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REDUCTION OF A FAMILY OF IDEALS

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Abstract

In the paper we prove that there exists a simultaneous reduction of one-parameter family of $\mathfrak{m}_n$-primary ideals in the ring of germs of holomorphic functions. Moreover, we generalize the result of A. Płoski [8] on the semicontinuity of the Łojasiewicz exponent in a multiplicity-constant deformation.

1. Introduction

Let R be a ring and I an ideal. We say that an ideal J is a *reduction of I* if it satisfies the following condition:

$$J \subset I, \quad \text{and for some } r > 0 \text{ we have } I^{r+1} = JI^r.$$

The notion of reduction is closely related to the notions of *Hilbert-Samuel multiplicity* and *integral closure* of an ideal.

Recall that if $(R, \mathfrak{m})$ is a Noetherian local ring of dimension n and I is an $\mathfrak{m}$-primary ideal of R, then the *Hilbert-Samuel multiplicity* of I is given by the formula

$$e(I) = \lim_{k\to\infty} \frac{n!}{k^n} \operatorname{length}_R R/I^k.$$

For the multiplicity theory in local rings see for example [7] or [4].

Let I be an ideal in a ring R. An element $x \in R$ is said to be *integral over I* if there exists an integer n and elements $a_k \in I^k$, $k = 1, \dots, n$, such that

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

The set of all elements of R that are integral over I is called the *integral closure of I*, and is denoted $\bar{I}$. If $I = \bar{I}$ then I is called *integrally closed*. It is well known that $\bar{I}$ is an ideal.

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The relationship between the above notions is given in the following Theorem due to D. Rees:

Theorem 1 (Rees, [4, Cor. 1.2.5, Thm. 11.3.1]). *Let* $(R, \mathfrak{m})$ *be a formally equidimensional Noetherian local ring and let* $J \subset I$ *be two* $\mathfrak{m}$*-primary ideals. Then the following conditions are equivalent*:

1. J *is a reduction of* I;
2. $e(I) = e(J)$;
3. $\bar{I} = \bar{J}$.

If $R/\mathfrak{m}$ is infinite, $\dim R = d$ and I is an $\mathfrak{m}$-primary ideal of R then any d "sufficiently general" elements of I form a reduction of I. More precisely we have the following result

Theorem 2 ([7, Theorem 14.14]). *Let* $(R, \mathfrak{m})$ *be a* d*-dimensional Noetherian local ring, and suppose that* $k = R/\mathfrak{m}$ *is an infinite field*; *let* $I = (u_1, \dots, u_s)$ *be an* $\mathfrak{m}$*-primary ideal. Then there exist a finite number of polynomials* $D_\alpha \in k[Z_{ij}; 1 \leqslant i \leqslant d, 1 \leqslant j \leqslant s]$, $1 \leqslant \alpha \leqslant v$ *such that if* $y_i = \sum a_{ij} u_j$, $i = 1, \dots, d$ *and at least one of* $D_\alpha(\bar{a}_{ij}; 1 \leqslant i \leqslant d, 1 \leqslant j \leqslant s) \neq 0$, *then the ideal* $(y_1, \dots, y_d)R$ *is a reduction of* I *and* $\{y_1, \dots, y_d\}$ *is a system of parameters of* R.

In fact, the above theorem could be generalized to arbitrary ideals in R. Recall that if $(R, \mathfrak{m})$ is a Noetherian local ring, the *analytic spread* of I (denoted $\ell(I)$) is the Krull dimension of the fiber cone of I:

$$\frac{R[It]}{\mathfrak{m}R[It]} \approx \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \frac{I^2}{\mathfrak{m}I^2} \oplus \cdots$$

where t is a variable over R.

Theorem 3 ([4, Theorem 8.6.6]). *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring with infinite residue field and* I *an ideal of analytic spread at most* l. *There exists a non-empty Zariski-open subset* U *of* $(I/\mathfrak{m}I)^l$ *such that whenever* $x_1, \dots, x_l \in I$ *with* $(x_1 + \mathfrak{m}I, \dots, x_l + \mathfrak{m}I) \in U$, *then* $(x_1, \dots, x_l)R$ *is a reduction of* I.

