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OPIAL AND LYAPUNOV INEQUALITIES ON TIME SCALES AND THEIR APPLICATIONS TO DYNAMIC EQUATIONS

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Abstract

We prove some weighted inequalities for delta derivatives acting on products and compositions of functions on time scales and apply them to obtain generalized dynamic Opial-type inequalities. We also employ these inequalities to establish some new dynamic Lyapunov-type inequalities, which are essential in studying disfocality, disconjugacy, lower bounds of eigenvalues, and distance between generalized zeros for half-linear dynamic equations. In particular, we solve an open problem posed by Saker in [Math. Comput. Modelling **58** (2013), 1777–1790]. Moreover, the results presented in this paper generalize, improve, extend, and unify most of known results not only in the discrete and continuous analysis but also on time scales.

1. Introduction

Since its discovery more than five decades ago, Opial's inequality has been receiving non-diminishing attention and a large number of papers dealing with new proofs, extensions, generalizations, variants, and discrete analogues have appeared in the literature. Inequalities of Opial-type turn out to be useful tools in the study of oscillation theory, disfocality, disconjugacy, eigenvalue problems, and numerous other applications in the theory of both differential and difference equations. A nice summary of continuous and discrete Opial-type inequalities and their applications can be found in the book [3] by Agarwal and Pang.

The calculus of time scales has been introduced by Hilger [12] in order to unify discrete and continuous analysis. Since then, many authors have been concerned with the theory of inequalities on time scales. The study of dynamic inequalities of Opial-type was initiated by Bohner and Kaymakçalan [5] (see

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also [1]), in which they showed that if $f : [0, c]_{\mathbf{T}} \to \mathbf{R}$ is Δ-differentiable with $f(0) = 0$, then

$$\int_0^c |[f(x) + f^\sigma(x)]f^\Delta(x)|\Delta x \le c \int_0^c |f^\Delta(x)|^2 \Delta x. \tag{1.1}$$

Afterwards, numerous authors have studied variants of (1.1). Two most natural extensions are the weighted Opial-type inequalities

$$\int_a^b |f(x)|^\gamma |f^\Delta(x)|^\beta \varphi(x)\Delta x \le C_1 \left(\int_a^b |f^\Delta(x)|^p \tau(x)\Delta x \right)^{(\gamma+\beta)/p} \tag{1.2}$$

and

$$\int_a^b |f(x) + f^\sigma(x)|^\gamma |f^\Delta(x)|^\beta \varphi(x)\Delta x \le C_2 \left(\int_a^b |f^\Delta(x)|^p \tau(x)\Delta x \right)^{(\gamma+\beta)/p}, \tag{1.3}$$

when $f(a) = 0$ or/and $f(b) = 0$, where $p > 1$, $\beta > 0$, $\gamma > 0$, C_1 and C_2 are constants. For contributions to inequalities (1.2) and (1.3) we refer the readers to [6, 14, 15, 17, 22, 24, 25, 26, 29, 30, 31, 35, 36]. The best reference here is the book by Agarwal, O'Regan and Saker [2, Chapter 3], where the most popular articles on this subject are collected.

Motivated by these works, in this paper we consider inequalities of Opial-type in the most general situation which demonstrate the usefulness in the field of dynamic equations. We are concerned with two following extensions.

Notice first that the term $(f + f^\sigma)f^\Delta$ in the left-hand side of (1.1) can be written as $(f^2)^\Delta$. So, it is more natural and general to replace the terms $|f(x)|^\gamma |f^\Delta(x)|^\beta$ and $|f(x) + f^\sigma(x)|^\gamma |f^\Delta(x)|^\beta$ in the left-hand sides of (1.2) and (1.3), respectively, by $|(G \circ f)^\Delta|^\beta$ and $|(G' \circ f)f^\Delta|^\beta$, where G belongs to a suitable class of functions. Of course, our setting brings in complications and it is much harder to handle than other inequalities obtained previously. The main difficulty in carrying on this construction is that for an arbitrary time scale $\mathbf{T}$, it is difficult to give an explicit formula for $(G \circ f)^\Delta$. Fortunately, by modifying the technique suggested by the authors in [19, 20], utilizing the chain rules [7, Theorems 1.87 and 1.88] and Hölder's inequality [7, Theorem 6.13], we establish surprising results, which are essentially new, contain both the continuous case [19] and the discrete case [20].

Next, in the time scale calculus the concept of a zero of a function is replaced by the so-called generalized zero (GZ for short). Hence, for wider applicability of the results, we consider Opial-type inequalities in the case when the endpoints are not necessarily zeros but GZs.

In addition to their intrinsic interest, our extensions will be proven essential in applications to dynamic equations. For illustration, we consider the following Δ-differential equation with a damping term

$$L_p f = (\tau G_p(f^\Delta))^\Delta + \psi G_p(f^\Delta) + \varphi G_p(f^\sigma) = 0, \tag{1.4}$$

where $p > 1$, $G_p(x) = |x|^{p-1} \operatorname{sign}(x)$, τ, ψ, and φ are real-valued rd-continuous functions on $\mathbf{T}$ with $\tau > 0$. By a solution of (1.4), we mean a function

$f : \mathbf{T} \to \mathbf{R}$ such that all delta derivatives involved in $L_p f$ exist and are rd-continuous at each point in $\mathbf{T}$, that satisfies equation (1.4). We say that f has a GZ at some point $c \in \mathbf{T}$ provided that $f(c) = 0$ or $f(c)f^\sigma(c) < 0$. Equation (1.4) is called disconjugate on $[a,b]_\mathbf{T}$ if there is no non-trivial solution of (1.4) with at least two GZs in $[a,b]_\mathbf{T}$, and (1.4) is said to be disfocal on $[a,\sigma(b)]_\mathbf{T}$ provided there is no non-trivial solution f of (1.4) with a GZ in $[a,\sigma(b)]_\mathbf{T}$ followed by a GZ of f^Δ in $[a,b]_\mathbf{T}$.

Two problems of interest associated with (1.4) are: (i) determining a lower bound for the distance between the GZs of f and f^Δ, i.e., obtaining sufficient conditions for disfocality of (1.4); and (ii) obtaining sufficient conditions for disconjugacy of (1.4). From this we are able to prove some new Lyapunov-type inequalities, which provide some useful tools in the study of disfocality, disconjugacy, counting number of GZs, lower bounds of eigenvalues, and distance between GZs for the half-linear dynamic equation

$$(\tau G_p(f^\Delta))^\Delta + \varphi G_p(f^\sigma) = 0. \tag{1.5}$$

In [27], Saker considered a special case of (1.4) when $p \geq 2$ is a quotient of odd positive integers and posed an open problem for the case when $1 < p < 2$. Hence, our results, in particular, solve this problem. Moreover, we note that (1.4) in its general form covers several different types of differential and difference equations depending on the choice of the time scale $\mathbf{T}$. For example, when $\mathbf{T} = \mathbf{R}$, (1.4) becomes

$$(\tau G_p(f'))' + \psi G_p(f') + \varphi G_p(f) = 0. \tag{1.6}$$

Some special cases of (1.6) have been studied by some authors, we refer to the papers by Brown and Hinton [8], Harris and Kong [11], Hong, Lian and Yeh [13], Lee et al. [16], Lian, Yeh and Li [18], Saker [23], Saker, Agarwal and O'Regan [28], Yan [33], and Yang [34]. Our results for the case when $\mathbf{T} = \mathbf{R}$ cover most of results given in these works and are essentially new for the other cases.

The rest of this paper is organized as follows. Section 2 contains some definitions and preliminary lemmas of time scale calculus. Section 3 is devoted to inequalities for products and compositions of functions on time scales, while Section 4 is intended to motivate our investigations of dynamic Opial-type inequalities. In Section 5, we proceed with the study of Lyapunov-type inequalities and give some answers for problems (i) and (ii) presented above. The last section works with solutions of equation (1.5).

2. Preliminaries

In this section, a brief list of essential lemmas, which are necessary for our results, are given. For the most part, the reader is expected to be familiar with the notion of time scales. See [7], [9], and [12] containing a lot of information on time scale calculus. Nevertheless, we state some time scale concepts here since they are frequently used in the sequel. Let $\mathbf{T}$, σ, f^σ, μ, f^Δ, and $\int_a^b f(x)\Delta x$ stand for time scale, forward jump operator, $f \circ \sigma$, graininess, delta derivative of f, and delta integral of f from a to b, respectively. The notations $[a,b]_{\mathbf{T}}$, $(a,b]_{\mathbf{T}}$, and so on, will denote time scale intervals, for example, $(a,b]_{\mathbf{T}} = (a,b] \cap \mathbf{T}$. In particular, $[1,n]_{\mathbf{N}} = \{1,\dots,n\}$, where n is a positive integer.

LEMMA 2.1. *If $f : \mathbf{T} \to \mathbf{R}$ is Δ-differentiable at $x \in \mathbf{T}^\kappa$, then*

$$f^\sigma(x) = f(x) + \mu(x) f^\Delta(x). \tag{2.1}$$

For $f : \mathbf{T} \to \mathbf{R}$ and $a \in \mathbf{T}^\kappa$, we have

$$\int_a^{\sigma(a)} f(x)\Delta x = \mu(a) f(a). \tag{2.2}$$

LEMMA 2.2 (Leibniz formula). *If f_j^Δ exists for $j \in [1,n]_{\mathbf{N}}$, then*

$$\left(\prod_{j=1}^n f_j\right)^\Delta = \sum_{j=1}^n \left(\prod_{i=1}^{j-1} f_i^\sigma\right) f_j^\Delta \left(\prod_{k=j+1}^n f_k\right). \tag{2.3}$$

LEMMA 2.3 (Integration by parts). *If $f^\Delta g^\sigma$ and fg^Δ are Δ-integrable on $[a,b)_{\mathbf{T}}$, then*

$$\int_a^b f^\Delta(x) g^\sigma(x)\Delta x + \int_a^b f(x) g^\Delta(x)\Delta x = f(b)g(b) - f(a)g(a). \tag{2.4}$$

A function $f : \mathbf{T} \to \mathbf{R}$ is said to be absolutely continuous on $\mathbf{T}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\{[a_k,b_k)_{\mathbf{T}}\}_{k=1}^n$, with $a_k, b_k \in \mathbf{T}$, is a finite pairwise disjoint family of subintervals of $\mathbf{T}$ satisfying $\sum_{k=1}^n (b_k - a_k) < \delta$, then $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$. Let us denote by $\mathscr{AC}([a,b]_{\mathbf{T}})$ the class of all real-valued absolutely continuous functions on $[a,b]_{\mathbf{T}}$.

