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GLOBAL UNIQUENESS RESULTS FOR GROUND STATES FOR A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS

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Abstract

In this paper, we are concerned with the uniqueness of ground states for a class of quasilinear elliptic equations which arise in the study of plasma physics. We obtain *global* uniqueness results in the sense that we don't require any assumptions on the parameter.

1. Introduction and main results

In this paper, we consider the following quasilinear Schrödinger equation:

$$i\frac{\partial z}{\partial t} = -\Delta z - |z|^{p-1}z - \kappa\Delta(|z|^2)z, \quad (t,x) \in (0,\infty) \times \mathbf{R}^N, \tag{1.1}$$

where $\kappa > 0$, $N \geq 1$ and $p > 1$. Our aim of this paper is to prove the uniqueness and the non-degeneracy of ground states of (1.1).

Recently there has been a lot of studies on the quasilinear Schrödinger equation (1.1). This problem arises in the study of superfluid film equation in plasma physics, and also this type of quasilinear problems is known to be a more accurate model in various physical phenomena compared with the standard semilinear problem. For more physical backgrounds, see [4], [5], [13]. From a physical as well as mathematical point of view, the most important topic is the stability of standing waves of the form: $z(t,x) = u(x)e^{i\lambda t}$, $\lambda > 0$, where $u : \mathbf{R}^N \to \mathbf{R}$ is a real valued function. Substituting this form for (1.1), we obtain the following quasilinear elliptic problem:

$$-\Delta u + \lambda u - \kappa\Delta(u^2)u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N. \tag{1.2}$$

It is known that in the study of the stability of standing waves, the uniqueness and the non-degeneracy of the ground state of (1.2) plays an important role. (See [6], [9], [10] for the results on the (in)stability.)

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One of the main difficulty on the uniqueness and non-degeneracy of the ground state is the fact that $p = \frac{3N+2}{N-2}$ is a critical exponent for the existence of nontrivial solutions. More precisely, since the Sobolev critical exponent $\frac{N+2}{N-2}$ is in the range $\left(1, \frac{3N+2}{N-2}\right)$, the feature of the limit equation as $\kappa \to 0$ is drastically changed. Indeed taking $\kappa \to 0$ formally in (1.2), we see that the problem reduces to the following semilinear elliptic equation:

$$-\Delta u + \lambda u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N. \tag{1.3}$$

It is well-known that the ground state (positive solution) of (1.3) is unique and non-degenerate if $1 < p < \frac{N+2}{N-2}$. On the other hand, if $\frac{N+2}{N-2} < p$, then it is also well-known that there exists no nontrivial solution of (1.3). Thus in view point of small perturbation problem of (1.3) for small $\kappa > 0$, we have to consider the uniqueness and non-degeneracy of the ground state of (1.2) for the case $1 < p < \frac{N+2}{N-2}$ and $\frac{N+2}{N-2} < p < \frac{3N+2}{N-2}$ respectively. We refer for instance [3] and [22] for the case $1 < p < \frac{N+2}{N-2}$ and small $\kappa > 0$, [2] for the case $\frac{N+2}{N-2} < p < \frac{3N+2}{N-2}$ and small $\kappa > 0$.

The main purpose of this paper is to obtain a *global* result, which means that we prove the uniqueness and the non-degeneracy *without* any restriction on the parameter κ and λ. Especially our aim is to obtain the uniqueness of the ground state of (1.2) for *full range* of p and for *any* κ, λ.

Although our main interest is in the problem (1.2), we consider general quasilinear problems in this paper. More precisely as in [11], we study quasilinear elliptic problems of the form:

$$-\mathrm{div}(a(u)\nabla u) + \frac{1}{2}a'(u)|\nabla u|^2 + u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N \tag{1.4}$$

where $N \ge 3$, the quasilinear term $a(t)$ has the form:

$$a(t) = a_0|t|^{\ell} + \psi(t) \quad (\ell > 0,\ a_0 \ge 0,\ \psi \in C^2(\mathbf{R}))$$

and the exponent p satisfies $1 < p < \frac{(\ell+1)N+2}{N-2}$.

We suppose that the following conditions hold for $a(t)$ and $\psi(t)$:

(a1) $\inf_{t \ge 0} a(t) > 0$, $a'(t) \ge 0$ for $t \ge 1$ and $a(t) \le a(1)$ for $0 \le t \le 1$.

(a2) $\lim_{t\to\infty} \frac{\psi(t)}{t^{\ell}} = 0$.

(a3) $\ell\psi(t) - t\psi'(t) \ge 0$ for $t \ge 1$.

(a4) $(\ell-1)\psi'(t) - t\psi''(t) \le 0$ for $t \ge 1$.

(a5) $\frac{\ell+2}{2}\int_0^1 \sqrt{a(t)}\,dt - \frac{\ell+2}{2}\sqrt{a(1)} + \frac{a'(1)}{2\sqrt{a(1)}} \ge 0$.

Under assumptions (a1)–(a2), the problem (1.4) has a positive radial solution. (See Proposition 2.9 below.) In [11], it was shown that the positive radial solution is unique under some restriction on parameter, which is equivalent to the condition a_0 is sufficiently large. The main result of this paper is to show that the uniqueness holds *without any restrictions on parameter*. Indeed we have the following result.

THEOREM 1.1. *Suppose that* (a1)–(a5) *hold and* $1 < p < \frac{(\ell+1)N+2}{N-2}$. *Assume further* $p \geq \frac{\ell}{2}$ *if* $\ell > 2$. *Then the positive radial solution of* (1.4) *is unique.*

A typical example of $a(t)$ is given by $a(t) = 1 + \alpha\kappa|t|^{2\alpha-2}$ which can be obtained by taking $a_0 = \alpha\kappa$, $\ell = 2\alpha - 2$ and $\psi(t) \equiv 1$. We can easily see that for $a(t) = 1 + \alpha\kappa|t|^{2\alpha-2}$, (1.4) becomes

$$-\Delta u + u - \kappa\Delta(|u|^{\alpha})|u|^{\alpha-2}u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N.$$

We also note that since $p > 1$, the condition $p \geq \frac{\ell}{2}$ is automatically satisfied if $0 < \ell \leq 2$. This restriction on p has been already observed in [1], [2].

Now for a solution u of (1.2), we rescale $u(x) = \lambda^{1/(p-1)}\tilde{u}(\lambda^{1/2}x)$. Then we can see that $\tilde{u}$ satisfies

$$-\Delta\tilde{u} + \tilde{u} - \kappa\lambda^{2/(p-1)}\Delta(\tilde{u}^2)\tilde{u} = |\tilde{u}|^{p-1}\tilde{u} \quad \text{in } \mathbf{R}^N.$$

Applying Theorem 1.1 to the case $a(t) = 1 + 2\kappa\lambda^{2/(p-1)}t^2$, we obtain the following result for the ground state of (1.2).

THEOREM 1.2. *Suppose* $N \geq 3$ *and* $1 < p < \frac{3N+2}{N-2}$. *Then for any* $\kappa > 0$ *and* $\lambda > 0$, *the ground state of* (1.2) *is unique* (*up to translation*).

Recently in [23], the non-degeneracy of the ground state of (1.2) has shown for all κ and λ under the additional assumption $p \geq 3$. Our second purpose of this paper is to obtain the non-degeneracy for (1.4) under similar assumption $p \geq \ell + 1$. To this aim, we impose the following conditions on $a(t)$ instead of (a1):

(a1$'$) $\inf_{t\geq 0} a(t) > 0$ and $a'(t) \geq 0$ for $t \geq 0$.

In this setting, we have the following result.

THEOREM 1.3. *Suppose* (a1$'$), (a2)–(a5) *hold and* $\ell + 1 \leq p < \frac{(\ell+1)N+2}{N-2}$. *Then the kernel of the linearized operator around the unique positive radial solution* u *of* (1.4) *is given by*

$$\mathrm{Ker}(L) = \mathrm{span}\left\{\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right\}.$$

Especially u is non-degenerate in $H^1_{\mathrm{rad}}(\mathbf{R}^N)$, *that is, if* $L(\phi) = 0$ *and* $\phi \in H^1_{\mathrm{rad}}(\mathbf{R}^N)$, *then* $\phi \equiv 0$.

Here the linearized operator L of (1.4) *is defined by*

$$L(\phi) = -a(u)\Delta\phi - a'(u)\nabla u \cdot \nabla\phi - \frac{1}{2}a''(u)|\nabla u|^2\phi - a'(u)\Delta u\phi + \phi - pu^{p-1}\phi. \tag{1.5}$$

Finally we can also obtain the uniqueness and the non-degeneracy for *complex-valued* ground state of (1.2). See Section 5 below for the result.

