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GLOBAL ISOMETRIC EMBEDDINGS OF MULTIPLE WARPED PRODUCT METRICS INTO QUADRICS

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Abstract

In this paper, we construct smooth isometric embeddings of multiple warped product manifolds in quadrics of semi-Euclidean spaces. Our main theorem generalizes previous results as given by Blanuša, Rozendorn, Henke and Azov.

1. Introduction

By fundamental works of Nash [19], Gromov and Rokhlin [11] and Günther [12] we know that every n-dimensional smooth Riemannian manifold admits a smooth isometric embedding[1] in an N-dimensional Euclidean space $\mathbf{R}^N$, for some $N \le c(n) = \max\{n(n+5)/2, n(n+3)/2+5\}$. The estimate $c(n)$ was given by Günther [12]; Nash's and Gromov-Rokhlin's estimates are larger than this upper bound. Since then, the problem of finding the lowest possible codimension is one of the major open problems in the theory of isometric immersion. For books and surveys about this subject see Gromov and Rokhlin [11], Jacobowitz [18], Poznyak and Sokolov [22], Aminov [1], Dajczer [8], Borisenko [6] and Han and Hong [13]. On the other hand, since the results of Nash, Gromov and Rokhlin and Günther follow as a consequence of existence theorems for certain PDE's, it is also an interesting problem to give the explicit construction of isometric immersions of a given Riemannian metric M^n in $\mathbf{R}^m$, mainly if the attained codimension is strictly less than $c(n) - n$. This is the point of view of the present paper.

Blanuša [4, 5] gave a method to construct injective smooth isometric immersions of the hyperbolic plane $\mathbf{H}^2$ in $\mathbf{R}^6$ and in the standard spherical space $\mathbf{S}^8$. Poznyak [21] wrote about Blanuša's surface: *"There is no doubt that this result is one of the most elegant in the theory of immersion of two-dimensional*

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[1] We recall that an injective immersion is an embedding if it is a homeomorphism onto its image, by considering the image with the induced topology.

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manifolds in Euclidean space". Blanuša also constructed injective isometric immersions of $\mathbf{H}^n$ in $\mathbf{R}^{6n-5}$ and of an infinite Möbius band with hyperbolic metric in $\mathbf{R}^8$ and in $\mathbf{S}^{10}$. His method was used and modified in further works: (i) Rozendorn [23] constructed non-injective smooth isometric immersions of the plane $\mathbf{R}^2$ with the warped product metric of the form $d\sigma^2 = dt^2 + f(t)^2\, dx^2$ in $\mathbf{R}^5$ (this class of surfaces includes $\mathbf{H}^2$ with the metric $dt^2 + e^{2t}\, dx^2$). Note that a celebrated theorem of Hilbert [17] states that $\mathbf{H}^2$ cannot be isometrically immersed in $\mathbf{R}^3$. However, the existence of an isometric immersion of $\mathbf{H}^2$ in $\mathbf{R}^4$ or even an injective isometric immersion of $\mathbf{H}^2$ in $\mathbf{R}^5$ is still an open problem (a partial answer to the first problem was given by Sabitov [24]). (ii) Henke [14, 15] exhibited isometric immersions of $\mathbf{H}^n$ in $\mathbf{R}^{4n-3}$ and in $\mathbf{S}^{4n-3}$. (iii) Henke and Nettekoven [16] showed that $\mathbf{H}^n$ can be isometrically embedded in $\mathbf{R}^{6n-6}$ whose image is the graph of a smooth map $g : \mathbf{R}^n \to \mathbf{R}^{5n-6}$. (iv) Azov [2] considered the space $\mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$ with one of the following metrics: $d\sigma^2 = dt^2 + f(t)^2 \sum_{j=1}^{n-1} dx_j^2$ or $d\sigma^2 = g(x_1)^2 \sum_{j=1}^{n} dx_j^2$ and constructed isometric immersions in $\mathbf{R}^{4n-3}$ and $\mathbf{S}^{4n-3}$. He also announced in [3] the construction of isometric immersions of these classes of metrics in $\mathbf{R}^{4n-4}$ and $\mathbf{S}^{4n-4}$, if $n > 2$.

In this paper we deal with product manifolds $M^n = I \times \mathbf{R}^{n-1}$, where I is an open interval, endowed with a multiple warped product metric of the form:

$$d\sigma^2 = \rho(t)^2\, dt^2 + \eta_1(t)^2\, dx_1^2 + \cdots + \eta_{n-1}(t)^2\, dx_{n-1}^2 \tag{1}$$

where $\rho(t)$, $\eta_j(t)$, with $t \in I$ and $j = 1, \ldots, n-1$, are positive smooth functions and $dx_1, \ldots, dx_{n-1}$ are the canonical coframes of $\mathbf{R}^{n-1}$. This class of metrics includes both Azov's metrics. We will modify Blanuša's method to exhibit isometric immersions and, mainly, embeddings of this class of metrics in quadrics of semi-Euclidean spaces. It is worth to mention that, in general, the immersions obtained by Rozendorn, Henke and Azov are not injective. Based on this, we consider such embeddings the main contribution of the present work. There exists a wide literature about aspects of rigidity and nonimmersibility of these spaces (see for instance Nölker [20], Chen [7], Florit [10], Dajczer and Tojeiro [9] and references therein).

