the case where $b \geq 5$. The next lemma follows from [7, Theorems 1 and 2]:

LEMMA 7. *Let* P *be a fixed point of* σ. *If* $b \geq 5$, *then we have* $w(P) = \underline{w}$ *if and only if the gap sequence of* P *has the following form*:

CASE (i). b *is even* (*let* $l = b/2 - 1$).

$$\begin{aligned} &\{1,\ 2,\ \ldots,\ d-1, \\ &\ d+1,\ d+2,\ \ldots,\ 2d-1, \\ &\qquad\vdots \\ &\ (l-1)d+1,\ (l-1)d+2,\ \ldots,\ ld-1\} \end{aligned} \tag{3.1}$$

CASE (ii). b *is odd* (*let* $l = (b-3)/2$ *and* $q = (d-1)/2$).

$$\begin{aligned} &\{1,\ 2,\ \ldots,\ d-1, \\ &\ d+1,\ d+2,\ \ldots,\ 2d-1, \\ &\qquad\vdots \\ &\ (l-1)d+1,\ (l-1)d+2,\ \ldots,\ ld-1, \\ &\ ld+1,\ ld+2,\ \ldots,\ ld+q\} \end{aligned} \tag{3.2}$$

In fact, it is implicit in Perez Del Pozo's proof that when $b \geq 5$, if $w(P) = \underline{w}$, then the condition $(*)$ defined in §1 must be satisfied. Thus, we only discuss the curves given in the form (2.1). Firstly, from the proof of Lemma 2 and (2.2), we have seen that

$$\sum_{i=1}^{d-1} l_i = g = \frac{(d-1)(b-2)}{2}. \tag{3.3}$$

That is, if d and b are fixed, then the genus g is determined, and so is $\sum_{i=1}^{d-1} l_i$. We also see from (3.3) that when b is odd, then so is d.

Theorem 8. *Let V be a smooth curve V of genus $g \geq 2$ with an automorphism σ of order d such that $V/\langle\sigma\rangle$ has genus* 0. *Assume that σ has b fixed points and that $b \geq 6$ is even. If there is a fixed point P of σ such that $w(P) = \underline{w}$, then V is given in the form* (2.1). *Moreover, for a curve given in* (2.1) *with such d and b, the following conditions are equivalent*:

(a) *one of $P_{j,k}$'s has the weight $\underline{w}$*;

(b) *either $d = 2$, or $d \geq 3$ and $s_j = s_{d-j}$ for any $j \in N(d)$*;

Example 9. As an example, we take $d = 5$ and $b = 6$, and give all the curves satisfying the equivalent conditions of Theorem 8. By the definition of b and the s_i's, we have $s_1 + s_2 + s_3 + s_4 = b = 6$. By the condition (b), we have $s_1 = s_4$ and $s_2 = s_3$. Thus, we obtain $(s_1, s_2, s_3, s_4) = (3,0,0,3)$, $(2,1,1,2)$, $(1,2,2,1)$, or $(0,3,3,0)$. But a curve with $(s_1, s_2, s_3, s_4) = (3,0,0,3)$ (resp., $(2,1,1,2)$) can be transformed to a curve with $(s_1, s_2, s_3, s_4) = (0,2,2,0)$ (resp., $(1,2,2,1)$) by Lemma 1. Hence, when $d = 5$ and $b = 6$, there are two families of curves satisfying the equivalent conditions of Theorem 8:

$$y^5 = x(x^2-1)(x-\lambda_1)^4(x-\lambda_2)^4(x-\lambda_3)^4,$$
$$y^5 = x(x^2-1)^2(x-\lambda_1)^3(x-\lambda_2)^3(x-\lambda_3)^4.$$

Theorem 10. *Using the notations and assumptions in Theorem* 8, *for a curve given in* (2.1) *with such d and b, the following conditions are also equivalent to the condition* (a):

(a′) *all of $P_{j,k}$'s have the weight $\underline{w}$*;

(c) *all of the l_j's are equal to each other*;

(c′) *all of the l_j's are equal to $l = b/2 - 1$.*

Proof of Theorems 8 *and* 10. The part before "moreover" in Theorem 8 follows from the discussions above, so it only remains to show the equivalence of the conditions (a), (a′), (b), (c) and (c′).