To prove Theorem 1.1, we adapt *dual approach* as in [1], [8]. More precisely, we convert our quasilinear equation into a semilinear equation by using a suitable translation f. We will see that the set of positive radial solutions of (1.4) has one-to-one correspondence to that of the semilinear problem. This enables us to apply the uniqueness result [24] for semilinear elliptic equations. We can also show that there is a complete relation between the linearized operator of the original quasilinear equation and that of the converted semilinear equation. This enables us to reduce our analysis into the study of the non-degeneracy for the converted semilinear problem.

The main idea to obtain the global uniqueness result is rather simple. We have just shown an improvement inequality of a function related to the dual transformation. (See Remark 2.4 below.) In order to prove the uniqueness, we have to show that some function related to the nonlinear term is monotone. Using the improved inequality carefully, we can show this monotonicity holds. Once we could get the uniqueness, the non-degeneracy can be proved by ODE analysis and spherical harmonic decomposition.

This paper is organized as follows. In section 2, we introduce the dual approach of (1.4) and prepare some auxiliary lemmas. We prove the uniqueness of positive radial solutions in section 3. We show the non-degeneracy of the unique positive radial solution in section 4. In section 5, we state results on complex-valued ground states of (1.2). Finally in section 6, we give some comments on the quasilinear term $a(t)$ of (1.4).

2. Dual approach and auxiliary lemmas

2.1. Properties of functions related to (1.4)

First we begin with the following relations, which can be shown by a direct consequence of the definition of $a(t)$.

Lemma 2.1. *For* $t > 0$, *functions* a' *and* a'' *can be written as follows*:
(i) $ta'(t) = \ell a(t) - (\ell\psi(t) - t\psi'(t))$.
(ii) $t^2a''(t) = (\ell - 1)\ell a(t) - (\ell - 1)(\ell\psi(t) - t\psi'(t)) - t((\ell - 1)\psi'(t) - t\psi''(t))$.

Next let $f(s)$ be a unique solution of the following ODE:

$$f'(s) = \frac{1}{\sqrt{a(f(s))}} \quad \text{for } s > 0, \quad f(0) = 0. \tag{2.1}$$

We extend $f(s)$ as an odd function for $s < 0$. We can see that $f(s)$ is monotone and hence the inverse f^{-1} exists.

Now by the definition of $f(s)$, we have the followings.

Lemma 2.2. *For $s > 0$, f'' and f''' can be written as follows*:
(i) $f''(s) = -\frac{1}{2}a'(f(s))f'(s)^4$.
(ii) $f'''(s) = -\frac{1}{2}a''(f(s))f'(s)^5 + a'(f(s))^2 f'(s)^7$.

The next lemma is the key to prove our global uniqueness result, which is an improvement of already known inequality. (See Remark 2.4 below.)

Lemma 2.3. *Assume* (a1), (a3)–(a5). *Then the following inequalities hold for $s \geq f^{-1}(1)$*:
(i) $sf'(s) \leq f(s)$.
Moreover if in addition (a1$'$) *is satisfied*, *then the same inequality holds for $s \geq 0$.*
(ii) $f(s) + \dfrac{f(s)f'(s)^2}{2}(\ell\psi(f(s)) - f(s)\psi'(f(s))) \leq \dfrac{\ell+2}{2} sf'(s)$.

Proof. (i) Since f is monotone, it follows that $f(s) \geq 1$ for $s > f^{-1}(1)$. Then by (a1) and Lemma 2.2 (i), we have

$$(f - sf')' = -sf'' = \frac{s}{2}a'(f)f'^4 > 0.$$

Moreover by (a1), we also have $\int_0^1 \sqrt{a(t)}\,dt \leq \sqrt{a(1)}$. Putting $t = f(s)$, we get $f^{-1}(1) \leq \dfrac{1}{f'(f^{-1}(1))}$ and hence $f(s) - sf'(s) \geq 0$ at $s = f^{-1}(1)$. Thus the claim holds for $s \geq f^{-1}(1)$.

If (a1$'$) is satisfied, then $(f - sf')' \geq 0$ for $s \geq 0$. Since $(f - sf')|_{s=0} = 0$, we get the same inequality for $s \geq 0$. This completes the proof of (i).

(ii) Since $f' > 0$, it suffices to show that

$$H(s) := \frac{\ell+2}{2}s - \frac{f}{f'} - \frac{ff'}{2}(\ell\psi(f) - f\psi'(f)) \geq 0 \quad \text{for } s \geq f^{-1}(1).$$

First we observe by Lemma 2.1 that

$$H(s) = \frac{\ell+2}{2}s - \frac{f}{f'} - \frac{ff'}{2}(\ell a(f) - fa'(f)).$$

Then by $f^{-1}(1) = \int_0^1 \sqrt{a(t)}\,dt$, $f'(f^{-1}(1)) = \dfrac{1}{\sqrt{a(1)}}$ and (a5), it follows that $H(f^{-1}(1)) \geq 0$. Next by a direct calculation, we get

$$\begin{aligned} H'(s) &= \frac{\ell}{2} + \frac{ff''}{f'^2} - \frac{f'^2}{2}(\ell\psi(f) - f\psi'(f)) \\ &\quad - \frac{ff''}{2}(\ell\psi(f) - f\psi'(f)) - \frac{ff'^2}{2}((\ell-1)\psi'(f) - f\psi''(f)). \end{aligned}$$

By Lemmas 2.1 (i), 2.2 (i) and (2.1), it follows that

$$\begin{aligned} \frac{ff''}{f'^2} &= -\frac{1}{2}a'(f)ff'^2 = -\frac{1}{2}f'^2(\ell a(f) - (\ell\psi(f) - f\psi'(f))) \\ &= -\frac{\ell}{2} + \frac{f'^2}{2}(\ell\psi(f) - f\psi'(f)). \end{aligned}$$

Thus from (a1), (a3) and (a4), we obtain

$$H'(s) = \frac{a'(f)ff'^4}{4}(\ell\psi(f) - f\psi'(f)) - \frac{ff'^2}{2}((\ell-1)\psi'(f) - f\psi''(f)) \geq 0.$$

This completes the proof. □

Remark 2.4. When $a(t) = 1 + t^2$, we have $\ell = 2$ and $\psi(t) \equiv 1$. Then the inequality in Lemma 2.3 (ii) reduces to

$$f(s) + f(s)f'(s)^2 \leq 2sf'(s).$$

We notice that it is already known that the inequality $f(s) \leq 2sf'(s)$ holds (see [8]). Our inequality is a simple improvement of it but this improvement enables us to obtain the global uniqueness result.

Finally in this subsection, we give asymptotic behavior of $f(s)$ at infinity, which will be used in section 4. For the proof, we refer to [11].

LEMMA 2.5. *$f(s)$ satisfies the following properties*:

$$\lim_{s\to\infty} \frac{f(s)}{s^{2/(\ell+2)}} = \left(\frac{\ell+2}{2\sqrt{a_0}}\right)^{2/(\ell+2)}, \quad \lim_{s\to\infty} \frac{f'(s)}{s^{2/(\ell+2)-1}} = \frac{2}{\ell+2}\left(\frac{\ell+2}{2\sqrt{a_0}}\right)^{2/(\ell+2)},$$

$$\lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \frac{2}{\ell+2}.$$

2.2. Dual approach and its correspondence

Now we consider the following semilinear elliptic problem:

$$-\Delta v + f(v)f'(v) = |f(v)|^{p-1}f(v)f'(v) \quad \text{in } \mathbf{R}^N, \tag{2.2}$$

which we call a *dual problem* of (1.4). Then we can show the following relation between (1.4) and (2.2).

PROPOSITION 2.6. *$u \in X \cap C^2(\mathbf{R}^N)$ is a positive radial solution of* (1.4) *if and only if $v = f^{-1}(u) \in H^1 \cap C^2(\mathbf{R}^N)$ is a positive radial solution of* (2.2).

For the proof, we refer to [11]. Proposition 2.6 tells us that if (2.2) has a unique positive radial solution v, then $u = f(v)$ is a unique positive radial solution of (1.4). Thus we have only to study the uniqueness of the positive radial solution of the semilinear problem (2.2).

Next we study the correspondence of the linearized operators of (1.4) and (2.2). Now let $\tilde{L} : H^2(\mathbf{R}^N) \to L^2(\mathbf{R}^N)$ be a linearized operator of (2.2), which is defined by

$$\tilde{L}(\tilde{\phi}) := -\Delta\tilde{\phi} + (f'(v)^2 + f(v)f''(v))\tilde{\phi} - (pf(v)^{p-1}f'(v)^2 + f(v)^p f''(v))\tilde{\phi}. \tag{2.3}$$

Then we have the following relation, which was already obtained in [2] for the case $a(t) = 1 + |t|^{2\alpha-2}$.

