CORRECTION TO “SURFACES WITH PARALLEL MEAN CURVATURE VECTOR IN $P^2(C)$”

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Abstract

We describe a condition under which the claims in the paper cited above hold.

1. Correction

It has been pointed out by Hirakawa [3] that a previous paper by Ogata [5] contained a mistake. In fact, the claim made in line 3, page 401 in [5], which states that "$l$ is a real-valued function defined on $U$,” is not generally correct. We now give a geometric condition for the claim to hold. We follow the notation used in [5].

Lemma. Suppose that the immersion in [5] satisfies a condition $a = \bar{a}$ on $M$. Then, there exists a complex coordinate $w$ on a neighborhood of a point of $M$ such that $f = \mu dw$, where $\mu$ is real valued.

Using this lemma, we can state the following:

Correction. For the claims given in [5] to hold, we add the condition $a = \bar{a}$ to the immersion.

Since Kenmotsu and Zhou [4] and Hirakawa [2] used the results given by Ogata [5], those papers also need the additional assumption $a = \bar{a}$ for the immersion.

2. Proof of Lemma

Set $\phi = \lambda dz$, where $\lambda$ is a non-zero complex valued function on a simply connected domain $U$ with complex coordinate $z$. Although the lemma can be
proved using (2.4) in [5], we employ a slightly modified formula here. By (2.4) of [5], we have

\[ \lambda_z = -\lambda \overline{\lambda} (\overline{a} - b) \cot \alpha, \]
\[ a_z = \overline{\lambda} (\overline{a} + b), \]
\[ a_z = \overline{\lambda} \left( 2a(\overline{a} - b) + \frac{3\rho}{2} \sin^2 \alpha \right) \cot \alpha, \]
\[ c_z = 2\lambda c(a - b) \cot \alpha. \]

We note that (2.8) in [5] is not generally correct.

First, we prove the lemma for the case in which \( \alpha \) is constant on \( M \). By (2.2), we have \( a = -b = \overline{\alpha} = \text{constant} \). By (2.6) of [5], \( |c|^2 \) is constant. Set \( c = |c| \exp(i\theta) \), where \( \theta \) is a real-valued function on \( U \). Then, using (2.4), we have \( i\theta_x = -4b\lambda \cot \alpha \). If we take the partial derivative with respect to \( \overline{z} \), then (2.1) can be used to obtain \( 8b^2 \lambda \overline{\lambda} \cot^2 \alpha + i\theta_{xx} = 0 \). Since \( \theta_{xx} \) is real valued, this implies \( \cot \alpha = 0 \). Therefore, we have \( \lambda_x = 0 \) by (2.1). Hence, \( \lambda \) is holomorphic. Define the complex coordinate \( w \) as \( w = \int \lambda \, dz \). Then, we have \( \phi = \lambda \, dz = dw \), which proves the lemma for the case \( \alpha = \text{constant} \).

When \( \alpha \) is not constant, we need the following claim to prove the lemma:

**Claim.** Suppose that \( a = \overline{\alpha} \) on \( M \). If \( \alpha \) is not constant, then \( a \) is a function of \( \alpha \).

**Proof.** By the assumption, we see \( a_z = (\overline{\alpha})_z = \overline{\alpha}_z \). By (2.2) and (2.3), we have

\[ dx = (a + b)(\phi + \overline{\phi}), \]
\[ da = \left( 2a(a - b) + \frac{3\rho}{2} \sin^2 \alpha \right) \cot \alpha \cdot (\phi + \overline{\phi}). \]

Canceling out \( (\phi + \overline{\phi}) \) in the above formulas, we have a differential equation in \( a \) for \( \alpha \), which proves the claim.

**Proof of Lemma.** Using the above claim, we write \( a = a(\alpha) \), and define a real-valued function \( F(\alpha) \) as

\[ F(\alpha) = \frac{(a(\alpha) - b)^2 + 3\rho/2 \sin^2 \alpha}{(a(\alpha) + b)^2} \cot \alpha. \]

Taking the partial derivative of (2.2) with respect to \( z \) and using (2.1) and (2.3), we have a second-order partial differential equation \( \alpha_{zz} - F(\alpha) \alpha_z \alpha_z = 0 \). It follows
that \((a_z \exp(-\int F(x) \, dx))_z = 0\). Hence, there exists a holomorphic function \(G(z)\) on \(U\) such that \(a_z = G(z) \exp(\int F(x) \, dx)\). Setting

\[
w = \int G(z) \, dz, \quad \mu = \frac{\exp(\int F(x) \, dx)}{a(x) + b},
\]
the lemma is proved by the conjugate of (2.2).

**Remark.** Briefly, we explain the geometric meanings for these quantities used in (2.1)–(2.4). The real valued function \(a\) is the Kaehler angle of the immersion, the positive number \(b\) is two times of the length of the mean curvature vector, and the complex valued functions \(a\) and \(c\) determine the second fundamental tensors of the immersion. The ambient space is a complex 2-dimensional Kaehler manifold of constant holomorphic sectional curvature \(4\rho\). These were first introduced in Chern and Wolfson [1].

**References**


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