Firstly, we see that (c) $\Rightarrow$ (c′) follows from (3.3) directly, (c′) $\Rightarrow$ (a′) follows from Theorem 3, and (a′) $\Rightarrow$ (a) is obvious. Hence, it suffices to show (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). Moreover, when $d = 2$, the assertion is just the well-known fact about hyperelliptic curves. Hence, in what follows, we may assume that $d \geq 3$.

(a) $\Rightarrow$ (b). Suppose that $w(P_{j_0,k}) = \underline{w}$ for some $j_0 \in N(d)$. Then by Lemma 7, the gap sequence of $P_{j_0,k}$ must have the form (3.1). Review that the d-cyclic covering $\pi : V \to \mathbf{P}^1$ corresponds to an automorphism $\sigma : (x, y) \mapsto (x, \varepsilon y)$, where $\varepsilon = \exp(2\pi\sqrt{-1}/d)$. We want to observe the action of σ on $H^0(V, K)$, which is given by $\sigma(\omega) = \omega \circ \sigma^{-1}$. By a theorem of J. Lewittes ([4, Theorem 5]), this action can be expressed by the gap sequence of any fixed point of σ. More precisely, since σ^{-1} is expressed by $(x, y) \mapsto (x, \varepsilon^{-1} y)$, around the point $P_{j_0,k}$, there is a local coordinate ζ such that $\sigma^{-1}(\zeta) = \varepsilon^{j_0'}\zeta$, where $j_0 j_0' \equiv -1 \pmod d$ (cf. [6]). Note that $(\varepsilon^{j_0'}, \varepsilon^{2j_0'}, \ldots, \varepsilon^{(d-1)j_0'})$ is a permutation of $(\varepsilon, \varepsilon^2, \ldots, \varepsilon^{d-1})$. Hence, by taking the basis of $H^0(V, K)$ given in Lemma 2, the representation of σ on $H^0(V, K)$ is the following $g \times g$ diagonal matrix

$$M = \operatorname{diag}(\underbrace{\varepsilon^1, \varepsilon^1, \ldots, \varepsilon^1}_{l}, \underbrace{\varepsilon^2, \varepsilon^2, \ldots, \varepsilon^2}_{l}, \ldots, \underbrace{\varepsilon^{d-1}, \varepsilon^{d-1}, \ldots, \varepsilon^{d-1}}_{l}).$$

In particular, the trace of M is given by

$$\operatorname{tr}(M) = l(\varepsilon + \varepsilon^2 + \cdots + \varepsilon^{d-1}) = -l = 1 - \frac{1}{2} \sum_{j \in N(d)} s_j.$$

On the other hand, around every point $P_{j,k}$, there is a local coordinate ζ such that $\sigma^{-1}(\zeta) = \varepsilon^{j'}\zeta$, where j' is the integer with $1 \leq j' \leq d-1$ and $jj' \equiv -1 \pmod d$ (cf. [6]). Note that j' is also contained in $N(d)$. Hence, by the Eichler trace formula (see [2, V.2.9], for instance), we have

$$\operatorname{tr}(M) = 1 + \sum_{j \in N(d)} \frac{\varepsilon^{j'} s_j}{1 - \varepsilon^{j'}}.$$

Combining the two formulas above, we deduce that

$$0 = \sum_{j \in N(d)} \left(\frac{\varepsilon^{j'}}{1 - \varepsilon^{j'}} + \frac{1}{2} \right) s_j = \sum_{j \in N(d)} \frac{s_j(1 + \varepsilon^{j'})}{2(1 - \varepsilon^{j'})} = \frac{\sqrt{-1}}{2} \sum_{j \in N(d)} s_j \cot \frac{j'\pi}{d}.$$