We recall that the semi-Euclidean space $\mathbf{R}^n_a$, with $a \in \{0, \ldots, n\}$, is simply the space $\mathbf{R}^n$ with the inner product of signature $(a, n-a)$ given by

$$\langle\, ,\rangle = -dx_1^2 - \cdots - dx_a^2 + dx_{a+1}^2 + \cdots + dx_n^2, \tag{2}$$

where dx_j, with $j = 1, \ldots, n$, denote the canonical coframes of $\mathbf{R}^n$. For a given $c > 0$, let $S^n_a(c)$ and $H^n_a(-c)$ be the following quadratic hypersurfaces (or, simply, quadrics):

$$\mathbf{S}^n_a(c) = \{x \in \mathbf{R}^{n+1}_a \mid \langle x, x\rangle = 1/c\};$$

$$\mathbf{H}^n_a(-c) = \{x \in \mathbf{R}^{n+1}_{a+1} \mid \langle x, x\rangle = -1/c\}.$$

Both hypersurfaces are semi-Riemannian manifolds with signature $(a, n-a)$ and constant curvatures c and $-c$, respectively. If $a = 0$, then $\mathbf{S}^n_0(c) = \mathbf{S}^n(c)$ is the standard sphere and $\mathbf{H}^n_0(-c) = \mathbf{H}^n(-c)$ is the hyperbolic space. If $a = 1$, the semi-Riemannian universal covering spaces of $\mathbf{S}_1^n(1)$ and $\mathbf{H}_1^n(-1)$ are called de Sitter dS^n and anti-de Sitter AdS^n spaces, respectively.

Our main result is

THEOREM 1.1. *Let $M^n = I \times \mathbf{R}^{n-1}$ be as given in* (1). *Then, for all $c > 0$ and $a \in \{0, \dots, n-1\}$, the manifold M^n admits*:

(i) *isometric immersions in $\mathbf{R}_a^{4n-3-2a}$, $\mathbf{S}_a^{4n-3-2a}(c)$, and $\mathbf{H}_a^{4n-3-a}(-c)$;*

(ii) *isometric embeddings in $\mathbf{R}_a^{8n-7-6a}$, $\mathbf{S}_a^{8n-5-6a}(c)$ and $\mathbf{H}_a^{8n-7-5a}(-c)$.*

Moreover, all immersions and embeddings above are smooth and given explicitly.

In Remark 1 (see Section 3), we observe that all immersions referred in Item (i) of Theorem 1.1 are not injective, provided that $a < n - 1$.

Based on Theorem 1.1, it is natural to ask if every n-dimensional Riemannian manifold M^n can be isometrically immersed in a semi-Euclidean $\mathbf{R}_a^N$ with $a > 0$ and N strictly less than the dimension $c(n)$ obtained by Günther [12].

As an application of Theorem 1.1 we generalize Rozendorn's surfaces [23]. We have the following.

COROLLARY 1.1. *Let $M^2 = I \times \mathbf{R}$ be a warped product surface as given in* (1). *Then, for all $c > 0$, the surface M^2 admits*:

(i) *non-injective isometric immersions in $\mathbf{R}^5$, $\mathbf{H}^5(-c)$, $\mathbf{S}^5(c)$ and $dS^3(c)$;*

(ii) *isometric embeddings in $\mathbf{R}^9$, $\mathbf{H}^9(-c)$, $\mathbf{S}^{11}(c)$, $\mathbf{R}_1^3$, $AdS^4(-c)$ and $dS^5(c)$.*

Moreover, all immersions and embeddings above are smooth and given explicitly.

The space Sol_3 is a simply connected homogeneous 3-dimensional space whose isometry group has dimension 3. It is one of the eight models of the Thurston geometry and it can be viewed as $\mathbf{R}^3$ with the metric $ds^2 = dt^2 + e^{2t}\,dx^2 + e^{-2t}\,dy^2$. It follows directly from Theorem 1.1 the following

COROLLARY 1.2. *For all $c > 0$, the space Sol_3 admits*:

(i) *non-injective isometric immersions in $\mathbf{R}^9$, $\mathbf{H}^9(-c)$, $\mathbf{S}^9(c)$, $\mathbf{R}_1^7$, $dS^7(c)$, $AdS^8(-c)$;*

(ii) *isometric embeddings in $\mathbf{R}^{17}$, $\mathbf{H}^{17}(-c)$, $\mathbf{S}^{19}(c)$, $\mathbf{R}_1^{11}$, $AdS^{12}(-c)$ and $dS^{13}(c)$.*

Moreover, all immersions and embeddings above are smooth and given explicitly.

Let $f_l : M_l \to \mathbf{R}^{n_l}$, with $l = 1, \dots, k$, be smooth isometric immersions of the manifold (M_l, g_l) in $\mathbf{R}^{n_l}$. Let I be an open interval and $\rho(t)$, $\eta_l(t)$, with $t \in I$ and $l = 1, \dots, k$, positive smooth functions. It is simple to show that the product manifold $M = I \times M_1 \times \cdots \times M_k$ with the warped product metric

$$g = \rho(t)^2\, dt^2 + \eta_1(t)^2 g_1 + \cdots + \eta_k(t)^2 g_k \tag{3}$$

can be isometrically immersed in $I \times \mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_k}$ with the metric

$$d\sigma^2 = \rho(t)^2\, dt^2 + \eta_1(t)^2 \delta_1 + \cdots + \eta_k(t)^2 \delta_k,$$

where δ_l denotes the Euclidean metric of $\mathbf{R}^{n_l}$. Thus it follows as a consequence of Theorem 1.1 the following result.