LEMMA 2.7. *Suppose that $u \in X \cap C^2(\mathbf{R}^N)$ is a positive solution of* (1.4) *and put $v = f^{-1}(u)$. Let L and $\tilde{L} : H^2(\mathbf{R}^N) \to L^2(\mathbf{R}^N)$ be the linearized operators defined by* (1.5) *and* (2.3) *respectively. Finally for $\phi \in H^2(\mathbf{R}^N)$, we put $\tilde{\phi} = \sqrt{a(u)}\phi$. Then it follows that*

$$\tilde{L}(\tilde{\phi}) = \frac{1}{\sqrt{a(u)}} L(\phi). \tag{2.4}$$

Proof. By direct computations, we have

$$\nabla\tilde{\phi} = \sqrt{a(u)}\nabla\phi + \frac{a'(u)\phi}{2\sqrt{a(u)}}\nabla u,$$

$$\Delta\tilde{\phi} = \sqrt{a(u)}\Delta\phi + \frac{a'(u)}{\sqrt{a(u)}}\nabla u \cdot \nabla\phi + \frac{a'(u)\Delta u}{2\sqrt{a(u)}}\phi + \frac{a''(u)|\nabla u|^2}{2\sqrt{a(u)}}\phi - \frac{a'(u)^2|\nabla u|^2}{4(\sqrt{a(u)})^3}\phi.$$

Next by Lemma 2.2 and from (2.1), we also have

$$(f'(v)^2 + f(v)f''(v))\tilde{\phi} = \left(f'(v)^2 - \frac{1}{2}a'(f(v))f(v)f'(v)^4\right)\tilde{\phi} = \frac{1}{\sqrt{a(u)}}\phi - \frac{a'(u)u}{2(\sqrt{a(u)})^3}\phi,$$

$$(pf(v)^{p-1}f'(v)^2 + f(v)^p f''(v))\tilde{\phi} = \frac{pu^{p-1}}{\sqrt{a(u)}}\phi - \frac{a'(u)u^p}{2(\sqrt{a(u)})^3}\phi.$$

Thus from (1.4), (1.5) and (2.3), we obtain

$$\begin{aligned}\tilde{L}(\tilde{\phi}) &= -\Delta\tilde{\phi} + (f'^2 + ff'')\tilde{\phi} - (pf^{p-1}f'^2 + f^p f'')\tilde{\phi} \\ &= \frac{1}{\sqrt{a(u)}}\left(-a(u)\Delta\phi - a'(u)\nabla u\cdot\nabla\phi - \frac{1}{2}a''(u)|\nabla u|^2\phi \right.\\ &\qquad \left. - a'(u)\Delta u\phi + \phi - pu^{p-1}\phi\right) \\ &\quad - \frac{a'(u)}{2(\sqrt{a(u)})^3}\left(-a(u)\Delta u - \frac{1}{2}a'(u)|\nabla u|^2 + u - u^p\right)\phi \\ &= \frac{1}{\sqrt{a(u)}}L(\phi).\end{aligned}$$

This completes the proof. □

By Lemma 2.7, we obtain the following result.

Proposition 2.8. *Suppose that* $u \in X \cap C^2(\mathbf{R}^N)$ *is a positive solution of* (1.4) *and put* $v = f^{-1}(u)$. *Then*

(i) $\phi \in \mathrm{Ker}(L)$ *if and only if* $\tilde{\phi} = \sqrt{a(u)}\phi \in \mathrm{Ker}(\tilde{L})$.

(ii) *u is non-degenerate if and only if v is non-degenerate.*

(iii) $\mathrm{Ker}(L) = \mathrm{span}\left\{\frac{\partial u}{\partial x_i}\right\}_{i=1}^N$ *if and only if* $\mathrm{Ker}(\tilde{L}) = \mathrm{span}\left\{\frac{\partial v}{\partial x_i}\right\}_{i=1}^N$.

Proof. (i) From (2.4), we have $\tilde{L}(\tilde{\phi}) = 0 \Leftrightarrow L(\phi) = 0$. Thus the claim holds.

(ii) The claim follows from (i).

(iii) We assume that $\mathrm{Ker}(L) = \mathrm{span}\left\{\frac{\partial u}{\partial x_i}\right\}_{i=1}^N$. Suppose by contradiction that $\mathrm{span}\left\{\frac{\partial v}{\partial x_i}\right\}_{i=1}^N \neq \mathrm{Ker}(\tilde{L})$. Since $\frac{\partial v}{\partial x_i} \in \mathrm{Ker}(\tilde{L})$ for $i = 1, \dots, N$, we have $\mathrm{span}\left\{\frac{\partial v}{\partial x_i}\right\}_{i=1}^N \subsetneq \mathrm{Ker}(\tilde{L})$. Thus there exists $\tilde{\phi} \not\equiv 0$ such that

$$\tilde{\phi} \in \mathrm{Ker}(\tilde{L}) \Big\backslash \mathrm{span}\left\{\frac{\partial v}{\partial x_i}\right\}_{i=1}^N.$$

Since $\tilde{\phi} \in \mathrm{Ker}(\tilde{L})$, we have $\tilde{L}(\tilde{\phi}) = 0$. Putting $\tilde{\phi} = \sqrt{a(u)}\phi$, we obtain $L(\phi) = 0$ by Lemma 2.7. Then by the assumption $\mathrm{Ker}(L) = \mathrm{span}\left\{\frac{\partial u}{\partial x_i}\right\}_{i=1}^N$, there exist $c_1, \dots, c_N$ such that

$$\phi = c_1\frac{\partial u}{\partial x_1} + \cdots + c_N\frac{\partial u}{\partial x_N}.$$

Now since $u = f(v)$, it follows that

$$\frac{\partial u}{\partial x_i} = f'(v)\frac{\partial v}{\partial x_i} = \frac{1}{\sqrt{a(u)}}\frac{\partial v}{\partial x_i} \quad \text{for } i = 1, \dots, N.$$

Thus we have

$$\tilde{\phi} = c_1\frac{\partial v}{\partial x_1} + \cdots + c_N\frac{\partial v}{\partial x_N} \in \operatorname{span}\left\{\frac{\partial v}{\partial x_i}\right\}_{i=1}^{N}.$$

This is a contradiction and hence $\operatorname{Ker}(\tilde{L}) = \operatorname{span}\left\{\frac{\partial v}{\partial x_i}\right\}_{i=1}^{N}$.

We can show the converse in a similar way. □

By Proposition 2.8, we have only to study the non-degeneracy of the unique positive radial solution of the semilinear problem (2.2).

2.3. Existence of a positive radial solution

In this subsection, we study the existence of a positive radial solution of (1.4). Indeed we are going to show the existence of a ground state of (1.4). To this aim, we define the energy functional $I : X \to \mathbf{R}$ by

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^N} a(u)|\nabla u|^2 + u^2\,dx - \frac{1}{p+1}\int_{\mathbf{R}^N}|u|^{p+1}\,dx,$$

where

$$X := \{u \in H^1(\mathbf{R}^N); a(u)|\nabla u|^2 \in L^1(\mathbf{R}^N)\}.$$

A solution u of (1.4) is called a ground state if it satisfies

$$I(u) = \inf\{I(w); I'(w) = 0, w \in X\backslash\{0\}\}.$$

Then we have the following result.

Proposition 2.9. *Assume* (a1)–(a2) *and* $1 < p < \frac{(\ell+1)N+2}{N-2}$. *Then the equation* (1.4) *has a ground state. Moreover any ground state is* C^2, *positive, radially symmetric* (*up to translation*), *decreasing and exponentially decaying up to second derivatives.*

For the proof, we adapt arguments in [9]. (See also [11].)

3. Uniqueness of the positive radial solution

In this section, we study the uniqueness of the positive radial solution of (2.2). For simplicity, we put

$$(3.1) \qquad g(s) := f(s)^p f'(s) - f(s)f'(s) \quad \text{for } s \geq 0 \quad \text{and} \quad K_g(s) := \frac{sg'(s)}{g(s)}.$$

We apply the following uniqueness result due to Serrin and Tang [24].

Proposition 3.1 [24]. *Suppose that there exists* $b > 0$ *such that*

(i) g *is continuous on* $(0, \infty)$, $g(s) \le 0$ *on* $(0, b]$ *and* $g(s) > 0$ *for* $s > b$.