Let $\lceil d/2 \rceil$ be the smallest integer greater than $d/2$, and let $M(d) = \{j \in \mathbf{N} : 1 \leq j \leq \lceil d/2 \rceil - 1, \gcd(d, j) = 1\}$. Since $j \in N(d)$ if and only if $d - j \in N(d)$, and since $d/2 \notin N(d)$ when d is even, the set $M(d)$ consists of a half of the integers in $N(d)$, and we have

$$N(d) = M(d) \cup \{d - j : j \in M(d)\}.$$

If $jj' \equiv -1 \pmod d$, then $(d - j)(d - j') \equiv jj' \equiv -1 \pmod d$, so we have $(d - j)' = d - j'$. From the above equation, we deduce that

$$
\begin{aligned}
(3.4) \qquad 0 &= \sum_{j \in M(d)} s_j \cot \frac{j'\pi}{d} + \sum_{j \in M(d)} s_{d-j} \cot \frac{(d-j)'\pi}{d} \\
&= \sum_{j \in M(d)} s_j \cot \frac{j'\pi}{d} + s_{d-j} \cot \frac{(d-j')\pi}{d} \\
&= \sum_{j \in M(d)} (s_j - s_{d-j}) \cot \frac{j'\pi}{d}.
\end{aligned}
$$

One knows that when j runs over $N(d)$, so does j'. Furthermore, we have $j_1 + j_2 = d$ if and only if $j_1' + j_2' = d$, and if and only if $\cot(j_1\pi/d) + \cot(j_2\pi/d) = 0$. Since the set $M(d)$ does not contain two integers with the sum d, neither does the set $\{j' : j \in M(d)\}$. Hence, for every $j_1, j_2 \in N(d)$ with $j_1 + j_2 = d$, exactly one of $\cot(j_1\pi/d)$ and $\cot(j_2\pi/d)$ is contained in the set $\{\cot(j'\pi/d)\}_{j \in M(d)}$. Namely, the set $\{|\cot(j'\pi/d)|\}_{j \in M(d)}$ is a permutation of the set $\{\cot(j\pi/d)\}_{j \in M(d)}$.

By Chowla' theorem (see [1] for the original statement, and see [3] and the references there for various generalizations), the set $\{\cot(j\pi/d)\}_{j \in M(d)}$ is **Q**-linear independent. Then from the discussions in the previous paragraph, we see that the set $\{\cot(j'\pi/d)\}_{j \in M(d)}$ is also **Q**-linear independent. Since the s_j's are integers, we conclude from (3.4) that $s_j - s_{d-j} = 0$ for any $j \in M(d)$, i.e., $s_j = s_{d-j}$ for any $j \in N(d)$.

(b) $\Rightarrow$ (c). By the definition of the l_i's, to prove that all of them are equal, it suffices to show that $\sum_{j \in N(d)} (r_{i_1,j} - r_{i_2,j}) s_j = 0$ for any $i_1, i_2 \in \{1, \ldots, d-1\}$. Since $s_j = s_{d-j}$, using the notation $M(d)$ as above, we have

$$
(3.5) \qquad \sum_{j \in N(d)} (r_{i_1,j} - r_{i_2,j}) s_j = \sum_{j \in M(d)} ((r_{i_1,j} - r_{i_2,j}) s_j + (r_{i_1,d-j} - r_{i_2,d-j}) s_j).
$$

Since $r_{i,d-j} = d - r_{i,j}$ for any integers i and j not divisible by d, we obtain

$$
\begin{aligned}
(3.6) \qquad \sum_{j \in N(d)} (r_{i_1,j} - r_{i_2,j}) s_j &= \sum_{j \in M(d)} (r_{i_1,j} - r_{i_2,j} + r_{i_1,d-j} - r_{i_2,d-j}) s_j \\
&= \sum_{j \in M(d)} (r_{i_1,j} - r_{i_2,j} + d - r_{i_1,j} - d + r_{i_2,j}) s_j = 0.
\end{aligned}
$$

Hence, all the l_i's are the same. □

THEOREM 11. *Let V be a smooth curve V of genus $g \geq 2$ with an automorphism σ of order d such that $V/\langle\sigma\rangle$ has genus* 0. *Assume that σ has b fixed points and that $b \geq 5$ is odd. If there is a fixed point P of σ such that $w(P) = \underline{w}$, then V is given in the form* (2.1), *and hence d is odd. Moreover, for a curve given in* (2.1) *with such d and b, the following conditions are equivalent*:

(a) *one of $P_{j,k}$'s has the weight $\underline{w}$*;