LEMMA 2.4 (Fundamental theorem of calculus). *For $f \in \mathscr{AC}([a,b]_{\mathbf{T}})$, we have*

$$f(x) = f(a) + \int_a^x f^\Delta(t)\Delta t, \quad x \in [a,b]_{\mathbf{T}} \tag{2.5}$$

and

$$f(x) = f(b) - \int_x^b f^\Delta(t)\Delta t, \quad x \in [a,b]_{\mathbf{T}}. \tag{2.6}$$

A function $\tau : [a,b]_{\mathbf{T}} \to \mathbf{R}$ is said to be a weight on $[a,b]_{\mathbf{T}}$ if τ is positive and rd-continuous on $[a,b]_{\mathbf{T}}$. Let $\mathscr{W}([a,b]_{\mathbf{T}})$ denote the set of all weights on $[a,b]_{\mathbf{T}}$. The Lebesgue Δ-measure is defined over the Lebesgue measurable subsets of $\mathbf{T}$, i.e., a set $E \subset \mathbf{T}$ is Δ-measurable if and only if E is Lebesgue measurable. For $p \ge 1$ and $\tau \in \mathscr{W}([a,b]_{\mathbf{T}})$, we denote by $L_\Delta^p([a,b]_{\mathbf{T}}, \tau)$ the set of all Δ-measurable functions f defined on $[a,b]_{\mathbf{T}}$ such that $\int_a^b |f(x)|^p \tau(x)\Delta x < \infty$ and by

$\mathscr{L}_a^p([a,b]_{\mathbf{T}},\tau)$ the set of all functions $f \in \mathscr{AC}([a,b]_{\mathbf{T}})$ for which $f^\Delta \in L_\Delta^p([a,b]_{\mathbf{T}},\tau)$ and that f has a GZ at a. From now on, $p > 1$ and $q > 1$ are conjugate exponents, i.e., $1/p + 1/q = 1$.

LEMMA 2.5 (Hölder's inequality). *Suppose that* $\tau \in \mathscr{W}([a,b]_{\mathbf{T}})$, $f \in L_\Delta^p([a,b]_{\mathbf{T}},\tau)$, *and* $g \in L_\Delta^q([a,b]_{\mathbf{T}},\tau)$. *Then* $fg \in L_\Delta^1([a,b]_{\mathbf{T}},\tau)$ *and*

$$\int_a^b |f(x)g(x)|\tau(x)\Delta x \le \left(\int_a^b |f(x)|^p \tau(x)\Delta x\right)^{1/p}\left(\int_a^b |g(x)|^q \tau(x)\Delta x\right)^{1/q}. \tag{2.7}$$

LEMMA 2.6. *For* $\tau \in \mathscr{W}([a,c]_{\mathbf{T}})$ *and* $f \in \mathscr{L}_a^p([a,c]_{\mathbf{T}},\tau)$, *there exists a* $\xi \in [0,1)$ *such that* $(1-\xi)f(a) + \xi f^\sigma(a) = 0$. *By setting* $\chi_\xi(x) = 1$ *if* $x \in [\sigma(a),c]_{\mathbf{T}}$ *and* $\chi_\xi(a) = 1-\xi$, $\tau_\xi(x) = \int_a^x \chi_\xi(t)\tau^{-q/p}(t)\Delta t$, *and* $F_\xi(x) = \int_a^x |f^\Delta(t)|^p \chi_\xi(t)\tau(t)\Delta t$, *we have*

$$|f(x)| \le F_\xi^{1/p}(x)\tau_\xi^{1/q}(x), \quad x \in [\sigma(a),c]_{\mathbf{T}}. \tag{2.8}$$

Similarly, for $\tau \in \mathscr{W}([c,\sigma(b)]_{\mathbf{T}})$ *and* $f \in \mathscr{L}_b^p([c,\sigma(b)]_{\mathbf{T}},\tau)$, *there exists an* $\eta \in [0,1)$ *such that* $(1-\eta)f(b) + \eta f^\sigma(b) = 0$. *Let* $\lambda_\eta(x) = 1$ *if* $x \in [c,b)_{\mathbf{T}}$ *and* $\lambda_\eta(b) = \eta$, $\hat{\tau}_\eta(x) = \int_x^{\sigma(b)} \lambda_\eta(t)\tau^{-q/p}(t)\Delta t$, *and* $\hat{F}_\eta(x) = \int_x^{\sigma(b)} |f^\Delta(t)|^p \lambda_\eta(t)\tau(t)\Delta t$. *Then*

$$|f(x)| \le \hat{F}_\eta^{1/p}(x)\hat{\tau}_\eta^{1/q}(x), \quad x \in [c,b]_{\mathbf{T}}. \tag{2.9}$$

Proof. Since $f(a) = 0$ or $f(a)f^\sigma(a) < 0$, it follows that there exists a $\xi \in [0,1)$ such that $(1-\xi)f(a) + \xi f^\sigma(a) = 0$. Hence, using (2.1), (2.2), and (2.5), we obtain

$$f(x) = \int_a^x \chi_\xi(t) f^\Delta(t)\Delta t, \quad x \in [\sigma(a),c]_{\mathbf{T}}. \tag{2.10}$$

Applying (2.7) to (2.10) yields (2.8) as required. The proof for (2.9) is similar. □

We now recall a class of functions introduced by the authors in [20], which will be required when we prove some inequalities for compositions of functions.

For $0 < R \le \infty$, let $\mathscr{G}_R^1$ stand for the class of all functions $G \in C^1(-R,R)$ satisfying the following conditions: $G(0) = 0$; $|G'(x)| \le G'(|x|)$ for all $x \in (-R,R)$; and if $x \le y^\alpha z^{1-\alpha}$, $\alpha \in [0,1]$, $0 < x,y,z < R$, then $G'(x) \le [G'(y)]^\alpha [G'(z)]^{1-\alpha}$, i.e., G' is geometrically convex on $(0,R)$.

We have $G(x) = |x|^p \in \mathscr{G}_\infty^1$ for $p > 1$. If $\sum_{k=0}^\infty a_k x^k$ is an absolutely convergent power series with radius of convergence R, then $G(x) = \sum_{k=0}^\infty |a_k| x^{k+1}/(k+1)$ belongs to $\mathscr{G}_R^1$. For example, $e^x - 1 \in \mathscr{G}_\infty^1$.

LEMMA 2.7 ([20]). *If* $G \in \mathscr{G}_R^1$, *then the following statements hold*:
(1) G' *is non-negative and increasing on* $(0,R)$;
(2) G *is increasing on* $(0,R)$ *and* $|G(x)| \le G(|x|)$ *for all* $x \in (-R,R)$;
(3) G *is geometrically convex on* $(0,R)$.

Lemma 2.8 (Chain rule). *Let $G \in \mathscr{G}_R^1$ and $f : \mathbf{T} \to (-R, R)$ be Δ-differentiable. Then $G \circ f$ is Δ-differentiable and*

$$(G \circ f)^{\Delta}(x) = f^{\Delta}(x) \int_0^1 G'(sf^{\sigma}(x) + (1-s)f(x))\, ds, \quad x \in \mathbf{T}^{\kappa}. \tag{2.11}$$

Moreover, if f is non-negative and increasing on $\mathbf{T}$, then

$$(G \circ f)^{\Delta}(x) \geq f^{\Delta}(x)(G' \circ f)(x), \quad x \in \mathbf{T}^{\kappa}. \tag{2.12}$$

3. Inequalities for products and compositions of functions

In what follows a, b and c belong to $\mathbf{T}$ such that $\sigma(a) < c < b$. If $\tau_j \in \mathscr{W}([a,c]_{\mathbf{T}})$ and $f_j \in \mathscr{L}_a^p([a,c]_{\mathbf{T}}, \tau_j)$ for $j \in [1,n]_{\mathbf{N}}$, then ξ_j, χ_{ξ_j}, τ_{ξ_j}, and F_{ξ_j} are defined as in Lemma 2.6. Similar considerations apply to the case when $\tau_j \in \mathscr{W}([c,\sigma(b)]_{\mathbf{T}})$ and $f_j \in \mathscr{L}_b^p([c,\sigma(b)]_{\mathbf{T}}, \tau_j)$. We first obtain weighted inequalities for a product of functions.