Corollary 1.3. *With the notations being as above, we consider* $n = n_1 + \cdots + n_k$ *and* $a \in \{0, \dots, n\}$. *For all* $c > 0$, *the manifold* M *admits*:

(i) *isometric immersions in* $\mathbf{R}_a^{4n+1-2a}$, $\mathbf{S}_a^{4n+1-2a}(c)$ *and* $\mathbf{H}_a^{4n+1-a}(-c)$;

(ii) *isometric embeddings in* $\mathbf{R}_a^{8n+1-6a}$, $\mathbf{S}_a^{8n+3-6a}(c)$ *and* $\mathbf{H}_a^{8n+1-5a}(-c)$, *provided that each* f_l *is an embedding.*

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2. Preliminaries

We recall Blanuša's functions $\hat{\psi}_1, \hat{\psi}_2 : \mathbf{R} \to \mathbf{R}$ defined by

$$\hat{\psi}_1(u) = \sqrt{\frac{1}{A}\int_0^{u+1} \xi(\tau)\, d\tau} \quad \text{and} \quad \hat{\psi}_2(u) = \sqrt{\frac{1}{A}\int_0^{u} \xi(\tau)\, d\tau},$$

where $A = \int_0^1 \xi(\tau)\, d\tau$ and $\xi(u) = \sin(\pi u) e^{-1/(\sin(\pi u))^2}$, if $u \in \mathbf{R} \backslash \mathbf{Z}$, and $\xi(u) = 0$, if $u \in \mathbf{Z}$. Blanuša proved in [4] that these functions are smooth, non-negative and satisfy:

(a) $\hat{\psi}_j$ is periodic with period 2, for all $j = 1, 2$;

(b) $\hat{\psi}_1^2 + \hat{\psi}_2^2 = 1$, everywhere;

(c) all the derivatives $\hat{\psi}_1^{(k)}(2l+1) = \hat{\psi}_2^{(k)}(2l) = 0$, for all $l \in \mathbf{Z}$.

The next two lemmas will be useful to prove Theorem 1.1. They are simple consequences of Items (a), (b) and (c) above. To state them, let I be an open interval and $\gamma : I \to \mathbf{R}$ an increasing smooth diffeomorphism. Consider the sequence $\mathbf{t}_k = \gamma^{-1}(k)$, with $k \in \mathbf{Z}$. The first lemma says the following.

Lemma 2.1. *The functions* $\psi_j = \hat{\psi}_j \circ \gamma : I \to \mathbf{R}$ *are smooth, non-negative and satisfy the following properties*:

$$(4) \qquad \begin{cases} \psi_1(t)^2 + \psi_2(t)^2 = 1, & \textit{everywhere in } I; \\ \psi_j((\gamma)^{-1}(u)) = \psi_j((\gamma)^{-1}(u+2)), & \textit{for all } u \in \mathbf{R} \textit{ and } j = 1, 2; \\ \psi_1^{(k)}(\mathbf{t}_{2l+1}) = \psi_2^{(k)}(\mathbf{t}_{2l}) = 0, & \textit{for all } k \geq 0 \textit{ and integers } l. \end{cases}$$

Let $S_1, S_2 : I \to (0, \infty)$ be any positive step functions satisfying

$$(5) \qquad \begin{cases} S_1 \text{ is constant on each interval } [\mathbf{t}_{2l+1}, \mathbf{t}_{2l+3}); \\ S_2 \text{ is constant on each interval } [\mathbf{t}_{2l}, \mathbf{t}_{2l+2}); \end{cases}$$

for each integer l.

The second lemma follows easily from Lemma 2.1.

Lemma 2.2. *For any* $\eta \in C^{\infty}(I)$, *the functions* $\dfrac{\eta(t)\psi_j(t)}{S_j(t)}$ *with* $t \in I$ *and* $j = 1, 2$, *are smooth and their derivatives satisfy*

$$\frac{d^k}{dt^k}\left(\frac{\eta(t)\psi_j(t)}{S_j(t)}\right) = \frac{\frac{d^k}{dt^k}(\eta(t)\psi_j(t))}{S_j(t)},$$

for all integers $k \geq 0$.

3. Proof of Theorem 1.1

First we consider the map $\eta(t) = (\eta_1(t), \ldots, \eta_{n-1}(t))$, with $t \in I$, where each function η_j is being as in (1). Consider the map $h : \mathbf{R} \to \mathbf{R}^2_1$ given by $h(u) = (\cosh(u), \sinh(u))$. Consider also the map $\varphi = (\varphi_1, \varphi_2) : I \times \mathbf{R} \to \mathbf{R}^4$ where each map $\varphi_j : I \times \mathbf{R} \to \mathbf{R}^2$, with $j = 1, 2$, is given by

$$\varphi_j(t, u) = \frac{\psi_j(t)}{S_j(t)}(\cos(S_j(t)u), \sin(S_j(t)u)). \tag{6}$$

The map φ is introduced in [16] for the case that $I = \mathbf{R}$ and γ is the identity function. By using Lemma 2.2, we obtain

$$\begin{aligned}
\frac{\partial(\eta_k(t)h(u))}{\partial t} &= \eta_k'(t)(\cosh(u), \sinh(u)); \\
\frac{\partial(\eta_k(t)h(u))}{\partial u} &= \eta_k(t)(\sinh(u), \cosh(u)); \\
\frac{\partial(\eta_k(t)\varphi_j(t,u))}{\partial t} &= \frac{(\eta_k(t)\psi_j(t))'}{S_j(t)}(\cos(S_j(t)u), \sin(S_j(t)u)); \\
\frac{\partial(\eta_k(t)\varphi_j(t,u))}{\partial u} &= \eta_k(t)\psi_j(t)(-\sin(S_j(t)u), \cos(S_j(t)u));
\end{aligned} \tag{7}$$

for all $j = 1, 2$, $t \in I$, $u \in \mathbf{R}$ and $k = 1, \ldots, n-1$.