(b) *when* $d = 3$, *we have* $s_1 = s_2 - 3$ *or* $s_2 = s_1 - 3$; *when* $d \geq 5$, *there exist* $j_1, j_2 \in N(d)$ *such that* $j_1 + 2j_2 = d$ *or* $2d$, *and we have* $s_{j_1} - 1 = s_{d-j_1}$, $s_{j_2} - 2 = s_{d-j_2}$ *and* $s_j = s_{d-j}$ *for* $j \in N(d) \backslash \{j_1, j_2, d-j_1, d-j_2\}$;

(b$'$) *when* $d = 3$, *after a transformation given in Lemma* 1, *we have* $s_1 = s_2 - 3$; *when* $d \geq 5$, *after a transformation given in Lemma* 1, *we have* $s_1 = s_{d-1} - 2$, $s_2 = s_{d-2} + 1$, *and* $s_j = s_{d-j}$ *for any* $j \in N(d) \backslash \{1, 2, d-2, d-1\}$.

Remark 12. Since d is odd, the set $N(d)$ contains 1, 2, $d-2$, $d-1$, which makes (b$'$) of Theorem 11 meaningful. Moreover, when $d = 3$ and $s_1 = s_2 - 3$ (resp., $s_2 = s_1 - 3$), the minimum weight $\underline{w}$ is taken by the $P_{2,k}$'s (resp., $P_{1,k}$'s). When $d \geq 5$, if V is given in the form of (b), then $\underline{w}$ is taken by the $P_{j_2,k}$'s; and if V is given in the form of (b$'$), then $\underline{w}$ is taken by the $P_{d-1,k}$'s.

Remark 13. When the conditions of Theorem 11 hold, we have

$$\underline{w} \leq w(P_{j,k}) \leq \underline{w} + \frac{(d-1)^2}{4}$$

for any ramification point $P_{j,k}$. If the curve is written in the form of condition (b$'$), then the right equality holds when $s_1 > 0$ and $j = 1$.

Example 14. As another example, we take $d = 5$ and $b = 5$, and give all the curves satisfying the equivalent conditions of Theorem 11. By the definition of b and the s_i's, we have $s_1 + s_2 + s_3 + s_4 = b = 5$. By the condition (b$'$), we have $s_1 = s_4 - 2$ and $s_2 = s_3 + 1$. Thus, we obtain $(s_1, s_2, s_3, s_4) = (1, 1, 0, 3)$ or $(0, 2, 1, 2)$. Hence, when $d = 5$ and $b = 5$, there are two families of curves satisfying the equivalent conditions of Theorem 11:

$$y^5 = x^4(x^2-1)^4(x-\lambda_1)(x-\lambda_2)^2,$$
$$y^5 = x^3(x^2-1)^4(x-\lambda_1)^2(x-\lambda_2)^2,$$

where the first family of curves takes the weight $\underline{w} = 2$ at $(\pm 1, 0)$ and $(0, 0)$, and the second one at $(\pm 1, 0)$. And the second family of curves takes the weight $\underline{w} + (d-1)^2/4 = 6$ at $(\lambda_1, 0)$.

Theorem 15. *Using the notations and assumptions in Theorem* 11, *for a curve given in* (2.1) *with such d and b*, *the following conditions are also equivalent to the condition* (a):

(a$'$) *at least two of* $P_{j,k}$*'s have the weight* $\underline{w}$;

(c) *there exists* $k \in N(d)$ *such that* $l_{r_{1,k}} = \cdots = l_{r_{q,k}} = l + 1$ *and* $l_{r_{q+1,k}} = \cdots = l_{r_{d-1,k}} = l$, *where* $l = (b-3)/2$ *and* $q = (d-1)/2$;

(c$'$) *after a transformation given in Lemma* 1, *we have* $l_1 = \cdots = l_q = l + 1$ *and* $l_{q+1} = \cdots = l_{d-1} = l$.