Theorem 3.1. *Let $\tau_j \in \mathscr{W}([a,c]_{\mathbf{T}})$, $f_j \in \mathscr{L}_a^p([a,c]_{\mathbf{T}}, \tau_j)$ and $\upsilon = [(\prod_{j=1}^n \tau_{\xi_j})^{\Delta}]^{-p/q}$. Then υ is a weight on $[\sigma(a), c]_{\mathbf{T}}$ and moreover,*

$$\int_{\sigma(a)}^{c} \left| \left(\prod_{j=1}^{n} f_j \right)^{\Delta}(x) \right|^p \upsilon(x) \Delta x \leq \prod_{j=1}^{n} F_{\xi_j}(c) - \prod_{j=1}^{n} F_{\xi_j}^{\sigma}(a). \tag{3.1}$$

Likewise, let $\tau_j \in \mathscr{W}([c,\sigma(b)]_{\mathbf{T}})$ and $f_j \in \mathscr{L}_b^p([c,\sigma(b)]_{\mathbf{T}}, \tau_j)$. Then the function $\hat{\upsilon} = [-(\prod_{j=1}^n \hat{\tau}_{\eta_j})^{\Delta}]^{-p/q}$ is a weight on $[c,b]_{\mathbf{T}}$ and

$$\int_{c}^{b} \left| \left(\prod_{j=1}^{n} f_j \right)^{\Delta}(x) \right|^p \hat{\upsilon}(x) \Delta x \leq \prod_{j=1}^{n} \hat{F}_{\eta_j}(c) - \prod_{j=1}^{n} \hat{F}_{\eta_j}(b). \tag{3.2}$$

Proof. Since $\tau_j \in \mathscr{W}([a,c]_{\mathbf{T}})$, we have $\tau_{\xi_j}, \tau_{\xi_j}^{\Delta} \in \mathscr{W}([\sigma(a),c]_{\mathbf{T}})$ for $j \in [1,n]_{\mathbf{N}}$. By Leibniz formula (2.3), $\upsilon \in \mathscr{W}([\sigma(a),c]_{\mathbf{T}})$. Also, for $x \in [\sigma(a),c]_{\mathbf{T}}$, we have

$$\left| \left(\prod_{j=1}^{n} f_j \right)^{\Delta}(x) \right| \leq \sum_{j=1}^{n} \left(\prod_{i=1}^{j-1} |f_i^{\sigma}(x)| \right) |f_j^{\Delta}(x)| \left(\prod_{k=j+1}^{n} |f_k(x)| \right). \tag{3.3}$$

Using $|f_j^{\Delta}(x)| = [F_{\xi_j}^{\Delta}(x)]^{1/p}[\tau_{\xi_j}^{\Delta}(x)]^{1/q}$ and (2.8) in (3.3) and then applying Hölder's inequality for the sum, we can assert that

$$\left| \left(\prod_{j=1}^{n} f_j \right)^{\Delta}(x) \right|^p \upsilon(x) \leq \left(\prod_{j=1}^{n} F_{\xi_j} \right)^{\Delta}(x), \quad x \in [\sigma(a),c]_{\mathbf{T}}. \tag{3.4}$$

Integrating (3.4) over $[\sigma(a), c]_{\mathbf{T}}$ and using (2.5), we get (3.1). The proof for (3.2) is similar. □

Remark 3.2. If $f_j(a) = 0$ for $j \in [1, n]_{\mathbf{N}}$, then inequality (3.1) also holds true when we replace $\sigma(a)$ by a. By taking $\mathbf{T} = \mathbf{R}$, inequality (3.1) reduces to [10, Theorem 1.6].

Next, let us consider some weighted inequalities for the transform $f \mapsto G \circ f$, where $G \in \mathscr{G}_R^1$ and $0 < R \leq \infty$.

THEOREM 3.3. *Let $\tau \in \mathscr{W}([a, c]_{\mathbf{T}})$ and $f \in \mathscr{L}_a^p([a, c]_{\mathbf{T}}, \tau)$ be such that $F_\xi(c) < R$ and $\tau_\xi(c) < R$. Then $\vartheta = [(G \circ \tau_\xi)^\Delta]^{-p/q} \in \mathscr{W}([\sigma(a), c]_{\mathbf{T}})$ and*

$$\int_{\sigma(a)}^{c} |(G \circ f)^\Delta(x)|^p \vartheta(x) \Delta x \leq (G \circ F_\xi)(c) - (G \circ F_\xi^\sigma)(a). \tag{3.5}$$

Similarly, let $\tau \in \mathscr{W}([c, \sigma(b)]_{\mathbf{T}})$ and $f \in \mathscr{L}_b^p([c, \sigma(b)]_{\mathbf{T}}, \tau)$ satisfy $\hat{F}_\eta(c) < R$ and $\hat{\tau}_\eta(c) < R$. Then $\hat{\vartheta} = [-(G \circ \hat{\tau}_\eta)^\Delta]^{-p/q} \in \mathscr{W}([c, b]_{\mathbf{T}})$ and furthermore,

$$\int_{c}^{b} |(G \circ f)^\Delta(x)|^p \hat{\vartheta}(x) \Delta x \leq (G \circ \hat{F}_\eta)(c) - (G \circ \hat{F}_\eta)(b). \tag{3.6}$$

Proof. We give the proof only for the case when $f \in \mathscr{L}_a^p([a, c]_{\mathbf{T}}, \tau)$; the proof of the other case is similar and therefore omitted. From (2.11) we observe that $\vartheta \in \mathscr{W}([\sigma(a), c]_{\mathbf{T}})$ and

$$|(G \circ f)^\Delta(x)| \leq |f^\Delta(x)| \int_0^1 |G'(sf^\sigma(x) + (1-s)f(x))| \, ds, \quad x \in [\sigma(a), c]_{\mathbf{T}}. \tag{3.7}$$

According to Lemma 2.6, properties of G', and Hölder's inequality, we have

$$\begin{aligned} &|G'(sf^\sigma(x) + (1-s)f(x))| \\ &\quad \leq [G'(sF_\xi^\sigma(x) + (1-s)F_\xi(x))]^{1/p} [G'(s\tau_\xi^\sigma(x) + (1-s)\tau_\xi(x))]^{1/q} \end{aligned} \tag{3.8}$$

for $x \in [\sigma(a), c]_{\mathbf{T}}$ and $s \in [0, 1]$. Using $|f^\Delta(x)| = [F_\xi^\Delta(x)]^{1/p} [\tau_\xi^\Delta(x)]^{1/q}$ and (3.8) in (3.7) and then applying (2.7), we arrive at

$$|(G \circ f)^\Delta(x)|^p \vartheta(x) \leq (G \circ F)^\Delta(x), \quad x \in [\sigma(a), c]_{\mathbf{T}}. \tag{3.9}$$

Integrate inequality (3.9) over $[\sigma(a), c]_{\mathbf{T}}$ and use (2.5) to obtain (3.5). □

Combining Theorems 3.1 and 3.3 yields the following corollary.

COROLLARY 3.4. *Let $G_j \in \mathscr{G}_R^1$, $\tau_j \in \mathscr{W}([a, c]_{\mathbf{T}})$ and $f_j \in \mathscr{L}_a^p([a, c]_{\mathbf{T}}, \tau_j)$ be such that $F_{\xi_j}(c) < R$ and $\tau_{\xi_j}(c) < R$ for $j \in [1, n]_{\mathbf{N}}$. Then $v = [(\prod_{j=1}^n G_j \circ \tau_{\xi_j})^\Delta]^{-p/q}$ is a* *weight on* $[\sigma(a), c]_{\mathbf{T}}$ *and*

$$\int_{\sigma(a)}^{c}\left|\left(\prod_{j=1}^{n} G_j \circ f_j\right)^{\Delta}(x)\right|^{p} v(x)\Delta x \le \prod_{j=1}^{n}(G_j \circ F_{\xi_j})(c) - \prod_{j=1}^{n}(G_j \circ F_{\xi_j}^{\sigma})(a). \tag{3.10}$$

Likewise, if $\tau_j \in \mathscr{W}([c, \sigma(b)]_{\mathbf{T}})$ *and* $f_j \in \mathscr{L}_b^p([c, \sigma(b)]_{\mathbf{T}}, \tau_j)$ *satisfying* $\hat{F}_{\eta_j}(c) < R$ *and* $\hat{\tau}_{\eta_j}(c) < R$ *for* $j \in [1, n]_{\mathbf{N}}$, *then* $\hat{v} = [-(\prod_{j=1}^{n} G_j \circ \hat{\tau}_{\eta_j})^{\Delta}]^{-p/q} \in \mathscr{W}([c, b]_{\mathbf{T}})$ *and*

$$\int_{c}^{b}\left|\left(\prod_{j=1}^{n} G_j \circ f_j\right)^{\Delta}(x)\right|^{p} \hat{v}(x)\Delta x \le \prod_{j=1}^{n}(G_j \circ \hat{F}_{\eta_j})(c) - \prod_{j=1}^{n}(G_j \circ \hat{F}_{\eta_j})(b). \tag{3.11}$$

Remark 3.5. As special cases when $\mathbf{T} = \mathbf{R}$ and $\mathbf{T} = \mathbf{Z}$, the results contained in Corollary 3.4 reduce to the corresponding ones stated in [19] and [20], respectively.

4. Opial-type inequalities on time scales

This section is devoted to establish generalized Opial-type inequalities on time scales by using weighted inequalities obtained above. It is a new argument that yields the most general version and we include it here since the technique may be useful in the proof of other inequalities. Here and subsequently, α and β are positive real numbers such that $1/p + 1/\alpha = 1/\beta$.

Theorem 4.1. *Let* $G_j \in \mathscr{G}_R^1$, $\tau_j \in \mathscr{W}([a, c]_{\mathbf{T}})$ *and* $f_j \in \mathscr{L}_a^p([a, c]_{\mathbf{T}}, \tau_j)$ *be such that* $F_{\xi_j}(c) < R$ *and* $\tau_{\xi_j}(c) < R$ *for* $j \in [1, n]_{\mathbf{N}}$. *For* $\varphi \in \mathscr{W}([a, c]_{\mathbf{T}})$ *such that*

$$K := \left\{\int_{\sigma(a)}^{c} \varphi^{\alpha/\beta}(x)\left[\left(\prod_{j=1}^{n} G_j \circ \tau_{\xi_j}\right)^{\Delta}(x)\right]^{-\alpha/q} \Delta x\right\}^{\beta/\alpha} < \infty,$$

we have

$$\int_{\sigma(a)}^{c}\left|\left(\prod_{j=1}^{n} G_j \circ f_j\right)^{\Delta}(x)\right|^{\beta} \varphi(x)\Delta x \le K\left[\prod_{j=1}^{n}(G_j \circ F_{\xi_j})(c) - \prod_{j=1}^{n}(G_j \circ F_{\xi_j}^{\sigma})(a)\right]^{\beta/p}. \tag{4.1}$$