Now, set $a \in \{0, \ldots, n-1\}$ and let $b = n - 1 - a$. First we consider $b > 0$. We will see that the case $b = 0$ is easier. We write the semi-Euclidean space $\mathbf{R}^{4n-4-2a}_a = \mathbf{R}^{2a+4b}_a$ isometrically as the following form

$$\mathbf{R}^{2a+4b}_a = (\mathbf{R}^2_1)^a \times \mathbf{R}^{4b} = \underbrace{\mathbf{R}^2_1 \times \cdots \times \mathbf{R}^2_1}_{a \text{ times}} \times \mathbf{R}^{4b}.$$

We denote by $x = (x_1, \ldots, x_{a+b})$ the coordinates of $\mathbf{R}^{n-1} = \mathbf{R}^{a+b}$. Let $P_1 : \mathbf{R}^{a+b} \to \mathbf{R}^a$ and $P_2 : \mathbf{R}^{a+b} \to \mathbf{R}^b$ be the standard orthogonal projections

$$\begin{aligned}
\tilde{x} &= P_1(x_1, \ldots, x_{a+b}) = (x_1, \ldots, x_a) \\
\bar{x} &= P_2(x_1, \ldots, x_{a+b}) = (x_{a+1}, \ldots, x_{a+b}).
\end{aligned}$$

Consider the maps

$$\tilde{\eta}(t) = P_1(\eta(t)) \quad \text{and} \quad \bar{\eta}(t) = P_2(\eta(t)), \tag{8}$$

with $t \in I$ and let $\tilde{\eta} \star h : I \times \mathbf{R}^{n-1} \to (\mathbf{R}_1^2)^a$ and $\bar{\eta} \star \varphi : I \times \mathbf{R}^{n-1} \to \mathbf{R}^{4b}$ be the maps given by

$$\begin{aligned} &(\tilde{\eta} \star h)(t, x_1, \dots, x_{a+b}) = (\eta_1(t)h(x_1), \dots, \eta_a(t)h(x_a)) \in (\mathbf{R}_1^2)^a; \\ &(\bar{\eta} \star \varphi)(t, x_1, \dots, x_{a+b}) = (\eta_{a+1}(t)\varphi(t, x_{a+1}), \dots, \eta_{a+b}(t)\varphi(t, x_{a+b})) \in \mathbf{R}^{4b}. \end{aligned} \tag{9}$$

Since $h(u) \in \mathbf{R}_1^2$ and $\varphi(t, u) \in \mathbf{R}^4$, by using (7), the pull-back symmetric tensors by $\tilde{\eta} \star h$ and $\bar{\eta} \star \varphi$ must satisfy

$$\begin{aligned} &(\tilde{\eta} \star h)^*(\langle\, , \rangle) = -|\tilde{\eta}'(t)|^2\, dt^2 + \eta_1(t)^2\, dx_1^2 + \cdots + \eta_a(t)^2\, dx_a^2; \\ &(\bar{\eta} \star \varphi)^*(\langle\, , \rangle) = \varepsilon(t)^2\, dt^2 + \eta_{a+1}(t)^2\, dx_{a+1}^2 + \cdots + \eta_{a+b}(t)^2\, dx_{a+b}^2, \end{aligned} \tag{10}$$

where $|\tilde{\eta}'(t)|^2 = \eta_1'(t)^2 + \cdots + \eta_a'(t)^2$ and $\varepsilon : I \to [0, \infty)$ is given by

$$\varepsilon(t)^2 = \sum_{r=a+1}^{a+b} \left[\frac{((\eta_r(t)\psi_1(t))')^2}{S_1(t)^2} + \frac{((\eta_r(t)\psi_2(t))')^2}{S_2(t)^2} \right]. \tag{11}$$

For the step functions $S_1, S_2 : I \to (0, \infty)$ defined as in (5), we can choose the steps $S_1|_{[\mathbf{t}_{2l+1}, \mathbf{t}_{2l+3})}$ and $S_2|_{[\mathbf{t}_{2l}, \mathbf{t}_{2l+2})}$, with integer l, sufficiently large so that, for all $r = a+1, \dots, a+b$, it holds

$$((\eta_r(t)\psi_j(t))')^2 < \frac{1}{4b} S_j(t)^2 \rho(t)^2, \tag{12}$$

for all $t \in I$ and $j = 1, 2$. We obtain $\rho(t)^2 - \varepsilon(t)^2 \geq \rho(t)^2 - 2\varepsilon(t)^2 > 0$, for all $t \in I$.

Let $f : I \times \mathbf{R}^{n-1} \to \mathbf{R}_a^{4n-3-2a} = \mathbf{R} \times (\mathbf{R}_1^2)^a \times \mathbf{R}^{4b}$ be the map

$$f(t, x) = \left(\int_{t_0}^{t} \sqrt{\rho(\tau)^2 + |\tilde{\eta}'(\tau)|^2 - \varepsilon(\tau)^2}\, d\tau, \tilde{\eta} \star h(t, x), \bar{\eta} \star \varphi(t, x) \right).$$

If $b = 0$, we define $f(t, x)$ simply by omitting $\varepsilon(t)$ and $\bar{\eta} \star \varphi(t, x)$ in the expression of $f(t, x)$ above. By using (10), we obtain

$$\begin{aligned} f^*(\langle\, , \rangle) &= (\rho(t)^2 + |\tilde{\eta}'(t)|^2 - \varepsilon(t)^2)\, dt^2 + (\tilde{\eta} \star h)^*(\langle\, , \rangle) + (\bar{\eta} \star \varphi)^*(\langle\, , \rangle) \\ &= \rho(t)^2\, dt^2 + \eta_1(t)^2\, dx_1^2 + \cdots + \eta_{a+b}(t)^2\, dx_{a+b}^2 \\ &= d\sigma^2. \end{aligned}$$

This implies that $f : M^n \to \mathbf{R}_a^{4n-3-2a}$ is a smooth isometric immersion.