Remark 16. The condition (c) or (c′) of Theorem 15 implies that one half of the l_i's are equal to l, and the other half are equal to $l+1$. But unlike Theorem 8, the converse is not true. For instance, let V be the curve with $d=7$ and $(s_1,\dots,s_6)=(2,1,0,1,0,1)$. We can compute that $(l_1,\dots,l_6)=(1,1,2,1,2,2)$, and thereby the five fixed points have the weight 4, 4, 4, 4, 11, respectively. But $\underline{w}=3$ when $d=7$ and $b=5$.

Proof of Theorems 11 *and* 15. As Theorems 8 and 10, we only need to show the equivalence of the conditions (a), (a′), (b), (b′), (c) and (c′).

Firstly, we see that (a′) ⇒ (a), (b′) ⇒ (b), and (c′) ⇒ (c) are obvious, and (c′) ⇒ (a) follows from Theorem 3 directly. Note that the transformation given in Lemma 1 fixes every $P_{j,k}$. Hence, when $d=3$, by taking $e=1$ or 2 in Lemma 1, we get (b) ⇒ (b′). Similarly, when $d\ge 5$, by taking e to be the integer such that $ej_2\equiv d-1 \pmod d$ in Lemma 1, we still obtain (b) ⇒ (b′). By taking e to be the integer such that $ek\equiv d-1 \pmod d$ in Lemma 1, we obtain (c) ⇒ (c′). Hence, it suffices to show (a) ⇒ (b′), (b′) ⇒ (c′), and (a) ⇒ (a′).

(a) ⇒ (b′). If some $w(P_{j_0,k})=\underline{w}$ for some $j_0\in N(d)$, then by taking the e in Lemma 1 to be the integer such that $ej_0\equiv -1 \pmod d$, we can transform the exponent of this $P_{j_0,k}$ from j_0 to $d-1$. We know that a birational transformation does not change the gap sequence and the weight of a point. Hence, we may suppose that $w(P_{d-1,k})=\underline{w}$ on V. Then by Lemma 7, the gap sequence at this $P_{d-1,k}$ must have the form (3.2).

We still write σ for the automorphism $(x,y)\mapsto(x,\varepsilon y)$ on V. Note that the local expression of σ^{-1} around $P_{d-1,k}$ is $\sigma^{-1}(\zeta)=\varepsilon\zeta$ (cf. [6]). Similarly as the "(a) ⇒ (b)" part in the proof of Theorem 8, by taking the basis of $H^0(V,K)$ given in Lemma 2, the representation of σ on $H^0(V,K)$ is the following $g\times g$ diagonal matrix

$$M=\mathrm{diag}(\underbrace{\varepsilon^1,\dots,\varepsilon^1}_{l+1},\dots,\underbrace{\varepsilon^q,\dots,\varepsilon^q}_{l+1},\underbrace{\varepsilon^{q+1},\dots,\varepsilon^{q+1}}_{l},\dots,\underbrace{\varepsilon^{2q},\dots,\varepsilon^{2q}}_{l}).$$

In particular, the trace of M is equal to

$$\mathrm{tr}(M)=l(\varepsilon+\varepsilon^2+\cdots+\varepsilon^{d-1})+\varepsilon+\varepsilon^2+\cdots+\varepsilon^q=\frac{3}{2}-\frac{1}{2}\sum_{j\in N(d)}s_j+\sum_{j=1}^{q}\varepsilon^j.$$

On the other hand, by the Eichler trace formula, we have

$$\mathrm{tr}(M)=1+\sum_{j\in N(d)}\frac{\varepsilon^{j'}s_j}{1-\varepsilon^{j'}},$$

where j' expresses the integer such that $1\le j'\le d-1$ and $jj'\equiv -1 \pmod d$. Combining the two above formulas, we deduce that

$$0=-\frac{1}{2}-\sum_{j=1}^{q}\varepsilon^j+\frac{\sqrt{-1}}{2}\sum_{j\in N(d)}s_j\cot\frac{j'\pi}{d}. \tag{3.7}$$

When $d = 3$, the equation (3.7) yields the condition (b′) directly. When $d \geq 5$, we note that

$$\sum_{j=1}^{q} \varepsilon^j = \frac{\varepsilon}{1-\varepsilon} - \frac{\varepsilon^{q+1}}{1-\varepsilon} = -\frac{1}{2} + \frac{\sqrt{-1}}{2}\left(\cot\frac{\pi}{d} + \csc\frac{\pi}{d}\right).$$