Similarly, let $\tau_j \in \mathscr{W}([c, \sigma(b)]_{\mathbf{T}})$ *and* $f_j \in \mathscr{L}_b^p([c, \sigma(b)]_{\mathbf{T}}, \tau_j)$ *satisfy* $\hat{F}_{\eta_j}(c) < R$ *and* $\hat{\tau}_{\eta_j}(c) < R$ *for* $j \in [1, n]_{\mathbf{N}}$. *If* $\varphi \in \mathscr{W}([c, \sigma(b)]_{\mathbf{T}})$ *such that*

$$\hat{K} := \left\{\int_{c}^{b} \varphi^{\alpha/\beta}(x)\left[-\left(\prod_{j=1}^{n} G_j \circ \hat{\tau}_{\eta_j}\right)^{\Delta}(x)\right]^{-\alpha/q} \Delta x\right\}^{\beta/\alpha} < \infty,$$

then

$$\int_c^b \left| \left(\prod_{j=1}^n G_j \circ f_j \right)^\Delta (x) \right|^\beta \varphi(x)\Delta x \tag{4.2}$$

$$\leq \hat{K} \left[\prod_{j=1}^n (G_j \circ \hat{F}_{\eta_j})(c) - \prod_{j=1}^n (G_j \circ \hat{F}_{\eta_j})(b) \right]^{\beta/p}.$$

Proof. Let $v = [(\prod_{j=1}^n G_j \circ \tau_{\xi_j})^\Delta]^{-p/q}$. Then v is a weight on $[\sigma(a), c]_{\mathbf{T}}$. Hence, (4.1) is derived by using (2.7) and (3.10). The same proof works for (4.2). □

COROLLARY 4.2. *Let $\tau, \varphi \in \mathscr{W}([a,c]_{\mathbf{T}})$ and $f \in \mathscr{L}_a^p([a,c]_{\mathbf{T}}, \tau)$ be such that*

$$L := \left[\int_{\sigma(a)}^c \varphi^q(x) (\tau_\xi^p)^\Delta (x) \Delta x \right]^{1/q} < \infty.$$

Then one has

$$\int_{\sigma(a)}^c |(|f|^p)^\Delta (x)| \varphi(x) \Delta x \leq L[F_\xi^p(c) - F_\xi^p(\sigma(a))]^{1/p}. \tag{4.3}$$

Similarly, if $\tau, \varphi \in \mathscr{W}([c, \sigma(b)]_{\mathbf{T}})$ and $f \in \mathscr{L}_b^p([c, \sigma(b)]_{\mathbf{T}}, \tau)$, then

$$\int_c^b |(|f|^p)^\Delta (x)| \varphi(x) \Delta x \leq \left[\int_c^b \varphi^q(x) (-\hat{\tau}_\eta^p)^\Delta (x) \Delta x \right]^{1/q} [\hat{F}_\eta^p(c) - \hat{F}_\eta^p(b)]^{1/p} \tag{4.4}$$

as long as the right-hand side exists and is finite.

Remark 4.3. When $f(a) = 0$ or/and $f(b) = 0$, Corollary 4.2 derives various known results in the literature:

(1) If $p = 2$, $a = 0$, $\varphi = \tau \equiv 1$, then (4.3) reduces to (1.1), while (4.4) is new;
(2) Inequalities (4.3) and (4.4) are the same as inequalities given in Theorem 4.1 and Corollary 4.4 in [36], respectively, if we take $p = 2$, $a = 0$ and $\varphi \equiv 1$;
(3) For $p = 2$ and $\tau \equiv 1$, (4.3) becomes a sharper version of [14, Theorem 3.1]. Similarly, (4.4) reduces to a sharpened version of [14, Theorem 3.2];
(4) Let $p = 2$, $a = 0$, $\tau = \omega\psi$, $\varphi = \psi^\sigma$, where ω and ψ are two weights on $[0, c]_{\mathbf{T}}$ such that $\int_0^c \Delta t / \omega(t) < \infty$ and ψ is decreasing. Then inequality (4.3) implies [5, Theorem 4.1]. In this case, inequality (4.4) is essentially new;
(5) Inequality (4.3) reduces to [6, Corollary 3.2].

Remark 4.4. In [24, Theorem 1], Saker proved an inequality similar to (4.3) in the case when $p = 2$ and $f(a) = 0$ with the term L being replaced by

$$M := \left[2\int_a^c \left(\int_a^x \frac{\Delta t}{\tau(t)}\right)\frac{\varphi^2(x)}{\tau(x)}\Delta x\right]^{1/2} + \sup_{a\le x\le c} \mu(x)\frac{\varphi(x)}{\tau(x)}.$$

In some cases we see that $M \ge L$, i.e., (4.3) gives a sharpened version of Saker's result. For example, if φ is increasing on $[a, c]_{\mathbf{T}}$, then

$$\int_a^c \mu(x)\frac{\varphi^2(x)}{\tau^2(x)}\Delta x \le \left(\sup_{x\in[a,c]_{\mathbf{T}}} \mu(x)\frac{\varphi(x)}{\tau(x)}\right)L,$$

which yields $M \ge L$. The same conclusion can be drawn for (4.4) which corresponds to [24, Theorem 2].

The following corollary can be proved in view of Theorem 4.1 and the well-known inequality of arithmetic and geometric means:

$$\prod_{j=1}^n a_j \le \left(\frac{1}{n}\sum_{j=1}^n a_j\right)^n, \quad a_j \ge 0 \quad \text{for } j \in [1, n]_{\mathbf{N}}. \tag{4.5}$$

Corollary 4.5. *Let $m \in \mathbf{N}$, $\varphi, \tau_j \in \mathscr{W}([a, c]_{\mathbf{T}})$ for $j \in [1, n]_{\mathbf{N}}$. If $f_j \in \mathscr{L}_a^p([a, c]_{\mathbf{T}}, \tau_j)$ with $f_j(a) = 0$ for $j \in [1, n]_{\mathbf{N}}$ and*

$$N := \left[\int_a^c \varphi^q(x)\left(\prod_{j=1}^n \left(\int_a^x \tau_j^{-q/p}(t)\Delta t\right)^m\right)^{\Delta}\Delta x\right]^{1/q} < \infty,$$

then

$$\int_a^c \left|\left(\prod_{j=1}^n f_j^m\right)^{\Delta}(x)\right|\varphi(x)\Delta x \le N\left(\frac{1}{n}\int_a^c \sum_{j=1}^n |f_j^{\Delta}(x)|^p\tau_j(x)\Delta x\right)^{mn/p}. \tag{4.6}$$

Likewise, if $\varphi, \tau_j \in \mathscr{W}([c, b]_{\mathbf{T}})$, $f_j \in \mathscr{L}_b^p([c, b]_{\mathbf{T}}, \tau_j)$ with $f_j(b) = 0$ for $j \in [1, n]_{\mathbf{T}}$, and

$$\hat{N} := \left[\int_c^b \varphi^q(x)\left(-\prod_{j=1}^n \left(\int_x^b \tau_j^{-q/p}(t)\Delta t\right)^m\right)^{\Delta}\Delta x\right]^{1/q} < \infty,$$

then

$$\int_c^b \left|\left(\prod_{j=1}^n f_j^m\right)^{\Delta}(x)\right|\varphi(x)\Delta x \le \hat{N}\left(\frac{1}{n}\int_c^b \sum_{j=1}^n |f_j^{\Delta}(x)|^p\tau_j(x)\Delta x\right)^{mn/p}. \tag{4.7}$$

Remark 4.6. (1) Note that when $n = 2$, $m = 1$, $p = 2$, $\tau_1 = \tau_2 = \psi\omega$, $\varphi = \psi^\sigma$, where $\psi, \omega \in \mathscr{W}([a, c]_{\mathbf{T}})$ and ψ is decreasing on $[a, c]_{\mathbf{T}}$, inequality (4.6) improves [29, Theorem 3.5]. Similarly, [29, Theorem 3.6] can be derived from (4.7).

(2) Inequality (4.6) implies [15, Theorem 3.2], and so is [15, Theorem 3.1] if we take $\tau_j \equiv 1$ for all $j \in [1, n]_{\mathbf{N}}$, $\varphi \equiv 1$ and $p = nm$. Moreover, let $\varphi = \psi^\sigma$ and $\tau = \psi^\sigma\omega$, where $\psi, \omega \in \mathscr{W}([a, c]_{\mathbf{T}})$ with ψ decreasing on $[a, b]_{\mathbf{T}}$. Then (4.6) improves the results contained in [15, Theorems 4.1 and 4.2] if we choose $m = 1$, $p = n$ and $p = mn - 1$, respectively.

We point out that it is not easy to give an explicit formula for $(G \circ f)^\Delta$ in general. So, for wider applicability of the results, we would like to replace $(G_j \circ f_j)^\Delta$ appeared in Theorem 4.1 by $f_j^\Delta(G_j' \circ f_j)$ and obtain the following theorem whose proof is based on Hölder's inequality (2.7), Lemmas 2.6, 2.7 and 2.8, and a modification of the method used in the proof of Theorem 3.3.