Now, let $(\mathcal{O}_n, \mathfrak{m}_n)$ be the ring of germs of holomorphic functions $(\mathbf{C}^n, 0) \to \mathbf{C}$. From Theorem 3 we see that, if $I = (f_1, \dots, f_m)\mathcal{O}_n$ and l denotes the integer $\ell(I)$, then the ideal

$$\left(\sum a_{1j} f_j, \dots, \sum a_{lj} f_j\right)\mathcal{O}_n,$$

is a reduction of I for generic coefficients $a_{ij} \in \mathbf{C}$.

If $g = (g_1, \dots, g_m) : (\mathbf{C}^n, 0) \to (\mathbf{C}^m, 0)$ is an analytic map germ, then we denote by $(g)\mathcal{O}_n$ the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_m$. The aim of this note is to study the following

Question 4. Let $I := (F)\mathcal{O}_{n+k}$ and $I_t := (F_t)\mathcal{O}_n \subset \mathfrak{m}_n$ be a family of ideals given by a holomorphic map $F = F_t(x) = F(x,t) : (\mathbf{C}^n \times \mathbf{C}^k, 0) \to (\mathbf{C}^m, 0)$. Assume that the analytic spread of I_t is constant in some neighbourhood of $0 \in \mathbf{C}^k$. Denote this constant value l. Does there exist a linear map $\pi : \mathbf{C}^m \to \mathbf{C}^l$ such that $J_t := (\pi \circ F_t)\mathcal{O}_n$ is a reduction of I_t for t close to $0 \in \mathbf{C}^k$?

By Theorem 3, the answer is immediate if $\ell(I) = \ell(I_t)$ in some neighbourhood of $0 \in \mathbf{C}^k$. It turns out that the above condition is fulfilled in multiplicity-constant families of $\mathfrak{m}_n$-primary ideals. This fact is implicitly stated in the proof of the principle of specialization of integral dependence given by B. Teissier in [11]. We will here recall this argument.

Proposition 5. *Let I be as in Question 4. If I_t are $\mathfrak{m}_n$-primary ideals and the function $t \mapsto e(I_t)$ is constant, then $\ell(I) = \ell(I_t) = n$ in some neighbourhood of $0 \in \mathbf{C}^k$. In particular, the answer to Question 4 is positive in this case.*

Proof. By [4, Corollary 8.3.9] we have $\mathrm{ht}(I) \leqslant \ell(I) \leqslant \dim R$ for any ideal I in Noetherian local ring $(R, \mathfrak{m})$. Thus $\ell(I_t) = n$ since I_t are $\mathfrak{m}_n$-primary. Let $\pi : \mathbf{C}^m \to \mathbf{C}^n$ be a linear map such that $(\pi \circ F_0)\mathcal{O}_n$ is a reduction of I_0. Put $J_t := (\pi \circ F_t)\mathcal{O}_n$, for all t. We have $e(J_t) \leqslant e(J_0)$, since the multiplicity $e(\cdot)$ is upper semicontinuos. Moreover $e(I_t) \leqslant e(J_t)$, because $J_t \subset I_t$. Summing up

$$e(I_0) = e(J_0) \geqslant e(J_t) \geqslant e(I_t) = e(I_0).$$

Therefore $\overline{J_t} = \overline{I_t}$ by Rees theorem. From this and [11, Corollaire I.2, p. 132] we deduce that $\bar{J} = \bar{I}$ and consequently $\ell(I) = n$. □

In the next section we get as a corollary that in a multiplicity-constant family of ideals the Łojasiewicz exponent is a lower semicontinuos function. A. Płoski proved this result under the additional restriction $m = n$.