We consider the twice of the imaginary part of the right side of (3.7). Using a similar calculation as in (3.4), we obtain

$$\begin{aligned} 0 &= \sum_{j \in N(d)} s_j \cot\frac{j'\pi}{d} - \left(\cot\frac{\pi}{d} + \csc\frac{\pi}{d}\right) \\ &= \sum_{j \in M(d)} (s_j - s_{d-j}) \cot\frac{j'\pi}{d} - \left(\cot\frac{\pi}{d} + \csc\frac{\pi}{d}\right), \end{aligned}$$

where $M(d)$ is defined in the proof of Theorem 8. Since $d \geq 5$, both 1 and 2 are contained in $M(d)$, so we deduce from the above equation that

$$\begin{aligned} 0 = {} & \sum_{j \geq 3, j \in M(d)} (s_j - s_{d-j}) \cot\frac{j'\pi}{d} + \left((s_1 - s_{d-1}) \cot\frac{1'\pi}{d} - 2\cot\frac{\pi}{d}\right) \\ & + \left((s_2 - s_{d-2}) \cot\frac{2'\pi}{d} + \cot\frac{\pi}{d} - \csc\frac{\pi}{d}\right). \end{aligned} \tag{3.8}$$

Since $d - 1 \equiv -1 \pmod d$, i.e., $1 = (d-1)'$, we have

$$\begin{aligned} & (s_1 - s_{d-1}) \cot\frac{1'\pi}{d} - 2\cot\frac{\pi}{d} \\ & \quad = (s_1 - s_{d-1}) \cot\frac{1'\pi}{d} + 2\cot\frac{(d-1)\pi}{d} = (s_1 - s_{d-1} + 2) \cot\frac{1'\pi}{d}. \end{aligned}$$

Since $2q = d - 1 \equiv -1 \pmod d$, i.e., $2' = q$, we have

$$\begin{aligned} & (s_2 - s_{d-2}) \cot\frac{2'\pi}{d} + \cot\frac{\pi}{d} - \csc\frac{\pi}{d} \\ & \quad = (s_2 - s_{d-2} - 1) \cot\frac{2'\pi}{d} + \cot\frac{q\pi}{d} + \left(\cot\frac{\pi}{d} - \csc\frac{\pi}{d}\right) \\ & \quad = (s_2 - s_{d-2} - 1) \cot\frac{2'\pi}{d} + \cot\frac{q\pi}{d} - \tan\frac{\pi}{2d} = (s_2 - s_{d-2} - 1) \cot\frac{2'\pi}{d}. \end{aligned}$$

Substituting the above two equations into (3.8), we conclude that

$$0 = (s_1 - s_{d-1} + 2) \cot\frac{1'\pi}{d} + (s_2 - s_{d-2} - 1) \cot\frac{2'\pi}{d} + \sum_{j \geq 3, j \in M(d)} (s_j - s_{d-j}) \cot\frac{j'\pi}{d}.$$

Now similarly as the "(a) $\Rightarrow$ (b)" part in the proof of Theorem 8, the **Q**-linear independence of $\{\cot(j'\pi/d)\}_{j \in M(d)}$ yields the condition (b′).

(b′) ⇒ (c′). When $d = 3$, this can be checked directly, so we assume $d \geq 5$. Similarly as the "(b) ⇒ (c)" part in the proof of Theorem 10, for two distinct $i_1, i_2 \in N(d)$, we need to compute

$$\Delta := (l_{i_1} - l_{i_2})d = \sum_{j \in N(d)} (r_{i_1,j} - r_{i_2,j})s_j.$$

Since $s_j = s_{d-j}$ for any $j \in \{3, \dots, d-3\}$, using a similar calculation as (3.5) and (3.6), we deduce that

$$\sum_{j \in N(d) \setminus \{1,2,d-2,d-1\}} (r_{i_1,j} - r_{i_2,j})s_j = 0.$$

Since $s_{d-1} = s_1 + 2$ and $r_{i,d-1} = d - i$, we have

$$\begin{aligned}(r_{i_1,1} - r_{i_2,1})s_1 &+ (r_{i_1,d-1} - r_{i_2,d-1})s_{d-1} \\ &= (i_1 - i_2)s_1 + (i_2 - i_1)(s_1 + 2) = 2i_2 - 2i_1.\end{aligned}$$