Theorem 4.7. *Let $G_j \in \mathscr{G}_R^1$, $\tau_j \in \mathscr{W}([a, c]_{\mathbf{T}})$ and $f_j \in \mathscr{L}_a^p([a, c]_{\mathbf{T}}, \tau_j)$ be such that $F_{\xi_j}(c) < R$ and $\tau_{\xi_j}(c) < R$ for $j \in [1, n]_{\mathbf{N}}$. For $\varphi \in \mathscr{W}([a, c]_{\mathbf{T}})$ such that*

$$P := \left[\int_{\sigma(a)}^{c} \left(\sum_{j=1}^{n} [\tau_{\xi_j}^\Delta(x)(G_j' \circ \tau_{\xi_j})(x)] \prod_{k \neq j} (G_k \circ \tau_{\xi_k})(x)\right)^{\alpha/q} \varphi^{\alpha/\beta}(x)\Delta x\right]^{\beta/\alpha} < \infty,$$

we obtain

$$\int_{\sigma(a)}^{c} \left(\sum_{j=1}^{n} |f_j^\Delta(x)(G_j' \circ f_j)(x)| \prod_{k \neq j} |(G_k \circ f_k)(x)|\right)^{\beta} \varphi(x)\Delta x \le P\left[\prod_{j=1}^{n}(G_j \circ F_{\xi_j})(c) - \prod_{j=1}^{n}(G_j \circ F_{\xi_j}^\sigma)(a)\right]^{\beta/p}. \tag{4.8}$$

Likewise, let $\tau_j \in \mathscr{W}([c, \sigma(b)]_{\mathbf{T}})$ and $f_j \in \mathscr{L}_b^p([c, \sigma(b)]_{\mathbf{T}}, \tau_j)$ satisfy $\hat{F}_{\eta_j}(c) < R$ and $\hat{\tau}_{\eta_j}(c) < R$ for $j \in [1, n]_{\mathbf{N}}$. One has

$$\int_{c}^{b} \left(\sum_{j=1}^{n} |f_j^\Delta(x)(G_j' \circ f_j)(x)| \prod_{k \neq j} |(G_k \circ f_k)(x)|\right)^{\beta} \varphi(x)\Delta x \le \hat{P}\left[\prod_{j=1}^{n}(G_j \circ \hat{F}_{\eta_j})(c) - \prod_{j=1}^{n}(G_j \circ \hat{F}_{\eta_j})(b)\right]^{\beta/p} \tag{4.9}$$

if $\varphi \in \mathscr{W}([c, \sigma(b)]_{\mathbf{T}})$ such that

$$\hat{P} := \left[\int_{c}^{b} \left(\sum_{j=1}^{n} [-\hat{\tau}_{\eta_j}^\Delta(x)(G_j' \circ \hat{\tau}_{\eta_j})(x)] \prod_{k \neq j} (G_k \circ \hat{\tau}_{\eta_k})(x)\right)^{\alpha/q} \varphi^{\alpha/\beta}(x)\Delta x\right]^{\beta/\alpha} < \infty.$$

Proof. We only prove (4.8), as the proof of (4.9) is similar. By Lemmas 2.6 and 2.7,

$$|(G_j \circ f_j)(x)| \le [(G_j \circ F_{\xi_j})(x)]^{1/p}[(G_j \circ \tau_{\xi_j})(x)]^{1/q}$$

and

$$|f_j^{\Delta}(x)(G_j' \circ f_j)(x)| \le [F_{\xi_j}^{\Delta}(x)(G_j' \circ F_{\xi_j})(x)]^{1/p}[\tau_{\xi_j}^{\Delta}(x)(G_j' \circ \tau_{\xi_j})(x)]^{1/q}$$

for $x \in [\sigma(a), c]_{\mathbf{T}}$ and $j \in [1, n]_{\mathbf{N}}$. Thus, by Hölder's inequality,

$$\left(\sum_{j=1}^{n} |f_j^{\Delta}(x)(G_j' \circ f_j)(x)| \prod_{k \neq j} |(G_k \circ f_k)(x)|\right)^{\beta} \varphi(x) \le \left[\sum_{j=1}^{n} [F_{\xi_j}^{\Delta}(x)(G_j' \circ F_{\xi_j})(x)] \prod_{k \neq j} (G_k \circ F_{\xi_k})(x)\right]^{\beta/p} \times \left[\sum_{j=1}^{n} [\tau_{\xi_j}^{\Delta}(x)(G_j' \circ \tau_{\xi_j})(x)] \prod_{k \neq j} (G_k \circ \tau_{\xi_k})(x)\right]^{\beta/q} \varphi(x), \tag{4.10}$$

for $x \in [\sigma(a), c]_{\mathbf{T}}$. Integrating both sides of (4.10) over $[\sigma(a), c]_{\mathbf{T}}$ and using Hölder's inequality with indices p/β and α/β, we get

$$\begin{aligned} &\int_{\sigma(a)}^{c} \left(\sum_{j=1}^{n} |f_j^{\Delta}(x)(G_j' \circ f_j)(x)| \prod_{k \neq j} |(G_k \circ f_k)(x)|\right)^{\beta} \varphi(x)\Delta x \\ &\le P\left[\int_{\sigma(a)}^{c} \sum_{j=1}^{n} [F_{\xi_j}^{\Delta}(x)(G_j' \circ F_{\xi_j})(x)] \prod_{k \neq j} (G_k \circ F_{\xi_k})(x)\Delta x\right]^{\beta/p} \\ &\le P\left[\prod_{j=1}^{n} (G_j \circ F_{\xi_j})(c) - \prod_{j=1}^{n} (G_j \circ F_{\xi_j}^{\sigma})(a)\right]^{\beta/p}, \end{aligned}$$

where we have used Leibniz formula (2.3) and the fact that

$$(G_j \circ F_{\xi_j})(x) \le (G_j \circ F_{\xi_j}^{\sigma})(x) \quad \text{and} \quad F_{\xi_j}^{\Delta}(x)(G_j' \circ F_{\xi_j})(x) \le (G_j \circ F_{\xi_j})^{\Delta}(x)$$

for $x \in [\sigma(a), c]_{\mathbf{T}}$ and $j \in [1, n]_{\mathbf{N}}$. Hence, (4.8) is verified. □

Remark 4.8. Taking $n = 2$, $\beta = 1$, $G_1(x) = G_2(x) = x$, $f_1(a) = f_2(a) = 0$ or/and $f_1(b) = f_2(b) = 0$, $\varphi = \psi^{\sigma}$, $\tau_1 = \tau_2 = \omega[\psi^{\sigma}]^{p/2}$, where $\omega, \psi \in \mathscr{W}([a, b]_{\mathbf{T}})$ and ψ is decreasing on $[a, b]_{\mathbf{T}}$, Theorem 4.7 improves and generalizes the results given in [35].

Corollary 4.9. *If* $\tau, \varphi \in \mathscr{W}([a, c]_{\mathbf{T}})$, $f \in \mathscr{L}_a^p([a, c]_{\mathbf{T}}, \tau)$ *and*

$$Q := \left(\frac{\beta}{\gamma + \beta}\right)^{\beta/p} \left[\int_{\sigma(a)}^{c} [\tau_{\xi}^{\Delta}(x)\tau_{\xi}^{\gamma/\beta}(x)]^{\alpha/q} \varphi^{\alpha/\beta}(x)\Delta x\right]^{\beta/\alpha} < \infty,$$

then

$$\int_{\sigma(a)}^{c} |f(x)|^{\gamma}|f^{\Delta}(x)|^{\beta}\varphi(x)\Delta x \le Q[F_{\xi}^{(\gamma+\beta)/\beta}(c) - F_{\xi}^{(\gamma+\beta)/\beta}(\sigma(a))]^{\beta/p}. \tag{4.11}$$

Similarly, suppose that $\tau, \varphi \in \mathscr{W}([c, \sigma(b)]_{\mathbf{T}})$, $f \in \mathscr{L}_b^p([c, \sigma(b)]_{\mathbf{T}}, \tau)$ *and*

$$\hat{Q} := \left(\frac{\beta}{\gamma+\beta}\right)^{\beta/p}\left[\int_c^b [-\hat{\tau}_{\eta}^{\Delta}(x)\hat{\tau}_{\eta}^{\gamma/\beta}(x)]^{\alpha/q}\varphi^{\alpha/\beta}(x)\Delta x\right]^{\beta/\alpha} < \infty.$$

We have

$$\int_{c}^{b} |f(x)|^{\gamma}|f^{\Delta}(x)|^{\beta}\varphi(x)\Delta x \le \hat{Q}[\hat{F}_{\eta}^{(\gamma+\beta)/\beta}(c) - \hat{F}_{\eta}^{(\gamma+\beta)/\beta}(b)]^{\beta/p}. \tag{4.12}$$

Remark 4.10. If $f(a) = 0$, then inequality (4.11) implies that

$$\int_{a}^{c} |f(x)|^{\gamma}|f^{\Delta}(x)|^{\beta}\varphi(x)\Delta x \le Q_1\left[\int_a^c |f^{\Delta}(x)|^p\tau(x)\Delta x\right]^{(\gamma+\beta)/p}, \tag{4.13}$$

where

$$Q_1 := \left(\frac{\beta}{\gamma+\beta}\right)^{\beta/p}\left[\int_a^c \frac{\varphi^{\alpha/\beta}(x)}{\tau^{\alpha/p}(x)}\left(\int_a^x \frac{\Delta t}{\tau^{q/p}(t)}\right)^{\gamma\alpha/(\beta q)}\Delta x\right]^{\beta/\alpha}.$$

From (4.13) we can deduce some existed Opial-type inequalities:

(1) If one sets $p = \gamma + \beta$ and $\varphi = \tau$, where τ is decreasing on $[a, c]_{\mathbf{T}}$, then (4.13) improves [17, Theorem 3.1] and [30, Theorem 1];

(2) By setting $\beta = 1$ and $\tau = \omega\varphi^{p/(\gamma+1)}$, where $\omega \in \mathscr{W}([a, c]_{\mathbf{T}})$ and φ is decreasing on $[a, c]_{\mathbf{T}}$, (4.13) reduces to [2, Theorem 3.2.4] (see also [31]);

(3) If $p = \gamma + \beta$, then inequality (4.13) becomes [29, Theorem 3.1] which reduces to [22, Theorem 2.4].

Similar consideration applying to (4.12) yields other results given in [17, 22, 29, 30], and [31].

5. Disfocal problems and disconjugacy conditions

In this section we establish sufficient conditions for disfocality and disconjugacy of equation (1.4). In the following, we assume that $\mu(x)|\psi(x)|/\tau(x) \le \delta < 1$ for all $x \in [a, \sigma(b)]_{\mathbf{T}}$. Notice that this assumption is trivial in the case when $\mathbf{T} = \mathbf{R}$ since $\mu \equiv 0$. We first formulate our main results related to the spacing between a GZ of the solution and a GZ of its derivative of equation (1.4), which yields sufficient conditions for disfocality of (1.4).