(ii) $g \in C^1(b, \infty)$ *and* $K_g'(s) < 0$ *on* (b, ∞).

Then the semilinear problem:

$$-\Delta v = g(v) \quad \text{in } \mathbf{R}^N,\ v > 0,\ v \to 0 \text{ as } |x| \to \infty, \quad v(0) = \max_{x \in \mathbf{R}^N} v(x)$$

has at most one positive radial solution.

Now we can see that g defined in (3.1) is of the class $C^1[0, \infty)$ and

$$g(s) = 0 \Leftrightarrow f^{p-1}(s) = 1 \Leftrightarrow s = f^{-1}(1).$$

We put $b := f^{-1}(1)$. Since $g(s) = ff'(f^{p-1} - 1)$, we can see that (i) of Proposition 3.1 holds.

Lemma 3.2. *Suppose that* $p \ge \frac{\ell}{2}$ *if* $\ell > 2$. *Then the function* g *defined in* (3.1) *satisfies* (ii) *of Proposition* 3.1.

Proof. First we see that

$$K_g'(s) = \frac{1}{g(s)^2}(g''(s)g(s)s + g'(s)g(s) - g'(s)^2 s).$$

Thus we have only to show that $sg''g + g'g - sg'^2 < 0$ for $s > b$.

Now by direct computations, it follows that

$$g(s) = f^p f' - ff' = ff'(f^{p-1} - 1), \tag{3.2}$$

$$g'(s) = (f^{p-1} - 1)(ff'' + f'^2) + (p-1)f^{p-1}f'^2, \tag{3.3}$$

$$g''(s) = (f^{p-1} - 1)(ff''' + 3f'f'') + (p-1)f^{p-2}(3ff'f'' + pf'^3). \tag{3.4}$$

Next we have

$$(f^{p-1} - 1)f^{p-1} = (f^{p-1} - 1)^2 + f^{p-1} - 1 = (f^{p-1} - 1)^2 + (f^{p-1} - 1),$$

$$f^{2p-2} = (f^{p-1} - 1)^2 + 2f^{p-1} - 1 = (f^{p-1} - 1)^2 + 2(f^{p-1} - 1) + 1.$$

From these equalities and (3.2)–(3.4), we can describe $sg''g + gg' - sg'^2$ as a polynomial of $f^{p-1} - 1$. Then we obtain

$$\begin{aligned} sg''g + g'g - sg'^2 &= (f^{p-1} - 1)^2(sf^2f'f''' + psff'^2f'' + f^2f'f'' - sf^2f''^2 - psf'^4 + pff'^3) \\ &\quad + (p-1)(f^{p-1} - 1)(sff'^2f'' - psf'^4 + ff'^3) - (p-1)^2sf'^4. \end{aligned}$$

By Lemma 2.2, we can write f'' and f''' by using f and f'. Then we have

$$
\begin{aligned}
&sf^2f'f''' + psff'^2f'' + f^2f'f''' - sf^2f''^2 - psf'^4 + pff'^3 \\
&\quad = sf^2f'\left(-\frac{1}{2}a''(f)f'^5 + a'(f)^2f'^7\right) + psff'^2\left(-\frac{1}{2}a'(f)f'^4\right) \\
&\qquad + f^2f'\left(-\frac{1}{2}a'(f)f'^4\right) - sf^2\left(-\frac{1}{2}a'(f)f'^4\right)^2 - psf'^4 + pff'^3 \\
&\quad = -\frac{1}{2}sa''(f)f^2f'^6 + \frac{3}{4}sa'(f)^2f^2f'^8 - \frac{p}{2}sa'(f)ff'^6 \\
&\qquad - \frac{1}{2}a'(f)f^2f'^5 - psf'^4 + pff'^3 \\
&\quad =: H_1(s)
\end{aligned}
$$

and

$$
sff'^2f'' - psf'^4 + ff'^3 = -\frac{1}{2}sa'(f)ff'^6 - psf'^4 + ff'^3 =: H_2(s).
$$

Thus we obtain

$$
\begin{aligned}
(3.5)\qquad &sg''g + g'g - sg'^2 \\
&\quad = (f^{p-1} - 1)^2H_1(s) + (p-1)(f^{p-1} - 1)H_2(s) - (p-1)^2sf'^4.
\end{aligned}
$$

By the definition of b, it follows that $f(s)^{p-1} - 1 > 0$ and $(p-1)^2sf'^4 > 0$ for $s > b$. Thus it suffices to show that $H_1(s) \le 0$, $H_2(s) \le 0$ for $s > b$ in order to prove that $sgg'' + g'g - sg'^2 < 0$.

First we estimate $H_1(s)$. To this aim, we rewrite a'' and a' by using ψ and a. Indeed by Lemma 2.1, it follows that

$$
\begin{aligned}
\frac{H_1(s)}{f'^3} &= -\frac{1}{2}sf'^3((\ell-1)\ell a - (\ell-1)(\ell\psi - f\psi') - f((\ell-1)\psi' - f\psi'')) \\
&\quad + \frac{3}{4}sf'^5(\ell a - (\ell\psi - f\psi'))^2 - \frac{p}{2}sf'^3(\ell a - (\ell\psi - f\psi')) \\
&\quad - \frac{1}{2}ff'^2(\ell a - (\ell\psi - f\psi')) - psf' + pf.
\end{aligned}
$$

Using $f' = \dfrac{1}{\sqrt{a}}$, we get

$$
\begin{aligned}
(3.6)\qquad \frac{H_1(s)}{f'^3} &= -\left(p - \frac{\ell}{2}\right)\frac{\ell+2}{2}sf' + \left(p - \frac{\ell}{2}\right)f + \frac{1}{2}ff'^2(\ell\psi - f\psi') \\
&\quad + \frac{p - 2\ell - 1}{2}sf'^3(\ell\psi - f\psi') + \frac{3}{4}sf'^5(\ell\psi - f\psi')^2 \\
&\quad + \frac{1}{2}sff'^3((\ell-1)\psi' - f\psi'').
\end{aligned}
$$

Now since $p \geq \frac{\ell}{2}$, we can apply Lemma 2.3 (ii) to the first and the second terms in the right hand side of (3.6). Then we have

$$\frac{H_1(s)}{f'^3} \leq -\left(p - \frac{\ell}{2} - 1\right)\frac{ff'^2}{2}(\ell\psi - f\psi') + (p - 2\ell - 1)\frac{sf'^3}{2}(\ell\psi - f\psi') + \frac{3}{4}sf'^5(\ell\psi - f\psi')^2 + \frac{1}{2}sff'^3((\ell - 1)\psi' - f\psi'').$$

Applying Lemma 2.3 (i) to the term sf'^5, we get

$$\frac{H_1(s)}{f'^3} \leq -\left(p - \frac{\ell}{2} - 1\right)\frac{ff'^2}{2}(\ell\psi - f\psi') + (p - 2\ell - 1)\frac{sf'^3}{2}(\ell\psi - f\psi') + \frac{3}{4}ff'^4(\ell\psi - f\psi')^2 + \frac{1}{2}sff'^3((\ell - 1)\psi' - f\psi'').$$

Next since $\ell\psi(f) - f\psi'(f) \geq 0$ for $s > b$ by (a3), we can use Lemma 2.3 (ii) to $\frac{3}{4}ff'^4(\ell\psi - f\psi')^2 = \frac{3}{2}f'^2(\ell\psi - f\psi') \cdot \frac{ff'^2}{2}(\ell\psi - f\psi')$. Thus we obtain

$$\frac{H_1(s)}{f'^3} \leq -\left(p - \frac{\ell}{2} + 2\right)\frac{ff'^2}{2}(\ell\psi - f\psi') + \left(p - \frac{\ell}{2} + 2\right)\frac{sf'^3}{2}(\ell\psi - f\psi') + \frac{1}{2}sff'^3((\ell - 1)\psi' - f\psi''). \tag{3.7}$$

Finally since $p - \frac{\ell}{2} + 2 > 0$ and $\ell\psi - f\psi' \geq 0$, we can apply Lemma 2.3 (i) to the first term of the right hand side of (3.7). Then we obtain

$$\frac{H_1(s)}{f'^3} \leq \frac{1}{2}sff'^3((\ell - 1)\psi'(f) - f\psi''(f)).$$

Thus by (a4), it follows that $H_1(s) \leq 0$ for $s > b$.