Since $s_{d-2} = s_2 - 1$ and $r_{i_*,2} + r_{i_*,d-2} = d$, we have

$$\begin{aligned}(r_{i_1,2} - r_{i_2,2})s_2 &+ (r_{i_1,d-2} - r_{i_2,d-2})s_{d-2} \\ &= (r_{i_1,2} - r_{i_2,2})s_2 + (r_{i_1,d-2} - r_{i_2,d-2})(s_2 - 1) = r_{i_1,2} - r_{i_2,2}.\end{aligned}$$

Combining the three formulas above, we obtain

$$\Delta = 2i_2 - 2i_1 + r_{i_1,2} - r_{i_2,2}.$$

Now taking $i_1 = 1$ and $i_2 = i$, we have

$$d(l_1 - l_i) = 2i - 2 + r_{1,2} - r_{i,2} = 2i - r_{i,2}.$$

Since $r_{i,2}$ is the remainder of $2i$ divided by d, we see that $l_1 - l_i = 0$ if $1 \leq i \leq q$ and $l_1 - l_i = 1$ if $q + 1 \leq i \leq 2q$. By (3.3), we must have $l_1 = \cdots = l_q = l + 1$ and $l_{q+1} = \cdots = l_{2q} = l$.

(a) ⇒ (a′). We have shown (a) ⇒ (b′) ⇒ (c′) ⇒ (a). By observing the condition (b′), we can see that the number of the $P_{j,k}$'s with the weight $\underline{w}$ is at least 2, which gives (a) ⇒ (a′). □

4. The case where $b = 1$

We still consider a curve V of genus $g \geq 2$ with an automorphism σ of order d such that $V/\langle\sigma\rangle$ has genus 0. In this section, we discuss the case where σ has a unique fixed point P. First of all, we conclude from [10, Theorem 1] that P must be a Weierstrass point, i.e., $w(P) \geq 1$.

Proposition 17. *Let V be a smooth curve of genus $g \geq 2$ with an automorphism σ of order d such that $V/\langle\sigma\rangle$ has genus 0. Assume that σ has a unique fixed point P.*

(1) *We have* $w(P) = 1$ *if and only if the curve* V *is given by the following equation* (*up to transformations in Lemma* 1)

$$V_0\colon \quad y^{10} = x(x-1)^4(x+1)^5.$$

(2) *We have* $w(P) = 2$ *if and only if the curve* V *is given by either of the following two equations* (*up to transformations in Lemma* 1)

$$V_1\colon \quad y^6 = x(x-1)^3(x+1)^4(x-\lambda)^4,$$

$$V_2\colon \quad y^{12} = x(x-1)^3(x+1)^8.$$

(3) *We have* $w(P) = 3$ *if and only if the curve* V *is given by either of the following two equations* (*up to transformations in Lemma* 1)

$$V_3\colon \quad y^{12} = x(x-1)^2(x+1)^9,$$

$$V_4\colon \quad y^{14} = x(x-1)^6(x+1)^7.$$

Proof. Suppose that $w(P) = 1$, i.e., P is a normal Weierstrass point. We see from [10, Theorem 2] that $g = 2$ and $d = 10$. Using the Riemann-Hurwitz formula and computing the rotation numbers, the only possible curve is V_0. We can show that the weight of the fixed point $(0,0)$ is indeed 1.

Now we suppose that $w(P) = 2$. We see from [11] that $(g,d) = (3,6)$ or $(3,12)$. Using the Riemann-Hurwitz formula, we can obtain the curves V_1 and V_2. It seems that the article [11] is not easily found, so we give a proof that $(g,d) = (3,6)$ or $(3,12)$.

Since $w(P) = 2$, the gap sequence at P have two possible types: $(1,2,\dots,g-1,g+2)$ and $(1,2,\dots,g-2,g,g+1)$. We can compute the trace $\mathrm{tr}(M)$ of matrix representation M of σ^{-1} on $H^0(V,K)$ for both types.