THEOREM 5.1. *If equation* (1.4) *has a non-trivial solution* $f \in \mathscr{L}_a^p([a, c]_{\mathbf{T}}, \tau)$ *such that* $f^{\Delta}(c)f(c) \le 0$, *then*

$$T_{\xi}(c, \psi) + W_{\xi}(a, c, \Phi_c) > 1 - \delta, \tag{5.1}$$

where $\Phi_c(x) := \int_x^c \varphi(t)\Delta t$,

$$T_\xi(c,\psi) := \left(\frac{1}{p}\right)^{1/p}\left[\int_{\sigma(a)}^{c} \tau_\xi^\Delta(x)\tau_\xi^{p-1}(x)|\psi(x)|^q\Delta x\right]^{1/q}$$

and

$$W_\xi(a,c,\Phi_c) := \left[\int_a^c |\Phi_c(x)|^q(\tau_\xi^p)^\Delta(x)\Delta x\right]^{1/q}.$$

Similarly, if $f \in \mathscr{L}_b^p([c,\sigma(b)]_{\mathbf{T}},\tau)$ *is a non-trivial solution of equation* (1.4) *such that* $f^\Delta(c)f(c) \geq 0$, *then*

$$\hat{T}_\eta(c,\psi) + \hat{W}_\eta(c,\sigma(b),\hat{\Phi}_c) > 1-\delta, \tag{5.2}$$

where $\hat{\Phi}_c(x) := \int_c^x \varphi(t)\Delta t$,

$$\hat{T}_\eta(c,\psi) := \left(\frac{1}{p}\right)^{1/p}\left[-\int_c^b \hat{\tau}_\eta^\Delta(x)\hat{\tau}_\eta^{p-1}(x)|\psi(x)|^q\Delta x\right]^{1/q}$$

and

$$\hat{W}_\eta(c,\sigma(b),\hat{\Phi}_c) := \left[\int_c^{\sigma(b)} |\hat{\Phi}_c(x)|^q(-\hat{\tau}_\eta^p)^\Delta(x)\Delta x\right]^{1/q}.$$

Proof. We only prove (5.1), as the proof of (5.2) is similar. Multiplying both sides of (1.4) by f^σ and integrating it over $[a,c]_{\mathbf{T}}$, we get

$$\begin{aligned} -\int_a^c (\tau G_p(f^\Delta))^\Delta(x)f^\sigma(x)\Delta x \\ = \int_a^c \psi(x)G_p(f^\Delta(x))f^\sigma(x)\Delta x + \int_a^c \varphi(x)|f^\sigma(x)|^p\Delta x. \end{aligned} \tag{5.3}$$

Using integration by parts formula (2.4) with $f^\Delta(c)f(c) \leq 0$ and $f(a) = -\xi\mu(a)f^\Delta(a)$, we obtain

$$-\int_a^c (\tau G_p(f^\Delta))^\Delta(x)f^\sigma(x)\Delta x \geq F_\xi(c). \tag{5.4}$$

By $\mu(x)|\psi(x)|/\tau(x) \leq \delta$ and (4.11),

$$\int_a^c \mu(x)\psi(x)|f^\Delta(x)|^p\chi_\xi(x)\Delta x \leq \delta F_\xi(c)$$

and

$$\int_{\sigma(a)}^c \psi(x)G_p(f^\Delta(x))f(x)\Delta x \leq T_\xi(c,\psi)F_\xi(c).$$

Therefore, due to (2.1),

$$\int_a^c \psi(x) G_p(f^\Delta(x)) f^\sigma(x) \Delta x \le (\delta + T_\xi(c,\psi)) F_\xi(c). \tag{5.5}$$

Since $\Phi_c^\Delta(x) = -\varphi(x)$ and $\Phi_c(c) = 0$, it follows from (2.4), (2.8), (4.3), and Hölder's inequality that

$$\int_a^c \varphi(x)|f^\sigma(x)|^p \Delta x = \Phi_c(a)|f^\sigma(a)|^p + \int_{\sigma(a)}^c \Phi_c(x)(|f|^p)^\Delta(x)\Delta x \tag{5.6}$$

$$< W_\xi(a,c,\Phi_c) F_\xi(c). \tag{5.7}$$

Considering (5.4), (5.5) and (5.7) in (5.3) and canceling $F_\xi(c)$, we get (5.1). □

Remark 5.2. When $f(a) = 0$ or/and $f(b) = 0$, Theorem 5.1 solves [27, Problem 1] and implies [14, Theorems 4.1 and 4.2]. In the special case when $\mathbf{T} = \mathbf{R}$, Theorem 5.1 generalizes [8, Theorem 3.1], [11, Theorems 2.1 and 2.2], [23, Theorem 1], and [28, Theorem 2.1].

By using the maximum of $|\psi|$, $|\Phi_c|$, and $|\hat{\Phi}_c|$ in (5.1) and (5.2), we obtain the following corollary.

COROLLARY 5.3. *Suppose that f is a non-trivial solution of equation* (1.4) *which belongs to $\mathscr{L}_a^p([a,c]_{\mathbf{T}},\tau)$ and $f^\Delta(c)f(c) \le 0$, then*

$$T_\xi(c,1)\left(\sup_{x\in[\sigma(a),c]_{\mathbf{T}}} |\psi(x)|\right) + \tau_\xi^{p-1}(c)\left(\sup_{x\in[a,c]_{\mathbf{T}}} |\Phi_c(x)|\right) > 1-\delta. \tag{5.8}$$

Likewise, if f is a non-trivial solution of (1.4) *which belongs to $\mathscr{L}_b^p([c,\sigma(b)]_{\mathbf{T}},\tau)$ and $f^\Delta(c)f(c) \ge 0$, then*

$$\hat{T}_\eta(c,1)\left(\sup_{x\in[c,b]_{\mathbf{T}}} |\psi(x)|\right) + \hat{\tau}_\eta^{p-1}(c)\left(\sup_{x\in[c,\sigma(b)]_{\mathbf{T}}} |\hat{\Phi}_c(x)|\right) > 1-\delta. \tag{5.9}$$

Remark 5.4. When $f(a) = 0$, (5.8) improves the results given in [24, Corollary 14], [26, Corollary 3.8] and [27, Corollary 2.5]. Similar considerations apply to (5.9). Also, if $\mathbf{T} = \mathbf{R}$, then Corollary 5.3 coincides with [23, Theorem 2].

THEOREM 5.5. *Equation* (1.4) *is disfocal on $[a,\sigma(b)]_{\mathbf{T}}$ if*

$$\max\{T_0(\sigma(b),\psi) + W_0(a,\sigma(b),\Phi_{\sigma(b)}), \hat{T}_0(a,\psi) + \hat{W}_0(a,\sigma(b),\hat{\Phi}_a)\} \le 1-\delta. \tag{5.10}$$

Proof. Assume, to the contrary, that (5.10) holds and equation (1.4) is not disfocal on $[a,\sigma(b)]_{\mathbf{T}}$. But then, by definition, there is a non-trivial solution f of (1.4) with a GZ in $[a,\sigma(b)]_{\mathbf{T}}$ followed by a GZ of f^Δ in $[a,b]_{\mathbf{T}}$. Without loss of generality, we may assume that a is a GZ of f, b is a GZ of f^Δ, and f

has no GZs on $(a, \sigma(b)]_{\mathbf{T}}$. Since $f^{\Delta}(b) f^{\Delta}(\sigma(b)) \leq 0$, we have $f(b) f^{\Delta}(b) \leq 0$ or $f(\sigma(b)) f^{\Delta}(\sigma(b)) \leq 0$. By (5.1),

$$1 - \delta < T_{\xi}(\sigma(b), \psi) + W_{\xi}(a, \sigma(b), \Phi_{\sigma(b)}) \leq T_0(\sigma(b), \psi) + W_0(a, \sigma(b), \Phi_{\sigma(b)}),$$

which contradicts (5.10). The proof is complete. □

Application of Theorem 5.1 enables us to establish some new Lyapunov-type inequalities on time scales which lead immediately to disconjugacy criteria for (1.4).

THEOREM 5.6. *Suppose that equation* (1.4) *has a non-trivial solution f with two consecutive GZs a and b, then there are $c, \sigma(d) \in [a, b]_{\mathbf{T}}$, $c \leq \sigma(d)$, for which*

$$T_{\xi}(c, \psi) + W_{\xi}(a, c, \Phi_c) > 1 - \delta \tag{5.11}$$

and

$$\hat{T}_{\eta}(d, \psi) + \hat{W}_{\eta}(d, \sigma(b), \hat{\Phi}_d) > 1 - \delta. \tag{5.12}$$

Proof. Since f has no GZs on $(a, b)_{\mathbf{T}}$, it follows that f does not change sign on $(a, b)_{\mathbf{T}}$. We can certainly assume that $f(x) > 0$ on $(a, b)_{\mathbf{T}}$, since otherwise we can replace f by $-f$. Let c and $\sigma(d)$ denote the least and the greatest extreme points of f on $[a, b]_{\mathbf{T}}$, respectively. If there is only one extreme point of f, then c and $\sigma(d)$ coincide. Since a is a GZ of f and $f(c) f^{\Delta}(c) \leq 0$, it follows that (5.11) holds due to (5.1). As it is shown in [7, Theorem 6.54] one can see that if $\sigma(d) = d$, then $f^{\Delta}(d) = 0$ and if $\sigma(d) > d$, we have $f(d) f^{\Delta}(d) \geq 0$. Hence, (5.12) follows by (5.2). □

COROLLARY 5.7. *Suppose that for all $c, \sigma(d) \in [a, b]_{\mathbf{T}}$, $c \leq \sigma(d)$, we have*

$$\max\{T_0(c, \psi) + W_0(a, c, \Phi_c), \hat{T}_0(d, \psi) + \hat{W}_0(d, \sigma(b), \hat{\Phi}_d)\} \leq 1 - \delta. \tag{5.13}$$

Then equation (1.4) *is disconjugate on $[a, b]_{\mathbf{T}}$.*

6. The distance between consecutive generalized zeros

In this section, we proceed with the study of disfocality, disconjugacy, lower bounds of eigenvalues, and the distance between GZs for dynamic equation (1.5).