Example 6. Let $F : (\mathbf{C}^2 \times \mathbf{C}, 0) \to (\mathbf{C}, 0)$ be given by $F(x, y, t) := (x^5, y^5, t^5 xy)$. By the main result of [2] we have $\ell((F)\mathcal{O}_3) = 3$. However, if $\pi : \mathbf{C}^3 \to \mathbf{C}^2$ is given by $\pi(u, v, w) = (u + w, v + w)$ then $\pi \circ F_t$ generate a reduction of $(F_t)\mathcal{O}_2$ for any t. Indeed, if we put $I_t := (F_t)\mathcal{O}_2$, $J_t := (\pi \circ F_t)\mathcal{O}_2$ then $I_t^2 = J_t I_t$.

Observe that the family $(F_t)\mathcal{O}_2$ is not multiplicity-constant. We have

$$e((F_t)\mathcal{O}_2) = \begin{cases} 10 & t \neq 0, \\ 25 & t = 0. \end{cases}$$

Our main result is given in the next theorem. It is a positive answer for Question 4 in case of one-parameter families of $\mathfrak{m}_n$-primary ideals.

Theorem 7. *Let $F = F_t(x) = F(x,t) : (\mathbf{C}^n \times \mathbf{C}, 0) \to (\mathbf{C}^m, 0)$ be a holomorphic map. Assume that $(F_t)\mathcal{O}_n$ is an $\mathfrak{m}_n$-primary ideal for all t. Then there exists* *a complex linear map* $\pi : \mathbf{C}^m \to \mathbf{C}^n$ *such that for all* t *the ideal* $(\pi \circ F_t)\mathcal{O}_n$ *is a reduction of* $(F_t)\mathcal{O}_n$.

We give the proof of Theorem 7 in Section 5. It is based on some geometric property of Hilbert-Samuel multiplicity given in Section 3. In Section 4 we recall the notion of elementary blowing-up.

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2. Semicontinuity of the Łojasiewicz exponent

Let $(R, \mathfrak{m})$ be a local ring and let I be an $\mathfrak{m}$-primary ideal. By the *Łojasiewicz exponent* $\mathcal{L}(I)$ of I we define the infimum of

$$\left\{\frac{p}{q} : \mathfrak{m}^p \subset \overline{I^q}\right\}.$$

It was proved in [5] that if $F : (\mathbf{C}^n, 0) \to (\mathbf{C}^m, 0)$ is a holomorphic map with an isolated zero at the origin and $I := (F)\mathcal{O}_n$, then $\mathcal{L}(I)$ is the optimal exponent ν in the inequality

$$|F(x)| \geqslant C|x|^\nu,$$

where C is some positive constant and x runs through sufficiently small neighbourhood of $0 \in \mathbf{C}^n$.

Lemma 8. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring. If* I *is an* $\mathfrak{m}$*-primary ideal of* R *and* J *is a reduction of* I *then* $\mathcal{L}(I) = \mathcal{L}(J)$.

Proof. Obviously $\mathcal{L}(I) \leqslant \mathcal{L}(J)$. Assume that $\mathfrak{m}^p \subset \overline{I^q}$. Since J is a reduction of I, then also J^q is a reduction of I^q [4, Prop. 8.1.5]. Thus $\overline{J^q} = \overline{I^q}$ by Theorem 1, which gives $\mathfrak{m}^p \subset \overline{J^q}$. This proves the inequality $\mathcal{L}(J) \leqslant \mathcal{L}(I)$ and ends the proof. □

Corollary 9 (A. Płoski for $m = n$, [8]). *Let* $F : (\mathbf{C}^n \times \mathbf{C}^k, 0) \to (\mathbf{C}^m, 0)$ *be a holomorphic map. Put* $I_t := (F_t)\mathcal{O}_n$. *If the function* $t \mapsto e(I_t)$ *is constant and finite then the function* $t \mapsto \mathcal{L}(I_t)$ *is lower semicontinuos.*

Proof. By Proposition 5 and Theorem 2 there exists a linear map $\pi : \mathbf{C}^m \to \mathbf{C}^n$ such that $J_t := (\pi \circ F_t)\mathcal{O}_n$ is a reduction of I_t for all t. Thus $\mathcal{L}(J_t) = \mathcal{L}(I_t)$ and $e(J_t) = e(I_t)$ by Theorem 1 and Lemma 8. Consequently $t \mapsto e(J_t)$ is constant and finite and the assertion follows from the case $m = n$ proved by A. Płoski. □

More direct proof of this result we will give in our forthcoming paper [10].