We fix $c > 0$. First we assume $b > 0$. We choose the step functions S_1, S_2 sufficiently large so that (12) holds. Let $\alpha : I \to [0, \infty)$ be the function given by

$$\alpha(t) = \sum_{r=a+1}^{a+b} \left(\frac{\eta_r(t)^2 \psi_1(t)^2}{S_1(t)^2} + \frac{\eta_r(t)^2 \psi_2(t)^2}{S_2(t)^2} \right). \tag{13}$$

Note that $\alpha(t) = \langle \bar{\eta} \star \varphi(t,x), \bar{\eta} \star \varphi(t,x) \rangle$.

Let $f_h : I \times \mathbf{R}^{n-1} \to \mathbf{R}^{4n-2-a}_{a+1} = \mathbf{R}^2_1 \times \mathbf{R}^a \times (\mathbf{R}^2_1)^a \times \mathbf{R}^{4b}$ be the map

$$f_h(t,x) = (\sqrt{1/c + \alpha(t)}\, h(\theta_h(t)), \tilde{\eta}(t), \tilde{\eta} \star h(t,x), \bar{\eta} \star \varphi(t,x)),$$

where $\theta_h : I \to \mathbf{R}$ is the function defined by

$$\theta_h(t) = \int_{t_0}^{t} \sqrt{\frac{1}{\frac{1}{c} + \alpha(\tau)} \left[\rho(\tau)^2 - \varepsilon(\tau)^2 + \frac{\alpha'(\tau)^2}{4\left(\frac{1}{c} + \alpha(\tau)\right)} \right]}\, d\tau.$$

If $b = 0$, we define $f_h(t,x)$ simply by omitting $\varepsilon(t)$, $\alpha(t)$ and $\bar{\eta} \star \varphi(t,x)$ in the expressions of $\theta_h(t)$ and $f_h(t,x)$ above. By (12), we have $\rho(t)^2 - \varepsilon(t)^2 > 0$. Thus, in both cases $b = 0$ and $b > 0$, we have that θ_h is well defined, smooth and increasing.

It is easy to see that $\langle f_h(t,x), f_h(t,x) \rangle = -(1/c + \alpha(t)) + |\tilde{\eta}(t)|^2 - |\tilde{\eta}(t)|^2 + \alpha(t) = -1/c$, hence the image of f_h is contained in $\mathbf{H}^{4n-3-a}_a(-c)$. By using (10),

$$\begin{aligned} f_h^*(\langle\, , \rangle) &= \left[-\frac{1}{4} \left(\frac{\alpha'(t)^2}{\frac{1}{c} + \alpha(t)} \right) + \left(\frac{1}{c} + \alpha(t) \right) \theta_h'(t)^2 + |\tilde{\eta}'(t)|^2 \right] dt^2 \\ &\quad + (\tilde{\eta} \star h)^*(\langle\, , \rangle) + (\bar{\eta} \star \varphi)^*(\langle\, , \rangle) \\ &= \rho(t)^2\, dt^2 + \eta_1(t)^2\, dx_1^2 + \cdots + \eta_{a+b}(t)^2\, dx_{a+b}^2. \end{aligned}$$

This implies that $f_h : M^n \to \mathbf{H}^{4n-3-a}_a(-c)$ is an isometric immersion.

Now, choose the step functions S_1, S_2 sufficiently large so that (12) is satisfied and, moreover, for all $r = a+1, \dots, a+b$, it holds

$$\eta_r(t)^2 \psi_j(t)^2 < \frac{1}{8bc} S_j(t)^2, \tag{14}$$

for all $t \in I$ and $j = 1, 2$. By (13) and (14), we obtain $0 \le \alpha(t) < \dfrac{1}{4c}$, for all $t \in I$.

Let $f_s : I \times \mathbf{R}^{n-1} \to \mathbf{R}^{4n-2-2a}_a = \mathbf{R}^2 \times (\mathbf{R}^2_1)^a \times \mathbf{R}^{4b}$ be the map defined by

$$f_s(t,x) = (\sqrt{1/c - \beta(t)}\, g(\theta_s(t)), \tilde{\eta} \star h(t,x), \bar{\eta} \star \varphi(t,x)),$$

where $\beta(t) = \alpha(t) - |\tilde{\eta}(t)|^2$, with $t \in I$, $g(u) = (\cos(u), \sin(u))$, with $u \in \mathbf{R}$, and $\theta_s : I \to \mathbf{R}$ is the function given by

$$\theta_s(t) = \int_{t_0}^{t} \sqrt{\frac{1}{\frac{1}{c} - \beta(\tau)} \left[\rho(\tau)^2 + |\tilde{\eta}'(\tau)|^2 - \varepsilon(\tau)^2 - \frac{\beta'(\tau)^2}{4\left(\frac{1}{c} - \beta(\tau)\right)} \right]}\, d\tau.$$

If $b = 0$, we define $f_s(t,x)$ by omitting $\varepsilon(t)$, $\alpha(t)$ and $\bar{\eta} \star \varphi(t,x)$ in the expressions of $\theta_s(t)$ and $f_s(t,x)$ above.