Next we estimate $H_2(s)$. By Lemma 2.1 and $f' = \frac{1}{\sqrt{a}}$, we have

$$\begin{aligned}\frac{H_2(s)}{f'^3} &= -\frac{1}{2}sf'^3(\ell a - (\ell\psi - f\psi')) - psf' + f \\ &= -\left(p + \frac{\ell}{2}\right)sf' + f + \frac{1}{2}sf'^3(\ell\psi - f\psi').\end{aligned}$$

By Lemma 2.3 (i), we also have

$$\frac{H_2(s)}{f'^3} \leq -\left(p + \frac{\ell}{2}\right)sf' + f + \frac{ff'^2}{2}(\ell\psi - f\psi').$$

Finally by Lemma 2.3 (ii), it follows that

$$\frac{H_2(s)}{f'^3} \le -\left(p+\frac{\ell}{2}\right)sf' + \frac{\ell+2}{2}sf' = -(p-1)sf' < 0.$$

This completes the proof. □

Now Theorem 1.1 follows by Propositions 2.6, 3.1 and Lemma 3.2.

4. Non-degeneracy of the unique positive radial solution

In this section, we show that the unique positive radial solution of (2.2) is non-degenerate. To this aim, we study the structure of radial solutions of the following ODE:

$$\begin{cases} v'' + \dfrac{N-1}{r}v' + g(v) = 0, & r \in (0,\infty), \\ v(0) = d > 0. \end{cases} \tag{4.1}$$

We define the energy E by

$$E(r) := \frac{1}{2}(v'(r))^2 + G(v(r)),$$

where $G(s) = \int_0^s g(t)\,dt = \frac{1}{p+1}f(s)^{p+1} - \frac{1}{2}f(s)^2$. From (4.1), we have

$$E'(r) = -\frac{N-1}{r}(v'(r))^2 \le 0.$$

Then we can show that for each $d > 0$, (4.1) has a unique solution $v(r,d)$.

As in [14], we classify the sets of initial values as follows:

$$N = \{d > 0;\ \text{there exists } r_0 = r_0(d) \in (0,\infty) \text{ such that } v(r_0,d) = 0\}.$$

$$G = \left\{d > 0;\ v(r,d) > 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} v(r,d) = 0\right\}.$$

$$P = \{d > 0;\ v(r,d) > 0 \text{ for all } r > 0$$
$$\text{but } v(r,d) \text{ does not converge to zero at infinity}\}.$$

First we prove the following properties on N.

Lemma 4.1. *The set N has the following properties*:

(i) *There exists $\hat{d} > 0$ such that $v(r,\hat{d})$ has a finite zero. Especially it follows that $N \ne \emptyset$.*

(ii) *N is an open set.*

Proof. (i) Let $R > 0$ be given. We consider the auxiliary problem:

$$\begin{cases} -\Delta v = g(v) & \text{in } B_R(0), \\ v > 0 & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0). \end{cases} \tag{4.2}$$

Then for large R, we can show that there exists a positive radial solution $v_R(x)$ of (4.2). (See Appendix below.) Putting $\hat{d} = v_R(0)$, we obtain $v(R, \hat{d}) = 0$ for a solution of (4.1).

(ii) The claim follows from the continuous dependence on the initial value. (see [14] Lemma 13, p. 253.) □

Next we show the following result on P. For simplicity, we write $v_d(r) := v(r, d)$ and $E_d(r) := E(r)$.

LEMMA 4.2. *The set P has the following properties*:
(i) *Let $s_1 > 0$ be a unique zero of $G(s)$. Then it follows that $(0, s_1) \subset P$.*
(ii) *P is an open set.*

Proof. (i) We take $d < s_1$. Then it follows from $v_d(0) = d$ and $v_d'(0) = 0$ that $E_d(0) = G(d)$. Since $G(s) < 0$ for $0 \le s < s_1$, we get

$$E_d(r) \le E_d(0) < 0 \quad \text{for all } r > 0. \tag{4.3}$$

Next we prove that $d \notin N \cup G$. First we show that $v_d(r)$ does not have a finite zero. To this aim, suppose by contradiction that $v_d(r_0) = 0$ for some $r_0 > 0$. Then from $G(0) = 0$ and by the definition of E_d, it follows that $E_d(r_0) = \frac{1}{2}(v_d'(r_0))^2 > 0$. This contradicts to (4.3).

Finally we show that $v_d(r)$ does not converges to zero as $r \to \infty$. If $v_d(r) \to 0$ as $r \to \infty$, then $v_d(r)$ decays exponentially up to the first derivative. Thus it follows that $E_d(r) \to 0$ as $r \to \infty$. This is a contradiction.

(ii) The claim follows by a similar argument as in [14] Lemma 13. □

Now by Proposition 3.1 and Lemma 3.2, we know that the positive radial solution of (2.2) is unique. This implies that there exists $d^* > 0$ such that $G = \{d^*\}$. Moreover by the proof of Lemma 4.2, we can see that $s_1 < d^*$. Since N and P are open, we obtain the following structure.

PROPOSITION 4.3. *There exists a unique $d^* > 0$ such that*

$$N = (d^*, \infty), \quad G = \{d^*\} \quad \text{and} \quad P = (0, d^*).$$

Next we consider the linearized equation of (4.1):

$$w'' + \frac{N-1}{r} w' + g'(v) w = 0, \quad r \in (0, \infty), \tag{4.4}$$

$$w(0) = 1, \quad w'(0) = 0.$$

Since $\dfrac{\partial v}{\partial d}(r, d^*)$ satisfies (4.4) with same initial values, it follows that $w = \dfrac{\partial v}{\partial d}$ by the uniqueness of the initial value problem. Moreover we have the following.

PROPOSITION 4.4. *Suppose* (a1′), (a2)–(a5) *hold and assume further* $\ell + 1 \leq p < \dfrac{(\ell+1)N+2}{N-2}$. *Then* $\dfrac{\partial v}{\partial d}(r, d^*)$ *does not belong to* $H^1(\mathbf{R}^N)$.

Proof. We claim that $\dfrac{\partial v}{\partial d}(r, d^*) \to -\infty$ as $r \to \infty$, which trivially implies $\dfrac{\partial v}{\partial d}(r, d^*) \notin H^1(\mathbf{R}^N)$. We can see that this claim is a direct consequence of the uniqueness proof in [19].

Firstly by Proposition 4.3, we know that $d^* = \inf(N \cup G)$. Then by Lemma 10 in [19], it follows that $\dfrac{\partial v}{\partial d}(r, d^*)$ has exactly one zero in $[0, \infty)$. (This property is called *admissible* in [14], [19].) Moreover by Lemma 8 in [19], we can conclude that $\dfrac{\partial v}{\partial d}(r, d^*) \to -\infty$ as $r \to \infty$, (this property is called *strictly admissible* in [14], [19]), *if* we could show that the hypotheses of Theorem 1 in [19] are satisfied. Thus it suffices to show that the following hypotheses hold for our nonlinear term $g(s)$.

Hypotheses of Theorem 1 in [19]:

(i) $g \in C^1[0, \infty)$, $g(0) = 0$ and $g'(0) = -m < 0$.

(ii) There exists $b > 0$ such that $g(s) < 0$ for $0 < s < b$, $g(s) > 0$ for $s > b$ and $g'(b) > 0$.

(iii) For $\beta > 0$, let

$$\Phi(s, \beta) = \beta s g'(s) - (\beta + 2) g(s).$$

(Φ is called *I-function* in [19].) Then for each $U > b$, there exists $\beta = \beta(U) > 0$ depending continuously on U such that

$$\Phi(s, \beta) \geq 0 \quad \text{for } 0 < s < U, \qquad \Phi(s, \beta) \leq 0 \quad \text{for } s > U.$$

As we have already observed in Section 3, we can see that g defined in (3.1) satisfies (i) and (ii). Moreover (iii) holds for $s = b$ because $g'(b) > 0$. Thus it remains to show that (iii) holds for $s \neq b$.