THEOREM 6.1. *Suppose that $f \in \mathscr{L}_a^p([a, c]_{\mathbf{T}}, \tau)$ is a non-trivial solution of equation* (1.5) *and $f^{\Delta}(c) f(c) \leq 0$, then*

$$\tau_{\xi}^{p-1}(c) \left(\sup_{x \in [a, c]_{\mathbf{T}}} \left| \int_x^c \varphi(t) \Delta t \right| \right) > 1. \tag{6.1}$$

Moreover, if there are no extreme values of f on $(a, c)_{\mathbf{T}}$, then

$$\tau_{\xi}^{p-1}(c) \left(\sup_{x \in [a, c]_{\mathbf{T}}} \int_x^c \varphi(t) \Delta t \right) > 1. \tag{6.2}$$

If instead $f \in \mathscr{L}_b^p([c,\sigma(b)]_{\mathbf{T}},\tau)$ *and* $f^{\Delta}(c)f(c) \geq 0$, *then*

$$\hat{\tau}_\eta^{p-1}(c)\left(\sup_{x\in[c,\sigma(b)]_{\mathbf{T}}}\left|\int_c^x \varphi(t)\Delta t\right|\right) > 1. \tag{6.3}$$

Moreover, if there are no extreme values of f *on* $(c,b]_{\mathbf{T}}$, *then*

$$\hat{\tau}_\eta^{p-1}(c)\left(\sup_{x\in[c,\sigma(b)]_{\mathbf{T}}}\int_c^x \varphi(t)\Delta t\right) > 1. \tag{6.4}$$

Proof. Inequalities (6.1) and (6.3) hold due to (5.8) and (5.9), respectively. We now prove (6.2). Suppose that there are no extreme values of f on $(a,c)_{\mathbf{T}}$. Since a is a GZ of f, we can assume that $f(c) = \max_{x\in[a,c]_{\mathbf{T}}} f(x)$. Then $f(x) > 0$, $f^{\Delta}(x) > 0$ and thus $(|f|^p)^{\Delta}(x) > 0$ for $x \in (a,c)_{\mathbf{T}}$. By (5.4) and (5.6), $\sup_{x\in[a,c]_{\mathbf{T}}}\int_x^c \varphi(t)\Delta t > 0$ and so,

$$F_\xi(c) < \left(\sup_{x\in[a,c]_{\mathbf{T}}}\int_x^c \varphi(t)\Delta t\right)\tau_\xi^{p-1}(c)F_\xi(c),$$

which yields (6.2). The proof for (6.4) follows in a way similar to the above. □

Remark 6.2. Observe that [11, Theorems 2.1 and 2.2], [13, Theorems 2.3 and 2.4], and [28, Theorem 2.3] are consequences of Theorem 6.1 if one sets $\mathbf{T} = \mathbf{R}$.

COROLLARY 6.3. *Equation* (1.5) *is disfocal on* $[a,\sigma(b)]_{\mathbf{T}}$ *if*

$$\max\left\{\sup_{x\in[a,b]_{\mathbf{T}}}\left|\int_x^{\sigma(b)} \varphi(t)\Delta t\right|, \sup_{x\in[a,\sigma(b)]_{\mathbf{T}}}\left|\int_a^x \varphi(t)\Delta t\right|\right\} \leq \left(\int_a^{\sigma(b)} \frac{\Delta x}{\tau^{q/p}(x)}\right)^{1-p}. \tag{6.5}$$

The following theorem gives more sufficient conditions for disconjugacy of (1.5) when φ is oscillatory and this behavior affects the bounds.

THEOREM 6.4. *Let* a *and* b *denote two consecutive GZs of a non-trivial solution* f *of* (1.5). *Then there exist two disjoint subintervals of* $[a,b]_{\mathbf{T}}$, I_1 *and* I_2, *satisfying*

$$\int_{I_1\cup I_2} \varphi(x)\Delta x > 2^p\left(\int_a^{\sigma(b)} \chi_\xi(x)\lambda_\eta(x)\tau^{-q/p}(x)\Delta x\right)^{1-p} \tag{6.6}$$

and

$$\int_{[a,\sigma(b)]_{\mathbf{T}}\setminus(I_1\cup I_2)} \varphi(x)\Delta x \leq 0. \tag{6.7}$$

Proof. Since f has no GZs on $(a,b)_{\mathbf{T}}$, we may assume that $f(x) > 0$ on $(a,b)_{\mathbf{T}}$. Let c and $\sigma(d)$ denote the least and the greatest extreme points of f on $[a,b]_{\mathbf{T}}$, respectively. If there is only one extreme point of f, then c and $\sigma(d)$ coincide.

Case 1. Suppose that $c < \sigma(d)$. Thus $c \le d$, and there exists $c_1 \in [a,c)_{\mathbf{T}}$ such that

$$\int_{c_1}^{c} \varphi(x)\Delta x > \tau_{\xi}^{1-p}(c) \quad \text{and} \quad \int_{c_1}^{c} \varphi(x)\Delta x \ge \int_{a}^{c} \varphi(x)\Delta x,$$

by (6.2). Similarly, we can choose $d_1 \in (d,\sigma(b)]_{\mathbf{T}}$ for which

$$\int_{d}^{d_1} \varphi(x)\Delta x > \hat{\tau}_{\eta}^{1-p}(d) \quad \text{and} \quad \int_{d}^{d_1} \varphi(x)\Delta x \ge \int_{d}^{\sigma(b)} \varphi(x)\Delta x.$$

Let $I_1 = [c_1,c]_{\mathbf{T}}$ and $I_2 = [d,d_1]_{\mathbf{T}}$. We thus get

$$\begin{aligned}\int_{I_1\cup I_2} \varphi(x)\Delta x &> \tau_{\xi}^{1-p}(c) + \hat{\tau}_{\eta}^{1-p}(d) \ge 2^p(\tau_{\xi}(c)+\hat{\tau}_{\eta}(d))^{1-p}\\ &\ge 2^p\left(\int_a^{\sigma(b)} \chi_{\xi}(x)\lambda_{\eta}(x)\tau^{-q/p}(x)\Delta x\right)^{1-p},\end{aligned}$$

where we have used Jensen's inequality for the convex function x^{1-p}. So, (6.6) is verified. Obviously, $\int_a^{c_1}\varphi(x)\Delta x \le 0$ and $\int_{d_1}^{\sigma(b)}\varphi(x)\Delta x \le 0$. To prove (6.7) it is sufficient to show that $\int_c^d \varphi(x)\Delta x \le 0$. Dividing both sides of (1.5) by $G_p(f^{\sigma}(x))$, integrating it over $[c,d]_{\mathbf{T}}$, and using (2.4) with $f^{\Delta}(d)f(d) \ge 0$ and $f^{\Delta}(c)f(c) \le 0$, we have

$$\int_c^d \varphi(x)\Delta x \le \int_c^d \tau(x)G_p(f^{\Delta}(x))\left(\frac{1}{G_p(f)}\right)^{\Delta}(x)\Delta x.$$

By [7, Theorem 1.20 (iv)] and (2.11),

$$G_p(f^{\Delta}(x))\left(\frac{1}{G_p(f)}\right)^{\Delta}(x) \le 0, \quad x \in [c,d]_{\mathbf{T}}.$$

Therefore, we conclude that $\int_c^d \varphi(x)\Delta x \le 0$.

Case 2. Assume that there is only one extreme point c of f. Then $f^{\Delta}(x) > 0$ for $x \in (a,c)_{\mathbf{T}}$ and $f^{\Delta}(x) < 0$ for $x \in (c,b]_{\mathbf{T}}$. We choose $\Phi(x) = C - \int_a^x \varphi(t)\Delta t$, where C is some constant. Set

$$M = \max_{x\in[a,\sigma(b)]}\int_a^x \varphi(t)\Delta t = \int_a^{d_1}\varphi(x)\Delta x,$$

$$m = \min_{x\in[a,\sigma(b)]}\int_a^x \varphi(t)\Delta t = \int_a^{c_1}\varphi(x)\Delta x,$$

and take $C = (M+m)/2$. Since $\int_a^x \varphi(t)\Delta t \in [m, M]$, we see that

$$|\Phi(x)| \leq \frac{1}{2}\int_{c_1}^{d_1} \varphi(x)\Delta x. \tag{6.8}$$

Multiply both sides of (1.5) by f^σ and integrate it over $[a,b]_{\mathbf{T}}$ to give

$$-\int_a^b (\tau G_p(f^\Delta))^\Delta(x) f^\sigma(x)\Delta x = \int_a^b \varphi(x)|f^\sigma(x)|^p \Delta x. \tag{6.9}$$

Since $f(a) = -\xi\mu(a)f^\Delta(a)$ and $f(b) = -\eta\mu(b)f^\Delta(b)$, integrating by parts yields

$$-\int_a^b (\tau G_p(f^\Delta))^\Delta(x) f^\sigma(x)\Delta x = F_\xi(c) + \hat{F}_\eta(c). \tag{6.10}$$

By integration by parts formula (2.4), Lemma 2.6 and Hölder's inequality,

$$\begin{aligned} \int_a^b \varphi(x)|f^\sigma(x)|^p \Delta x &\leq \left(\max_{x\in[a,\sigma(b)]_{\mathbf{T}}} |\Phi(x)|\right) 2|f(c)|^p \\ &< \left(\int_{c_1}^{d_1} \varphi(x)\Delta x\right) 2^{-p}(F_\xi(c) + \hat{F}_\eta(c))(\tau_\xi(c) + \hat{\tau}_\eta(c))^{p-1}. \end{aligned} \tag{6.11}$$

Considering (6.10) and (6.11) in (6.9) and dividing two sides by $F_\xi(c) + \hat{F}_\eta(c)$, we get

$$\int_{c_1}^{d_1} \varphi(x)\Delta x > 2^p \left(\int_a^{\sigma(b)} \chi_\xi(x)\lambda_\eta(x)\tau^{-q/p}(x)\Delta x\right)^{1-p},$$

which implies (6.6). Finally, inequality (6.7) follows by $\int_a^{c_1} \varphi(x)\Delta x \leq 0$ and $\int_{d_1}^{\sigma(b)} \varphi(x)\Delta x \leq 0$. □

Corollary 6.5. *Equation* (1.5) *is disconjugate on* $[a,b]_{\mathbf{T}}$ *if for every subintervals* I_1 *and* I_2 *of* $[a,\sigma(b)]_{\mathbf{T}}$,

$$\int_{I_1\cup I_2} \varphi(x)\Delta x \leq 2^p \left(\int_a^{\sigma(b)} \tau^{-q/p}(x)\Delta x\right)^{1-p}. \tag{6.12}$$

Remark 6.6. Our results contained in Theorem 6.4 and Corollary 6.5 reduce to the ones known not only on time scales (see [2, Theorems 4.2.1 and 4.2.2], [4, Theorems 1.3 and 3.6], [32, Theorem 2D], and [21]) but also in **R** (see [8, Corollary 4.1], [11, Theorem 2.3 and Corollary 2.2], [13, Theorem 2.6 and Corollary 2.8], [16, Lemma 1], and [34, Theorems 2.1 and 2.4]).