Claim 3.1. *We can choose steps functions S_1, S_2, sufficiently large so that the function θ_s is well defined and smooth.*

In fact, first we assume $b = 0$. In this case, by definition, we have $\beta(t) = -|\tilde{\eta}(t)|^2$. Hence, $\frac{1}{c} - \beta(t) > |\tilde{\eta}(t)|^2$. Moreover, $\beta'(t)^2 = 4\langle \tilde{\eta}'(t), \tilde{\eta}(t)\rangle^2 \leq 4|\tilde{\eta}'(t)|^2 |\tilde{\eta}(t)|^2$. Thus,

$$\frac{\beta'(t)^2}{4\left(\frac{1}{c} - \beta(t)\right)} \leq |\tilde{\eta}'(t)|^2.$$

By (12), we have $\rho(t)^2 - \varepsilon(t)^2 > 0$. Thus we conclude that θ_s is well defined and smooth. Now, assume $b > 0$. By (14), it holds $\frac{1}{c} - \beta(t) \geq \frac{1}{c} - \alpha(t) > 0$. Using Lemma 2.2, we have

$$\alpha'(t) = \sum_{r=a+1}^{a+b} \left[\frac{(\eta_r(t)^2 \psi_1(t)^2)'}{S_1(t)^2} + \frac{(\eta_r(t)^2 \psi_2(t)^2)'}{S_2(t)^2} \right]. \tag{15}$$

Since $\beta'(t) = \alpha'(t) - 2\langle \tilde{\eta}'(t), \tilde{\eta}(t)\rangle$, we obtain $\frac{\beta'(t)^2}{4} \leq \delta(t) + |\tilde{\eta}'(t)|^2 |\tilde{\eta}(t)|^2$, where $\delta : I \to [0, \infty)$ is the continuous function given by

$$\delta(t) = \left| \frac{\alpha'(t)^2}{4} - \alpha'(t)\langle \tilde{\eta}'(t), \tilde{\eta}(t)\rangle \right|. \tag{16}$$

Using that $\frac{1}{c} - \beta(t) = \frac{1}{c} - \alpha(t) + |\tilde{\eta}(t)|^2 > \frac{1}{2c} + |\tilde{\eta}(t)|^2$, it holds that

$$\frac{\beta'(t)^2}{4\left(\frac{1}{c} - \beta(t)\right)} \leq \frac{1}{\frac{1}{2c} + |\tilde{\eta}(t)|^2} (\delta(t) + |\tilde{\eta}(t)|^2 |\tilde{\eta}'(t)|^2).$$

This implies that

$$\rho(t)^2 + |\tilde{\eta}'(t)|^2 - \varepsilon(t)^2 - \frac{\beta'(t)^2}{4\left(\dfrac{1}{c} - \beta(t)\right)} \geq \rho(t)^2 - \varepsilon(t)^2 + |\tilde{\eta}'(t)|^2 \left(1 - \frac{|\tilde{\eta}(t)|^2}{\dfrac{1}{2c} + |\tilde{\eta}(t)|^2}\right) - \frac{\delta(t)}{\dfrac{1}{2c} + |\tilde{\eta}(t)|^2}$$

$$= \frac{1}{\dfrac{1}{2c} + |\tilde{\eta}(t)|^2} (\Gamma(t) - \delta(t)),$$

where $\Gamma : I \to \mathbf{R}$ is the continuous function given by

$$\Gamma(t) = \left(\frac{1}{2c} + |\tilde{\eta}(t)|^2\right)\left(\rho(t)^2 - \varepsilon(t)^2 + |\tilde{\eta}'(t)|^2 \left(1 - \frac{|\tilde{\eta}(t)|^2}{\dfrac{1}{2c} + |\tilde{\eta}(t)|^2}\right)\right).$$

Since the step functions S_1 and S_2 satisfy (12), we obtain $\Gamma(t) > 0$, for all $t \in I$. Furthermore, if $S_1(t)$ and $S_2(t)$ become larger, then $\Gamma(t) > 0$ increases and $\delta(t)$ is as smaller as we want. So, we choose each step of S_1 and S_2 sufficiently large so that $\delta(t) < \Gamma(t)$, for all $t \in I$. This implies that

$$\rho(t)^2 + |\tilde{\eta}'(t)|^2 - \varepsilon(t)^2 - \frac{\beta'(t)^2}{4\left(\dfrac{1}{c} - \beta(t)\right)} > 0, \tag{17}$$

for all $t \in I$. Claim 3.1 is proved.

It is easy to see that $\langle f_s(t,x), f_s(t,x)\rangle = 1/c$, hence the image of f_s is contained in $\mathbf{S}_a^{4n-3-2a}(c)$. By using (10),

$$f_s^*(\langle\,,\rangle) = \left(\frac{\beta'(t)^2}{4\left(\dfrac{1}{c} - \beta(t)\right)} + \left(\frac{1}{c} - \beta(t)\right)\theta_s'(t)^2\right) dt^2 + (\tilde{\eta} \star h)^*(\langle\,,\rangle) + (\bar{\eta} \star \varphi)^*(\langle\,,\rangle)$$

$$= \rho(t)^2\, dt^2 + \eta_1(t)^2\, dx_1^2 + \cdots + \eta_{a+b}(t)^2\, dx_{a+b}^2.$$

This implies that $f_s : M^n \to \mathbf{S}_a^{4n-3-2a}(c)$ is an isometric immersion. Item (i) is proved.

Remark 1. The immersions f, f_h and f_s are not injective, if $b > 0$. In fact, we take $t = \mathbf{t}_{2k}$, for some integer k. Let $x^1 = (x_1^1, \ldots, x_{a+b}^1)$ and $x^2 = (x_1^2, \ldots, x_{a+b}^2)$ be vectors satisfying the following.