Now we observe that

$$\frac{\Phi(s, \beta)}{\beta g(s)} = \frac{s g'(s)}{g(s)} - \frac{\beta + 2}{\beta} = K_g(s) - \frac{\beta + 2}{\beta} \quad \text{for } s \neq b,$$

where $K_g(s)$ is a function defined in (3.1). Then from (ii), the hypothesis (iii) is equivalent to showing that

$$K_g(s) - \frac{\beta+2}{\beta} \le 0 \quad \text{for } s > U, \tag{4.5}$$

$$K_g(s) - \frac{\beta+2}{\beta} \ge 0 \quad \text{for } b < s < U, \tag{4.6}$$

$$K_g(s) - \frac{\beta+2}{\beta} \le 0 \quad \text{for } 0 < s < b. \tag{4.7}$$

To this aim, we firstly show that

$$\lim_{s\to\infty} K_g(s) = \frac{2}{\ell+2}\left(p - \frac{\ell}{2}\right). \tag{4.8}$$

Indeed by Lemmas 2.1 and 2.2, we have

$$\begin{aligned} K_g(s) &= \frac{s(pf^{p-1}f'^2 + f^p f'' - f'^2 - ff'')}{f^p f' - ff'} \\ &= \frac{sf'}{f} \cdot \frac{1}{1-f^{1-p}} \left(p - \frac{\ell}{2} + \frac{1}{2} f'^2(\ell\psi(f) - f\psi'(f)) \right. \\ &\qquad \left. + \left(\frac{\ell}{2} - 1\right) f^{1-p} - \frac{1}{2} f^{1-p} f'^2 (\ell\psi(f) - f\psi'(f)) \right). \end{aligned}$$

Now from (a3) and (a4), it follows that

$$0 \le \ell\psi(f) - f\psi'(f) \le \ell\psi(1) - \psi'(1) \quad \text{for } s \ge f^{-1}(1).$$

Then by Lemma 2.5, we get (4.8).

Now by the assumption $p \ge \ell + 1$, it follows that $\frac{2}{\ell+2}\left(p - \frac{\ell}{2}\right) \ge 1$. Moreover by Lemma 3.2, we know that $K_g'(s) < 0$ for $s > b$. Thus we have $K_g(s) > 1$ for all $s > b$. Choosing $\beta = \frac{2}{K_g(U) - 1}$ for each $U > b$, we can see that (4.5) and (4.6) hold.

Finally we show that (4.7) is fulfilled. To this aim, we observe from (3.2), (3.3) and Lemma 2.2 that

$$sg'(s) - g(s) = (f^{p-1} - 1)\left(-\frac{1}{2} sa'(f) ff'^4 + sf'^2 - ff' \right) + (p-1) s f^{p-1} f'^2.$$

Then from (a1′) and Lemma 2.3 (i), we have

$$sg'(s) - g(s) \ge (p-1) f^{p-1} f'^2 > 0 \quad \text{for } 0 < s < b$$

and hence $K_g(s) < 1$ because $g(s) < 0$ for $0 < s < b$. Thus we obtain

$$K_g(s) < 1 < K_g(U) = \frac{\beta+2}{\beta} \quad \text{for } 0 < s < b$$

and hence (4.7) holds.

Since the hypotheses of Theorem 1 in [19] are all satisfied, the proof is complete. □

Proposition 4.4 implies that the unique positive radial solution v of (2.2) is non-degenerate in $H^1_{\mathrm{rad}}(\mathbf{R}^N)$. Finally we show the following result on the linearized operator $\tilde{L} = -\Delta - g'(v)$ of (2.2). We adopt similar arguments as in [18], [20] and [23].

Proposition 4.5. *The kernel of* $\tilde{L}$ *is given by*

$$\mathrm{Ker}(\tilde{L}) = \mathrm{span}\left\{\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N}\right\}.$$

Proof. First we observe that $\mathrm{span}\left\{\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N}\right\} \subset \mathrm{Ker}(\tilde{L})$. In fact, since v is a solution of (2.2), $\frac{\partial v}{\partial x_i}$ satisfies

$$-\Delta\left(\frac{\partial v}{\partial x_i}\right) - g'(v)\frac{\partial v}{\partial x_i} = 0 \quad \text{in } \mathbf{R}^N, \quad i = 1, \dots, N.$$

Moreover by the elliptic regularity theory, we can see that $\frac{\partial v}{\partial x_i} \in H^2(\mathbf{R}^N)$. Thus it follows that $\mathrm{span}\left\{\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N}\right\} \subset \mathrm{Ker}(\tilde{L})$.

To complete the proof, it suffices to show that $\dim \mathrm{Ker}(\tilde{L}) \le N$. To this aim, we apply the argument in [18], [20]. Suppose that $\tilde{\phi} \in \mathrm{Ker}(\tilde{L})$, that is, $\tilde{\phi} \in H^2(\mathbf{R}^N)$ and it satisfies

$$-\Delta\tilde{\phi} - g'(v)\tilde{\phi} = 0 \quad \text{in } \mathbf{R}^N.$$

Then by the elliptic regularity theory, it follows that $\tilde{\phi} \in C^2(\mathbf{R}^N)$.

Now let μ_i and $\psi_i(\theta)$ with $\theta \in S^{N-1}$ be the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on S^{N-1}. Then it follows that

$$0 = \mu_0 < \mu_1 = \cdots = \mu_N = (N-1) < \mu_{N+1} \cdots$$

and $\{\psi_i\}$ forms an orthonormal basis of $L^2(S^{N-1})$. For $\tilde{\phi} \in \mathrm{Ker}(\tilde{L})$, we define

$$\phi_i(r) := \int_{S^{N-1}} \tilde{\phi}(r,\theta)\psi_i(\theta)\, d\theta.$$

Then we have

$$\phi_i'' + \frac{N-1}{r}\phi_i' + \left(g'(v) - \frac{\mu_i}{r^2}\right)\phi_i = 0, \quad \phi_i'(0) = 0. \tag{4.9}$$

Moreover $\tilde{\phi} \in \mathrm{Ker}(\tilde{L})$ can be written as follows.

$$\tilde{\phi}(x) = \tilde{\phi}(r,\theta) = \sum_{i=0}^{\infty} \phi_i(r)\psi_i(\theta). \tag{4.10}$$

When $i = 0$, we have from $\mu_0 = 0$ that

$$\phi_0'' + \frac{N-1}{r}\phi_0' + g'(v)\phi_0 = 0, \quad \phi_0'(0) = 0.$$

Then ϕ_0 is a constant multiple of $\dfrac{\partial v}{\partial d}(r, d^*)$ and hence by Proposition 4.4, it follows that $\phi_0 \equiv 0$.

Next we show that $\phi_i \equiv 0$ for $i \geq N+1$. If $\phi_i \not\equiv 0$, then $\phi_i(0) \neq 0$. Thus we may assume that $\phi_i(0) > 0$. Let $r_i \in (0, \infty]$ be such that $\phi_i(r) > 0$ on $[0, r_i)$ and $\phi_i(r_i) = 0$.

First we suppose that $r_i < \infty$. Multiplying (4.9) by $r^{N-1}v'$ and integrating it over $[0, r_i]$, we get

$$\int_0^{r_i} r^{N-1}v'\phi_i'' + (N-1)r^{N-2}v'\phi_i' + r^{N-1}g'(v)v'\phi_i - \mu_i r^{N-3}v'\phi_i \, dr = 0.$$

By the integration by parts, it follows that

$$r_i^{N-1}v'(r_i)\phi_i'(r_i) - \int_0^{r_i} r^{N-1}v''\phi_i' \, dr + \int_0^{r_i} r^{N-1}g'(v)v'\phi_i - \mu_i r^{N-3}v'\phi_i \, dr = 0.$$

By the integration by parts again and combined with $\phi(r_i) = 0$, we obtain

$$r_i^{N-1}v'(r_i)\phi_i'(r_i) + \int_0^{r_i} (r^{N-1}v''' + (N-1)r^{N-2}v'' + r^{N-1}g'(v)v')\phi_i \, dr$$
$$- \int_0^{r_i} \mu_i r^{N-3}v'\phi_i \, dr = 0.$$

Moreover since v satisfies (4.1), we have

$$v''' + \frac{N-1}{r}v'' - \frac{N-1}{r^2}v' + g'(v)v' = 0.$$

Thus we obtain

$$r_i^{N-1}v'(r_i)\phi_i'(r_i) + (N-1-\mu_i)\int_0^{r_i} r^{N-3}v'\phi_i \, dr = 0.$$

Since $v'(r_i) < 0$ and $\phi_i'(r_i) < 0$, it follows that

$$(N-1-\mu_i)\int_0^{r_i} r^{N-3}v'\phi_i \, dr < 0.$$

On the other hand since $\phi_i(r) > 0$ on $(0, r_i)$ and $\mu_i > N-1$ for $i \geq N+1$, we also have

$$0 < (N-1-\mu_i)\int_0^{r_i} r^{N-3}v'\phi_i \, dr.$$

This is a contradiction.

Next suppose that $r_i = +\infty$. Since $v'(r)$ and $v''(r)$ decay exponentially as $r \to \infty$, we have

$$(N - 1 - \mu_i) \int_0^\infty r^{N-3} v' \phi_i \, dr = 0.$$

This implies again that $\phi_i \equiv 0$ for $i \geq N + 1$.