Case 1: the gap sequence at P is $(1,2,\dots,g-1,g+2)$. By replacing σ by some σ^k with k relatively prime to d, we may assume that the rotation number of σ at P is equal to 1. By the Eichler trace formula, we have $\mathrm{tr}(M) = 1/(1-\varepsilon)$, where $\varepsilon = \exp(2\pi\sqrt{-1}/d)$. On the other hand, by the theorem of J. Lewittes, we have

$$\mathrm{tr}(M) = \varepsilon + \varepsilon^2 + \cdots + \varepsilon^{g-1} + \varepsilon^{g+2} = (\varepsilon - \varepsilon^g)/(1-\varepsilon) + \varepsilon^{g+2}.$$

Combining the two formulas of $\mathrm{tr}(M)$, we obtain

$$\varepsilon^{g+3} - \varepsilon^{g+2} + \varepsilon^g - \varepsilon + 1 = 0.$$

By taking the complex conjugate ($\bar{\varepsilon} = \varepsilon^{-1}$), we have

$$\varepsilon^{g+3} - \varepsilon^{g+2} + \varepsilon^3 - \varepsilon + 1 = 0.$$

It follows that $\varepsilon^g = \varepsilon^3$, so we obtain $g \equiv 3 \pmod{d}$. Substituting this in the above equation, we have

$$\varepsilon^6 - \varepsilon^5 + \varepsilon^3 - \varepsilon + 1 = (\varepsilon^4 - \varepsilon^2 + 1)(\varepsilon^2 - \varepsilon + 1) = 0.$$

Considering the roots of the equation $(x^4 - x^2 + 1)(x^2 - x + 1) = 0$, we see that

(4.1) $\quad d = 6$ and $g \equiv 3 \pmod 6$, or $d = 12$ and $g \equiv 3 \pmod{12}$.

We remark that d is a non-gap at P, since $V/\langle\sigma\rangle \cong \mathbf{P}^1$. Since the gap sequence at P is $(1, 2, \dots, g-1, g+2)$, we have

(4.2) $\quad d = g$, or $d = g+1$, or $d \geq g+3$.

Combining (4.1) with (4.2), we obtain $(g, d) = (3, 6)$ or $(3, 12)$. Using the Riemann-Hurwitz formula, we see that there are only two curves V_1 and V_2. We can show that the weight of the fixed point $(0, 0)$ on V_1 and V_2 is indeed 2.

Case 2: the gap sequence at P is $(1, 2, \dots, g-2, g, g+1)$. Using the same calculation as in the previous case, we deduce that $(g, d) = (2, 12)$. But by the Riemann-Hurwitz formula, we see that there does not exist such a curve.

Finally we suppose that $w(P) = 3$. Either from [12] or from a similar calculation as that in the case where $w(P) = 2$, we can conclude that $(g, d) = (4, 12)$ or $(3, 14)$, then the curve must be V_3 or V_4. □

Remark 18. In the following table, we give some examples for the case in which $b = 1$ and $w(P) \geq 4$.

Curves	g	$w(P)$	Gap seq. at P
$y^{15} = x(x-1)^5(x+1)^9$ $y^5 = x(x^2-1)^2(x-\lambda)^3(x-\mu)^4$	4 5	4 4	$\{1, 2, 4, 7\}$ $\{1, 2, 4, 5, 7\}$
$y^{10} = x(x-1)^5(x+1)^6(x-\lambda)$	6	5	$\{1, 2, 3, 4, 7, 9\}$
$y^6 = x(x-1)^2(x+1)^3(x-\lambda)^3(x-\mu)^3$	4	6	$\{1, 3, 5, 7\}$
$y^{12} = x(x-1)^4(x+1)^9(x-\lambda)^{10}$	8	7	$\{1, 2, 3, 4, 5, 7, 10, 11\}$
$y^{16} = x(x-1)^6(x+1)^9$	7	8	$\{1, 2, 3, 4, 6, 9, 11\}$
$y^{14} = x(x-1)^7(x+1)^8(x-\lambda)^{12}$	9	9	$\{1, 2, 3, 4, 5, 6, 9, 11, 13\}$

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