(i) $(x_1^1, \ldots, x_a^1) = (x_1^2, \ldots, x_a^2)$;

(ii) $S_1(t)x_r^1 = S_1(t)x_r^2 + 2\pi l_r$, for some integer l_r, with $r = a+1, \ldots, a+b$ and $l_r \neq 0$ for some r.

Notice that $\tilde{\eta} \star h(t, x^1) = \tilde{\eta} \star h(t, x^2)$, since $(x_1^1, \ldots, x_a^1) = (x_1^2, \ldots, x_a^2)$. Since $\psi_2(t) = \psi_2(\mathbf{t}_{2k}) = 0$ and $(\cos(S_1(t)x_r^1), \sin(S_1(t)x_r^1)) = (\cos(S_1(t)x_r^2), \sin(S_1(t)x_r^2))$, we obtain

$$\psi_j(t)(\cos(S_j(t)x_r^1), \sin(S_j(t)x_r^1)) = \psi_j(t)(\cos(S_j(t)x_r^2), \sin(S_j(t)x_r^2)).$$

This implies that $\bar{\eta} \star \varphi(t, x^1) = \bar{\eta} \star \varphi(t, x^2)$. Since the first coordinates of f, f_h and f_s depend only on t, it follows that $f(t, x^1) = f(t, x^2)$, $f_h(t, x^1) = f_h(t, x^2)$, and $f_s(t, x^1) = f_s(t, x^2)$. Thus, the immersions f, f_h and f_s are not injective.

Now we will prove Item (ii). We will continue to assume the notations being as given in the proof of Item (i). Let $T_1 : \mathbf{R} \to \left(0, \frac{\pi}{2}\right)$ and $T_2 : \mathbf{R} \to \mathbf{R}$ be the smooth functions

$$T_1(u) = \frac{\pi}{4}(1 + \tanh(u)) \quad \text{and} \quad T_2(u) = \int_0^u \sqrt{1 - T_1'(\tau)^2}\, d\tau. \tag{18}$$

Note that T_2 is smooth since T_1 is analytic and $T_1'(u) = \frac{\pi}{4}\operatorname{sech}^2(u) \le \frac{\pi}{4} < 1$.

Consider the map $\hat{\varphi} = (\varphi_{11}, \varphi_{21}, \varphi_{12}, \varphi_{22}) : I \times \mathbf{R} \to \mathbf{R}^8$, where each map $\varphi_{ji} : I \times \mathbf{R} \to \mathbf{R}^2$, with $i, j = 1, 2$, is defined by

$$\varphi_{ji}(t, u) = \frac{\psi_j(t)}{S_j(t)}(\cos(T_i(S_j(t)u)), \sin(T_i(S_j(t)u))). \tag{19}$$

Consider the map

$$(\bar{\eta} \star \hat{\varphi})(t, x_1, \ldots, x_{a+b}) = (\eta_{a+1}(t)\hat{\varphi}(t, x_{a+1}), \ldots, \eta_{a+b}(t)\hat{\varphi}(t, x_{a+b})) \in \mathbf{R}^{8b}, \tag{20}$$

with $t \in I$ and $x \in \mathbf{R}^{n-1} = \mathbf{R}^{a+b}$. Since $T_1'(t)^2 + T_2'(t)^2 = \psi_1(t)^2 + \psi_2(t)^2 = 1$, by using Lemma 2.2, it follows similarly as in (10) that the pull-back symmetric tensor by the map $\bar{\eta} \star \hat{\varphi} : I \times \mathbf{R}^{n-1} \to \mathbf{R}^{8b}$ satisfies

$$(\bar{\eta} \star \hat{\varphi})^*(\langle\,,\rangle) = 2\varepsilon(t)^2\, dt^2 + \eta_{a+1}(t)^2\, dx_{a+1}^2 + \cdots + \eta_{a+b}(t)^2\, dx_{a+b}^2, \tag{21}$$

where $\varepsilon : I \to [0, \infty)$ is the smooth function defined as in (11). We choose the step functions S_1 and S_2 so that (12) is satisfied. This implies that $\rho(t)^2 - 2\varepsilon(t)^2 > 0$.

Let $\hat{f} : I \times \mathbf{R}^{n-1} \to \mathbf{R}_a^{8n-7-6a} = \mathbf{R} \times (\mathbf{R}_1^2)^a \times \mathbf{R}^{8b}$ be the map

$$\hat{f}(t, x) = \left(\int_{t_0}^t \sqrt{\rho(\tau)^2 + |\tilde{\eta}'(\tau)|^2 - 2\varepsilon(\tau)^2}\, d\tau, \tilde{\eta} \star h(t, x), \bar{\eta} \star \hat{\varphi}(t, x)\right),$$

where $\tilde{\eta} \star h : I \times \mathbf{R}^{n-1} \to (\mathbf{R}_1^2)^a$ is the map defined as in (9). If $b = 0$, we define $\hat{f}(t, x)$ by simply omitting $\varepsilon(t)$ and $\bar{\eta} \star \hat{\varphi}(t, x)$ in the definition of $\hat{f}(t, x)$ above. By using (10) and (21), it is easy to conclude that $\hat{f} : M^n \to \mathbf{R}_a^{8n-7-6a}$ is an isometric immersion.