Now since $\phi_0 \equiv 0$ and $\phi_i \equiv 0$ for $i \geq N + 1$, we have from (4.10) that

$$\tilde{\phi}(x) = \tilde{\phi}(r, \theta) = \sum_{i=1}^{N} c_i \phi_i(r) \psi_i(\theta).$$

This implies that $\dim \mathrm{Ker}(\tilde{L}) \leq N$ and hence the claim holds. □

We can see that Theorem 1.3 follows from Propositions 2.8 and 4.5.

Remark 4.6. The key of the proof of the non-degeneracy is to prove that $\phi_0 \equiv 0$ (ϕ_0 is called an ODE spherical harmonic), which is equivalent to showing that $\frac{\partial v}{\partial d}(r, d^*) \notin H^1(\mathbf{R}^N)$.

In [23], the author firstly showed that the function $\frac{\partial v}{\partial d}(r, d^*)$ changes sign at least once, and secondly proved that $\frac{\partial v}{\partial d}(r, d^*)$ is unbounded. This approach does not need the uniqueness. However in order to prove the first statement, the assumption $p \geq 3$ was necessary. (See Proposition 3.10 in [23].)

It is known that the uniqueness follows if we could show that $\frac{\partial v}{\partial d}(r, d^*)$ changes sign exactly once. (See [14], [19].) However we don't need to prove this property for the uniqueness, since the uniqueness proof in [24] is based on a different approach. This enables us to obtain the uniqueness for $p > 1$.

On the other hand we have to adopt same arguments as in [23] to prove the non-degeneracy. To this aim, we need a restriction on p. Although this restriction seems to be technical, we don't know how to remove it at present.

5. Results for the complex valued ground state

In this section, we consider a complex-valued ground state of (1.2), which is important in the study of the corresponding time-evolution Schrödinger equation (1.1).

Now for $u : \mathbf{R}^N \to \mathbf{C}$, we consider the following elliptic equation:

$$-\Delta u + \lambda u - \kappa u \Delta(|u|^2) = |u|^{p-1} u \quad \text{in } \mathbf{R}^N. \tag{5.1}$$

We define the energy functional and the energy space by

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + |u|^2 |\nabla |u||^2 + |u|^2 \, dx - \frac{1}{p+1}\int_{\mathbf{R}^N} |u|^{p+1} \, dx,$$

$$X_{\mathbf{C}} = \left\{ u \in H^1(\mathbf{R}^N, \mathbf{C}); \int_{\mathbf{R}^N} |u|^2 |\nabla |u||^2 \, dx < \infty \right\}.$$

As for the existence and properties of a complex-valued ground state, we have the following. For the proof, we refer to [9].

Proposition 5.1. *Suppose* $1 < p < \frac{3N+2}{N-2}$, $\lambda > 0$ *and* $\kappa > 0$. *Then the problem* (5.1) *has a ground state.*

Moreover let u *be a ground state of* (5.1). *Then there exists* $\theta \in \mathbf{R}$ *such that* $u(x) = e^{i\theta}|u(x)|$.

Proposition 5.1 tells us that up to a phase shift, we may assume that the ground state of (5.1) is real-valued.

Now let $\mathscr{G}$ be the set of ground states of (5.1). Since (5.1) is invariant under the translation and the phase shift, we have the following result.

Theorem 5.2. *Suppose* $1 < p < \frac{3N+2}{N-2}$, $\lambda > 0$ *and* $\kappa > 0$. *Let* u *be the unique* (*real-valued*) *ground state of* (1.2) (*obtained in Theorem* 1.2). *Then we have*

$$\mathscr{G} = \{ e^{i\theta} u(\cdot + y); y \in \mathbf{R}^N, \theta \in \mathbf{R} \}.$$

Moreover if $3 \le p < \frac{3N+2}{N-2}$, *we also have*

$$\mathrm{Ker}(L) = \mathrm{span}\left\{ iu(x), \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right\}.$$

Here L *is the linearized operator of* (5.1) *around the unique* (*real-valued*) *ground state* u, *which is given by*

$$\begin{aligned} L(\phi) = & -\Delta\phi + \lambda\phi - \kappa(2u\Delta u + 2|\nabla u|^2)\phi \\ & - \kappa u^2 \Delta(\phi + \bar{\phi}) - 2\kappa u \nabla u \cdot \nabla(\phi + \bar{\phi}) - \kappa u \Delta u(\phi + \bar{\phi}) \\ & - u^{p-1}\phi - \frac{p-1}{2} u^{p-1}(\phi + \bar{\phi}), \quad \phi \in H^2(\mathbf{R}^N, \mathbf{C}). \end{aligned}$$

Proof. To prove Theorem 5.2, we put $\phi = \phi_1 + i\phi_2$ with $\phi_1, \phi_2 \in H^2(\mathbf{R}^N, \mathbf{R})$ and decompose L into two operators L_1, L_2 acting on ϕ_1 and ϕ_2 respectively. By a direct computation, we have

$$L_1(\phi_1) = -\Delta\phi_1 + \lambda\phi_1 - \kappa(2u\Delta u + 2|\nabla u|^2)\phi_1$$
$$\qquad - 2\kappa u^2\Delta\phi_1 - 4\kappa u\nabla u\cdot\nabla\phi_1 - 2\kappa u\Delta u\phi_1 - pu^{p-1}\phi_1,$$
$$L_2(\phi_2) = -\Delta\phi_2 + \lambda\phi_2 - \kappa(2u\Delta u + 2|\nabla u|^2)\phi_2 - u^{p-1}\phi_2.$$

By Theorem 1.3 with $a(t) = 1 + 2\kappa\lambda^{2/(p-1)}t^2$, we know that

$$\mathrm{Ker}(L_1) = \mathrm{span}\left\{\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right\}.$$

Moreover we can see that $u \in \mathrm{Ker}(L_2)$. By the result in [23], we also have $\mathrm{Ker}(L_2) = \mathrm{span}\{u\}$. Thus Theorem 5.2 holds. □

6. Comments on the quasilinear term

In this section, we study some examples of the quasilinear term $a(t)$ and give some remarks.

First we observe that by (a1), $a(t)$ needs not to be monotone for $0 \le t \le 1$. This enables our result to cover a wider class of functions. Indeed we consider a function $a(t) = a_0t^\ell + e^{-a_1t^2}$ for $a_1 > 0$, that is, $\psi(t) = e^{-a_1t^2}$. By elementary calculations, we have

$$\ell\psi - t\psi' = e^{-a_1t^2}(\ell + 2a_1t^2).$$
$$(\ell - 1)\psi' - t\psi'' = 2a_1te^{-a_1t^2}(2 - \ell - 2a_1t^2).$$
$$a'(t) = a_0\ell t^{\ell-1} - 2a_1te^{-a_1t^2}.$$

Choosing a_1 smaller if $\ell \ge 2$ and larger if $0 < \ell < 2$ respectively, we can see that (a1)–(a4) hold. We notice that if $\ell > 2$, $a(t)$ cannot be monotone near zero. Similar statements hold for a function $a(t) = a_0t^\ell + \dfrac{1}{(1 + a_1t^2)^q}$ for $q > 0$.

Next we claim that the assumption (a5) also holds for a wide class of functions. First let us consider the typical case $a(t) = 1 + a_0t^2$, that is, $\ell = 2$ and $\psi(t) \equiv 1$. By a direct computation, we have

$$\frac{\ell + 2}{2}\int_0^1 \sqrt{a(t)}\,dt - \frac{\ell + 2}{2}\sqrt{a(1)} - \frac{a'(1)}{2\sqrt{a(1)}}$$
$$= \frac{\sinh^{-1}(\sqrt{a_0})}{\sqrt{a_0}} - \frac{1}{\sqrt{1 + a_0}} \ge 0 \quad \text{for any } a_0 \ge 0.$$

Here we used the inequality: $\sinh^{-1}(t) \ge \dfrac{t}{\sqrt{1 + t^2}}$ for $t \ge 0$. For the case $a(t) = a_0t^2 + e^{-t^2}$, we can observe by numerical calculations that the l.h.s. of (a5) ≈ 0.2778 for $a_0 = 1$, ≈ 0.3389 for $a_0 = 2$ and ≈ 0.3651 for $a_0 = 5$.

As we have observed above, some *decreasing* functions $\psi(t)$ may satisfy assumptions (a1)–(a5). On the other hand, our assumptions prevent us to treat *increasing* functions. Indeed let us consider the case $\ell > 2$ and suppose that some increasing function ψ satisfies (a4). Then solving the differential inequality, we can see that

$$\psi(t) \geq \psi(s) + \frac{\psi'(s)}{\ell} t^{\ell} \quad \text{for } t > s \geq 1.$$

This is incompatible with (a2) unless $\psi' \equiv 0$. We remark that the increasing case has been studied in [11]. In this case, the authors obtained the uniqueness when the parameter is sufficiently large.