3. Improper intersection multiplicity

Let V, Z be a pair of analytic sets defined in some neighbourhood of $p \in \mathbf{C}^N$ and assume that p is an isolated point of $V \cap Z$. If $\dim_p V + \dim_p Z = N$ then it is well known how to define the *intersection index* $i(V \cdot Z, p)$ of V and Z at p (see e.g. [3]). Now, assume that $\dim_p V + \dim_p Z < N$ and Z is smooth at p. In this case the *intersection index* of V and Z at p was defined in [1] by the formula $\tilde{i}(V \cdot Z; p) = \min_W i_p(V \cdot W; p)$, where W goes over all analytic sets defined in some neighbourhood of p such that

- $Z \cap U \subset W \cap U$ for some neighbourhood U of p,
- p is an isolated point of $V \cap W$,
- $\dim_p V + \dim_p W = N$.

Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^m, 0)$ be a holomorphic map with an isolated zero. Using the above definition one may define (see [9]) the so-called *improper intersection multiplicity* of f by the formula

$$i_0(f) := \tilde{i}(\operatorname{graph} f \cdot (\mathbf{C}^n \times \{0\}^m); (0,0)).$$

We recall one more definition. Let V be a germ of an analytic set at $p \in \mathbf{C}^n$. Then the (*Whitney*) *tangent cone* of V is the set of all $v \in \mathbf{C}^n$ such that there exist $\{p_n\} \subset V$, $\{t_n\} \subset \mathbf{C}$ with $p_n \to p$ and $t_n(p_n - p) \to v$. For the map f as above by C_f we will denote the tangent cone of the germ of the image of f at the origin.

The following observation is due to S. Spodzieja.

Theorem 10 ([9]). *If $\pi : \mathbf{C}^m \to \mathbf{C}^l$ is a linear map such that* $\ker \pi \cap C_f = \{0\}$, *then $\pi \circ f$ has an isolated zero in the origin and $i_0(f) = i_0(\pi \circ f)$. If additionally $l = n$ then $i_0(f) = e((\pi \circ f)\mathcal{O}_n)$. Moreover, the number $i_0(f)$ depends only on the ideal generated by the components of f.*

In what follows we will write $i_0(I) := i_0(f)$, where $f = (f_1, \dots, f_m)$ are any generators of an $\mathfrak{m}_n$-primary ideal I in $\mathcal{O}_n$.

Corollary 11. *If I is an $\mathfrak{m}_n$-primary ideal in $\mathcal{O}_n$, then $i_0(I) = e(I)$.*

Proof. Let $I = (f_1, \dots, f_m)\mathcal{O}_n$. By Theorems 2 and 10 there exist linear combinations $g_i = \sum a_{ij} f_j$, $i = 1, \dots, n$ such that $J = (g_1, \dots, g_n)\mathcal{O}_n$ is a reduction of I, $\{g_1, \dots, g_n\}$ is a system of parameters of $\mathcal{O}_n$ and $i_0(I) = i_0(J) = e(J)$. From Theorem 1 we get $e(I) = e(J)$. This ends the proof. □

Corollary 12. *If $\pi : \mathbf{C}^m \to \mathbf{C}^l$ is a linear map such that* $\ker \pi \cap C_f = \{0\}$, *then the ideal J generated by $\pi \circ f$ is a reduction of I.*

Proof. We have $J \subset I$ and $e(J) = e(I)$. This and Theorem 1 give the assertion. □

4. Elementary blowing-up

Here we recall the notion of an elementary blowing-up after [6].