Theorem 6.4 also allows for a counting of the number of GZs.

Theorem 6.7. *Assume that a non-trivial solution of* (1.5) *has* $(n+1)$ *GZs on* $[a,b]_{\mathbf{T}}$. *Then there exist* $2n$ *disjoint subintervals of* $[a,b]_{\mathbf{T}}$, I_{j1} *and* I_{j2}, *such that*

$$n < \frac{1}{2}\left(\int_a^{\sigma(b)} \tau^{-q/p}(x)\Delta x\right)^{1/p} \left(\int_I \varphi(x)\Delta x\right)^{1/p} \tag{6.13}$$

and

$$\int_{[a,\sigma(b)]_{\mathbf{T}}\setminus I} \varphi(x)\Delta x \leq 0, \tag{6.14}$$

where $I = \bigcup_{j=1}^{n}(I_{j1} \cup I_{j2})$.

Proof. If a solution f of (1.5) has consecutive GZs $a_1 < \cdots < a_{n+1}$ on $[a,b]_{\mathbf{T}}$, then Theorem 6.4 yields that for each $j \in [1,n]_{\mathbf{N}}$, there are two disjoint subintervals of $[a_j, a_{j+1}]_{\mathbf{T}}$, I_{j1} and I_{j2}, with

$$\int_{I_{j1}\cup I_{j2}} \varphi(x)\Delta x > 2^p \left(\int_{a_j}^{\sigma(a_{j+1})} \chi_{\xi_j}(x)\lambda_{\xi_{j+1}}(x)\tau^{-q/p}(x)\Delta x \right)^{1-p} \tag{6.15}$$

and

$$\int_{[a_j,\sigma(a_{j+1})]_{\mathbf{T}}\setminus(I_{j1}\cup I_{j2})} \varphi(x)\Delta x \leq 0, \tag{6.16}$$

where $\xi_i \in [0,1)$ such that $(1-\xi_i)f(a_i) + \xi_i f^\sigma(a_i) = 0$ for $i \in [1,n+1]_{\mathbf{N}}$. Sum (6.15) for j from 1 to n and use Jensen's inequality for the convex function x^{1-p} to obtain

$$\int_I \varphi(x)\Delta x > 2^p n^p \left(\sum_{j=1}^{n} \int_{a_j}^{\sigma(a_{j+1})} \chi_{\xi_j}(x)\lambda_{\xi_{j+1}}(x)\tau^{-q/p}(x)\Delta x \right)^{1-p}. \tag{6.17}$$

Since

$$\sum_{j=1}^{n} \int_{a_j}^{\sigma(a_{j+1})} \chi_{\xi_j}(x)\lambda_{\xi_{j+1}}(x)\tau^{-q/p}(x)\Delta x \leq \int_a^{\sigma(b)} \tau^{-q/p}(x)\Delta x,$$

inequality (6.17) yields (6.13) as required. Finally, from (6.16) it is easy to deduce (6.14). □

Theorem 6.4 also gives a clear relationship between Lyapunov-type inequalities and eigenvalue problems. Let us note that changing φ to $\lambda\varphi$ in (1.5) we easily obtain a lower bound for the eigenvalues using the fact that the eigenfunction f_n associated with the n^{th} eigenvalue, has exactly $(n+1)$ GZs.

THEOREM 6.8. *Let λ_n be the n^{th} eigenvalue of*

$$-(\tau G_p(f^\Delta))^\Delta(x) = \lambda\varphi(x)G_p(f^\sigma(x)), \quad x \in (a,b)_{\mathbf{T}}, \tag{6.18}$$

where a and b are GZs of f. Then there exist $2n$ disjoint subintervals of $[a,b]_{\mathbf{T}}$, I_{j1} and I_{j2}, such that

$$\lambda_n > 2^p n^p \left(\int_a^{\sigma(b)} \tau^{-q/p}(x)\Delta x \right)^{1-p} \left(\int_I \varphi(x)\Delta x \right)^{-1} \tag{6.19}$$

and

$$\int_{[a,\sigma(b)]_{\mathbf{T}}\setminus I} \varphi(x)\Delta x \le 0, \tag{6.20}$$

where $I = \bigcup_{j=1}^{n}(I_{j1} \cup I_{j2})$.

For any $x \in \mathbf{T}$, $\delta > 0$, $x+\delta \in \mathbf{T}$, we denote by $(I_1 \cup I_2)(x,\delta)$ the union of two disjoint subintervals of $[x, x+\delta]_{\mathbf{T}}$, I_1 and I_2. We can now obtain the distance between consecutive GZs of solutions of (1.5). A non-trivial solution of (1.5) is called oscillatory if it has infinitely many (isolated) GZs in $[a,\infty)_{\mathbf{T}}$.

Theorem 6.9. *If f is an oscillatory solution of* (1.5), $\tau \ge K > 0$ *on* $[a,\infty)_{\mathbf{T}}$, *and*

$$\limsup_{x\to\infty}\left(\frac{\delta^{p-1}}{K}\int_{(I_1\cup I_2)(x,\delta)} \varphi(t)\Delta t\right) < 2^p \tag{6.21}$$

for all $\delta > 0$ and for every two disjoint subintervals I_1 and I_2 of $[x, x+\delta]_{\mathbf{T}}$, then the distance between consecutive GZs of f is unbounded as $x \to \infty$.

Proof. Assume, for a contradiction, that equation (1.5) has an oscillatory solution f whose GZs contain a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $0 < \sigma(x_{n_{k+1}}) - x_{n_k} \le \delta$ for some δ and all k. By Theorem 6.4, there are disjoint subintervals $I_1(x_{n_k},\delta)$ and $I_2(x_{n_k},\delta)$ of $[x_{n_k}, x_{n_k}+\delta]_{\mathbf{T}}$ satisfying

$$\int_{(I_1\cup I_2)(x_{n_k},\delta)} \varphi(x)\Delta x > 2^p\left(\int_{x_{n_k}}^{\sigma(x_{n_{k+1}})} K^{-q/p}\Delta x\right)^{1-p} \ge 2^p K\delta^{1-p} > 0$$

for any k, since $\tau \ge K > 0$ on $[a,\infty)_{\mathbf{T}}$. We thus have

$$\limsup_{k\to\infty}\left(\frac{\delta^{p-1}}{K}\int_{(I_1\cup I_2)(x_{n_k},\delta)} \varphi(t)\Delta t\right) \ge 2^p,$$

which contradicts (6.21). □

Remark 6.10. For the time scale $\mathbf{T} = \mathbf{R}$, Theorem 6.9 generalizes [11, Theorem 3.1], [13, Theorem 3.1], [18, Theorem 2], and [33, Theorem 1].

Theorem 6.11. *Let f be an oscillatory solution of* (1.5). *If $\tau \ge K > 0$ on $[a,\infty)_{\mathbf{T}}$ and there exists a $\delta_0 > 0$ such that for every two disjoint subintervals of $[x, x+\delta_0]_{\mathbf{T}}$, I_1 and I_2,*

$$\lim_{x\to\infty}\left(\int_{(I_1\cup I_2)(x,\delta_0)} \varphi(t)\Delta t\right) = 0, \tag{6.22}$$

then the distance between consecutive GZs of f must become infinite as $x \to \infty$.

Proof. We first claim that for all $\delta > 0$ and any disjoint subintervals of $[x, x+\delta]_{\mathbf{T}}$, I_1 and I_2,

$$\lim_{x\to\infty}\left(\int_{(I_1\cup I_2)(x,\delta)} \varphi(t)\Delta t\right) = 0. \tag{6.23}$$

Let k denote the least integer with $k\delta_0 \geq \delta$, $x_j = x + j\delta_0$, $j = 0, 1, \ldots, k-1$, and $x_k = \delta$. By taking $I_{ji}(x_j, \delta_0) = I_i(x, \delta) \cap [x_j, x_{j+1}]_{\mathbf{T}}$ for $j = 0, 1, \ldots, k-1$ and $i = 1, 2$, we get

$$\int_{(I_1\cup I_2)(x,\delta)} \varphi(t)\Delta t = \sum_{j=0}^{k-1} \int_{(I_{j1}\cup I_{j2})(x_j,\delta_0)} \varphi(t)\Delta t. \tag{6.24}$$

Since $x_j \to \infty$ as $x \to \infty$, we have from (6.22) that

$$\lim_{x\to\infty}\left(\int_{(I_{j1}\cup I_{j2})(x_j,\delta_0)} \varphi(t)\Delta t\right) = 0 \quad \text{for } j = 0, 1, \ldots, k-1. \tag{6.25}$$

Combining (6.25) and (6.24) we obtain (6.23). As a consequence of (6.23), for $\delta > 0$ and any disjoint subintervals I_1 and I_2 of $[x, x+\delta]_{\mathbf{T}}$, we have

$$\lim_{x\to\infty}\left(\frac{\delta^{p-1}}{K}\int_{(I_1\cup I_2)(x,\delta)} \varphi(t)\Delta t\right) = 0.$$

Thus, the result follows by Theorem 6.9. $\square$

Remark 6.12. The results given in [11, Theorem 3.2], [13, Theorem 3.2], and [33, Theorem 2] are special cases of Theorem 6.11 obtained by setting $\mathbf{T} = \mathbf{R}$.

Remark 6.13. Most of our results are essentially new even in the well-studied difference equation setting, as far as the authors are aware.

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