Finally we cannot completely cover the case when $a(t)$ is a *bounded* function. Especially our result does not cover the case $a(t) = 1 + \dfrac{\kappa t^2}{2(1+t^2)}$, which also appears in the study of plasma physics. In this case, we may choose $\ell = 0$. However we can easily see that (a3) and (a4) are never satisfied. The uniqueness of this case has been obtained in [7] by assuming the parameter is sufficiently large.

7. Appendix

In this appendix, we show that the Dirichlet problem:

$$\begin{cases} -\Delta v = f(v)^p f'(v) - f(v)f'(v) & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0). \end{cases} \tag{7.1}$$

has a positive radial solution. By Lemma 2.3, we can see that the nonlinear term in (7.1) satisfies so-called Ambrosetti-Rabinowitz condition if $p > \ell + 1$. Then the existence of a positive solution follows by the standard Mountain Pass argument. Even if $1 < p \leq \ell + 1$, we can obtain the existence of a positive solution, provided that R is sufficiently large. Although the existence of a positive solution of (7.1) in this case may be obtained by general theory for Dirichlet problems with *sublinear* or *asymptotically linear nonlinearity*, we give the proof for the sake of completeness.

For this purpose, we recall the abstract result in [12] as follows. Let $(X, \|\cdot\|)$ be a Banach space and $J \subset \mathbf{R}$ be a compact interval.

Definition 7.1. A family of functionals $\mathscr{I} = \{I(\lambda, \cdot) \in C^1(X, \mathbf{R}); \lambda \in J\}$ is said to have mountain-pass geometry if there exist $v_1, v_2 \in X$ such that

$$c(\lambda) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\lambda, \gamma(t)) > \max\{I(\lambda, v_1), I(\lambda, v_2)\} \quad \text{for any } \lambda \in J,$$

where $\Gamma := \{\gamma \in C([0,1], X); \gamma(0) = v_1, \gamma(1) = v_2\}$.

Now we assume the following assumption holds for $I(\lambda, \cdot)$.

(H) Suppose that a sequence $\{(\lambda_n, u_n)\} \subset J \times X$ satisfies $\{\lambda_n\}$ is strictly increasing, $\lambda_n \nearrow \lambda_0 \in J$ and

$$-I(\lambda_0, u_n), I(\lambda_n, u_n), \frac{I(\lambda_n, u_n) - I(\lambda_0, u_n)}{\lambda_0 - \lambda_n} \quad \text{are bounded from above.}$$

Then $\{\|u_n\|\}$ is bounded and for $\varepsilon > 0$, there exists $N > 0$ such that

$$I(\lambda_0, u_n) \le I(\lambda_n, u_n) + \varepsilon \quad \text{for all } n \ge N.$$

Then the following result holds.

Proposition 7.2 ([12, Theorem 2.1]). *Suppose that* (H) *holds and* $\mathscr{I}$ *has mountain-pass geometry. Then there exists* $D \subset J$ *such that for each* $\lambda_0 \in D$, $I(\lambda_0, \cdot)$ *has a bounded Palais-Smale sequence* $\{v_n\} \subset X$ *at the level* $c(\lambda_0)$. *Moreover* $J \backslash D$ *has a zero Lebesgue measure.*

Now by using Proposition 7.2, we show the following.

Proposition 7.3. *There exists large* $R > 0$ *such that the problem* (7.1) *has a positive solution* $v_R(x)$.

Proof. For $\lambda > 0$, we put

$$B = B_1(0), \quad X = H^1_{0,\mathrm{rad}}(B), \quad \|u\| = \left(\int_B |\nabla u|^2 \, dx\right)^{1/2},$$

$$I(\lambda, u) = \frac{1}{2}\|u\|^2 + \lambda\left(\frac{1}{2}\int_B f(u)^2 \, dx - \frac{1}{p+1}\int_B f(u)_+^{p+1} \, dx\right).$$

Then $I(\lambda, \cdot) \in C^1(X, \mathbf{R})$ for any $\lambda > 0$. Moreover since $\lim_{s\to\infty} f(s) = \infty$, we can choose $\phi \in C_0^\infty(B)$ such that

$$\frac{1}{2}\int_B f(\phi)^2 \, dx - \frac{1}{p+1}\int_B f(\phi)_+^{p+1} \, dx < 0.$$

Thus there exists large $\underline{\lambda} > 0$ such that

$$I(\underline{\lambda}, \phi) = \frac{1}{2}\|\phi\|^2 + \underline{\lambda}\left(\frac{1}{2}\int_B f(\phi)^2 \, dx - \frac{1}{p+1}\int_B f(\phi)_+^{p+1} \, dx\right) \le 0$$

and $I(\lambda, \phi) \le I(\underline{\lambda}, \phi)$ for any $\lambda \ge \underline{\lambda}$.

To apply Proposition 7.2, we put

$$J = [\underline{\lambda}, \underline{\lambda} + 1], \quad v_1 = 0 \quad \text{and} \quad v_2 = \phi.$$

Then it holds that

$$\max\{I(\lambda, v_1), I(\lambda, v_2)\} = \max\{0, I(\lambda, \phi)\} \le 0 \quad \text{for } \lambda \in J.$$

Moreover by the Sobolev embedding theorem, we can show that $c(\lambda) > 0$. Hence a family of functionals $\mathscr{I}$ has mountain-pass geometry.

Next we prove that the condition (H) holds. Let $\{(\lambda_n, u_n)\}$ be a sequence in $J \times X$ which satisfies following properties:

$$\lambda_n \nearrow \lambda_0 \in J \quad \text{as } n \to \infty, \quad -I(\lambda_0, u_n) \le M,$$

$$I(\lambda_n, u_n) = \frac{1}{2}\|u_n\|^2 + \lambda_n \left(\frac{1}{2}\int_B f(u_n)^2 \, dx - \frac{1}{p+1}\int_B f(u_n)_+^{p+1} \, dx \right) \le M, \tag{7.2}$$

$$\frac{I(\lambda_n, u_n) - I(\lambda_0, u_n)}{\lambda_0 - \lambda_n} = -\left(\frac{1}{2}\int_B f(u_n)^2 \, dx - \frac{1}{p+1}\int_B f(u_n)_+^{p+1} \, dx \right) \le M \tag{7.3}$$

for some $M > 0$ independent of n. Then from (7.2) and (7.3) we have

$$\begin{aligned} \frac{1}{2}\|u_n\|^2 &\le M - \lambda_n \left(\frac{1}{2}\int_B f(u_n)^2 \, dx - \frac{1}{p+1}\int_B f(u_n)_+^{p+1} \, dx \right) \\ &\le M + \lambda_n M \\ &\le M(2 + \underline{\lambda}). \end{aligned}$$

Thus $\|u_n\|$ is bounded. Moreover from (7.2), we also have

$$\frac{1}{2}\int_B f(u_n)^2 \, dx - \frac{1}{p+1}\int_B f(u_n)_+^{p+1} \, dx \le \frac{M}{\lambda_n} \le \frac{M}{\underline{\lambda}}.$$

Thus we get

$$|I(\lambda_0, u_n) - I(\lambda_n, u_n)| \le (\lambda_0 - \lambda_n)\left(1 + \frac{1}{\underline{\lambda}}\right) M = o(1) \quad \text{as } n \to \infty.$$

Therefore the condition (H) holds.

Now Proposition 7.2 asserts that there exist $\lambda_0 \in J = [\underline{\lambda}, \underline{\lambda} + 1]$ and $\{u_n\} \subset X$ such that

$$\partial_u I(\lambda_0, u_n) = o(1) \quad \text{in } X^*, \qquad I(\lambda_0, u_n) = c(\lambda_0) + o(1) \quad \text{as } n \to \infty$$

and $\{u_n\}$ is bounded in X. By standard arguments, we can show that there exists $u \in X$ such that

$$\partial_u I(\lambda_0, u) = 0 \quad \text{and} \quad I(\lambda_0, u) = c(\lambda) > 0.$$

By the elliptic regularity theory and the maximal principle, it follows that u is a positive solution of

$$\begin{cases} -\Delta u = \lambda_0 (f(u)^p f'(u) - f(u) f'(u)) & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

Putting $v_R(x) = u(x/R)$ with $R = \sqrt{\lambda_0}$, v_R is a solution of (7.1). □

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