Let $U \subset \mathbf{C}^n$ be an open and connected neighbourhood of $0 \in \mathbf{C}^n$; let $f = (f_0, \dots, f_m) \neq 0$ be a sequence of holomorphic functions on U. Put $S = \{x \in U : f(x) = 0\}$ and

$$E(f) = \{(x, u) \in U \times \mathbf{P}^m : f_i(x)u_j = f_j(x)u_i,\ i, j = 0, \dots, m\},$$

where $u = [u_0 : \cdots : u_m] \in \mathbf{P}^m$.

Let Y be the closure of $E(f) \backslash (S \times \mathbf{P}^m)$ in $U \times \mathbf{P}^m$. The natural projection

$$\pi : Y \to U$$

is called the (*elementary*) *blowing-up of* U *by means of* $f_0, \dots, f_m$. The analytic subset S is called the *centre of the blowing-up* and its inverse image $\pi^{-1}(S) \subset Y$ is called the *exceptional set* of the blowing-up.

PROPOSITION 13. *Under above notations we have*:
1. Y *is an analytic subset of* $U \times \mathbf{P}^m$;
2. π *is proper, its range is* U *and the restriction* $\pi_{|Y \backslash \pi^{-1}(S)}$ *is a biholomorphism onto* $U \backslash S$;
3. Y *is irreducible*;
4. *The exceptional set* $\pi^{-1}(S)$ *is analytic in* $U \times \mathbf{P}^m$ *and it is of pure dimension* $n - 1$.

Proof. For items 1. and 2. see [6, VII.5.1]. Item 3. follows from our assumption that U is connected. For the proof of 4., let us consider the analytic map

$$F : U \times \mathbf{P}^m \ni (x, u) \mapsto (f(x), u) \in \mathbf{C}^{m+1} \times \mathbf{P}^m.$$

Let $y_0, \dots, y_m$ be coordinates in $\mathbf{C}^{m+1}$. If we denote by $\pi_{m+1} : \Pi_{m+1} \to \mathbf{C}^{m+1}$ the blowing-up of $\mathbf{C}^{m+1}$ by means of $y_0, \dots, y_m$ then for the restriction $\tilde{f} = F_{|Y}$ we get the following commutative diagram of analytic maps:

$$\begin{array}{ccc} Y & \xrightarrow{\ \tilde{f}\ } & \Pi_{m+1} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_{m+1}} \\ U & \xrightarrow{\ f\ } & \mathbf{C}^{m+1} \end{array}$$

Take $(x_0, u_0) \in \pi^{-1}(0)$. Let $\Omega \subset \Pi_{m+1}$ be a neighbourhood of $(0, u_0)$, $h : \Omega \to \mathbf{C}$ an analytic function such that

$$\pi_{m+1}^{-1}(0) \cap \Omega = \{(y, u) \in \Omega : h(y, u) = 0\}.$$

Let $\tilde{\Omega} \subset Y$ be a neighbourhood of (x_0, u_0) such that $\tilde{f}(\tilde{\Omega}) \subset \Omega$. Since $\tilde{f}^{-1}(\pi_{m+1}^{-1}(0)) = \pi^{-1}(S)$ we get

$$\pi^{-1}(S) \cap \tilde{\Omega} = \{(x, u) \in \tilde{\Omega} : h \circ \tilde{f}(x, u) = 0\}.$$

Thus there exists a neighbourhood $\Delta \subset U \times \mathbf{P}^m$ of (x_0, u_0) and an analytic set $V \subset \Delta$ of pure dimension $n + m - 1$ such that

$$\pi^{-1}(S) \cap \Delta = V \cap Y \cap \Delta.$$

This gives

$$\dim_{(x_0, u_0)} \pi^{-1}(S) \geqslant \dim_{(x_0, u_0)} Y - 1 = n - 1.$$

Since Y is irreducible and $\pi^{-1}(S) \subsetneq Y$ we get that $\dim_p \pi^{-1}(S) = n - 1$ for any $p \in \pi^{-1}(S)$. This ends the proof. □

5. Proof of Theorem 7

Lemma 14. *Let $F : (\mathbf{C}^n \times \mathbf{C}, 0) \to (\mathbf{C}^{m+1}, 0)$, $m \geqslant n$ be a holomorphic map. Assume that* 0 *is an isolated point of $F_t^{-1}(0)$ for $|t| < \delta$. Then there exists $\delta > \varepsilon > 0$ and a complex line $V \subset \mathbf{C}^{m+1}$, such that $V \cap C_{F_t} = \{0\}$ for $|t| < \varepsilon$.*

Proof. Let $F : U \to \mathbf{C}^{m+1}$, where $U \subset \mathbf{C}^n \times \mathbf{C}$ is a connected neighbourhood of the origin. Put $S = \{(z, t) \in U : F(z, t) = 0\}$ and let $\pi : U \times \mathbf{P}^m \supset Y \to U$ be the elementary blowing-up of U by F. By Proposition 13 its exceptional set $E := \pi^{-1}(S)$ is an analytic set of pure dimension n. Let $\mathscr{E}$ be the set of irreducible components W of E such that the origin in $\mathbf{C}^{n+1}$ is an accumulation point of $\pi(W) \cap (\{0\} \times \mathbf{C})$. Then $\mathscr{E}$ is finite. Denote by $\widetilde{C_{F_t}}$ the image of the cone C_{F_t} in $\mathbf{P}^m$. Observe that

$$\{(0, t)\} \times \widetilde{C_{F_t}} \subset \bigcup_{W \in \mathscr{E}} W, \quad |t| < \delta.$$

On the other hand, if $W \in \mathscr{E}$ then W is n-dimensional irreducible set and $W \not\subset \{0\} \times \mathbf{P}^m$. Consequently

$$\dim W \cap (\{0\} \times \mathbf{P}^m) \leqslant n - 1 < m.$$

Thus there exists $\varepsilon > 0$ and an open set $G \subset \mathbf{P}^m$ such that

$$(\{(0, t)\} \times G) \cap \left(\bigcup_{W \in \mathscr{E}} W \right) = \emptyset, \quad 0 < |t| < \varepsilon$$

As a result if V is a line in $\mathbf{C}^{m+1}$ corresponding to some point in G then $V \cap C_{F_t} = \{0\}$ for $0 < |t| < \varepsilon$. Since G is not a subset of C_{F_0} we get the assertion. □

Proof of Theorem 7. Induction on m. In the case $m = n$ there is nothing to prove. Let us assume that the assertion is true for some $m \geqslant n$ and let $F : (\mathbf{C}^n \times \mathbf{C}, 0) \to (\mathbf{C}^{m+1}, 0)$ be a holomorphic map such that the ideals $(F_t)\mathcal{O}_n$ are $\mathfrak{m}_n$-primary. By Lemma 14 there exists $\varepsilon > 0$ and a linear mapping $\pi' : \mathbf{C}^{m+1} \to \mathbf{C}^m$ such that $\ker \pi' \cap C_{F_t} = \{0\}$ for $|t| < \varepsilon$. Thus, by Corollary 12 the ideal $(\pi' \circ F_t)\mathcal{O}_n$ is a reduction of $(F_t)\mathcal{O}_n$, for all $t \in \mathbf{C}$ such that $|t| < \varepsilon$. On the other hand, by induction hypothesis, there exists a linear map $\pi'' : \mathbf{C}^m \to \mathbf{C}^n$ such that $(\pi'' \circ \pi' \circ F_t)\mathcal{O}_n$ is a reduction of $(\pi' \circ F_t)\mathcal{O}_n$ for small t. Thus if we put $\pi := \pi'' \circ \pi'$ we get the assertion. □

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