Claim 3.2. *The immersion $\hat{f}$ is injective.*

In fact, assume that $\hat{f}(t^1, x^1) = \hat{f}(t^2, x^2)$, for some $t^1, t^2 \in I$ and $x^1, x^2 \in \mathbf{R}^{n-1}$. We write $x^j = (x_1^j, \dots, x_{a+b}^j)$, with $j = 1, 2$. Using that the function

$$s(t) = \int_{t_0}^{t} \sqrt{\rho(\tau)^2 + |\tilde{\eta}'(\tau)|^2 - 2\varepsilon(\tau)^2}\, d\tau, \quad t \in I,$$

is increasing, we obtain $t^1 = t^2$. Since $\psi_1^2 + \psi_2^2 = 1$, we can assume, without loss of generality, that $\psi_1(t^1) \neq 0$. Using that $\eta_i(t) > 0$, for all $i = 1, \dots, a+b$ and $\hat{f}(t^1, x^1) = \hat{f}(t^1, x^2)$, we have $h(x_k^1) = h(x_k^2)$ and $\varphi_{11}(t^1, x_r^1) = \varphi_{11}(t^1, x_r^2)$, for all $k = 1, \dots, a$ and $r = a+1, \dots, a+b$. These imply that

$$\sinh(x_k^1) = \sinh(x_k^2) \quad \text{and} \quad \sin(T_1(S_1(t^1)x_r^1)) = \sin(T_1(S_1(t^1)x_r^2)),$$

for all $k = 1, \dots, a$ and $r = a+1, \dots, a+b$. Since $S_1(t^1) > 0$ and the functions $\sinh(u)$ and $\sin(T_1(u))$, with $u \in \mathbf{R}$, are injective, we obtain that $x^1 = x^2$. Claim 3.2 is proved.

Claim 3.3. $\hat{f} : M^n \to \mathbf{R}_a^{8n-7-6a}$ *is an isometric embedding.*

We just need to show that the inverse map $\hat{f}^{-1} : \hat{f}(I \times \mathbf{R}^{n-1}) \to I \times \mathbf{R}^{n-1}$ is continuous. In fact, let $y_m = \hat{f}(t_m, x_1^m, \dots, x_{n-1}^m)$ be a sequence that converges to a point $y_\infty = \hat{f}(t_\infty, x_1^\infty, \dots, x_{n-1}^\infty)$. Since the function $s(t)$ is the first coordinate of $\hat{f}(t, x)$, we obtain $\lim s(t_m) = s(t_\infty)$. This implies that $\lim t_m = t_\infty$, since $s : I \to \mathbf{R}$ is a diffeomorphism of I onto its image $s(I)$. Since the coordinates of the map $\eta(t) = (\eta_1(t), \dots, \eta_{n-1}(t))$ are positive and smooth, we obtain

(a) $\lim h(x_k^m) = h(x_k^\infty)$

(b) $\lim \varphi_{ji}(t_m, x_r^m) = \varphi_{ji}(t_\infty, x_r^\infty)$,

for all $i, j = 1, 2$, $k = 1, \dots, a$ and $r = a+1, \dots, n-1$. It follows from (a) that $\lim x_k^m = x_k^\infty$, for all $k = 1, \dots, a$, since $h(u) = (\cosh(u), \sinh(u))$ and $\sinh(u)$, with $u \in \mathbf{R}$, is a diffeomorphism. Now, using that $\psi_1(t_\infty)^2 + \psi_2(t_\infty)^2 = 1$, we can assume that $\psi_1(t_\infty) \neq 0$. Since $\psi_1(t_\infty) > 0$, we obtain that S_1 is a positive constant function in a neighborhood of t_∞. This implies that $S_1(t_m) = S_1(t_\infty) > 0$, for sufficiently large m. Since $\lim \psi_1(t_m) = \psi_1(t_\infty) > 0$, we obtain from (b) and (19) that

$$\begin{aligned} \lim \cos(T_1(S_1(t_\infty)x_r^m)) &= \lim \frac{S_1(t_m)}{\psi_1(t_m)} P(\varphi_{11}(t_m, x_r^m)) = \frac{S_1(t_\infty)}{\psi_1(t_\infty)} P(\varphi_{11}(t_\infty, x_r^\infty)) \\ &= \cos(T_1(S_1(t_\infty)x_r^\infty)), \end{aligned}$$

for all $r = a+1, \dots, n-1$, where $P : \mathbf{R}^2 \to \mathbf{R}$ is the projection $P(u, v) = u$. Again using that $S_1(t_\infty) > 0$ and since $\cos(T_1(u))$ is a diffeomorphism of $\mathbf{R}$ onto $(0, 1)$, it follows that $\lim x_r^m = x_r^\infty$, for all $r = a+1, \dots, n-1$. We conclude that $\hat{f}^{-1}$ is continuous. Claim 3.3 is proved.

Let $\hat{f}_h : I \times \mathbf{R}^{n-1} \to \mathbf{R}^{8n-6-5a}_{a+1} = \mathbf{R}^2_1 \times \mathbf{R}^a \times (\mathbf{R}^2_1)^a \times \mathbf{R}^{8b}$ be the map

$$\hat{f}_h(t,x) = \left(\sqrt{\frac{1}{c} + 2\alpha(t)}\, h(\hat{\theta}_h(t)), \tilde{\eta}(t), \tilde{\eta} \star h(t,x), \bar{\eta} \star \hat{\varphi}(t,x)\right), \tag{22}$$

where $\alpha : I \to [0,\infty)$ is as defined in (13) and $\hat{\theta}_h : I \to \mathbf{R}$ is the function given by

$$\hat{\theta}_h(t) = \int_{t_0}^{t} \sqrt{\frac{1}{\frac{1}{c} + 2\alpha(\tau)}\left(\rho(\tau)^2 - 2\varepsilon(\tau)^2 + \frac{\alpha'(\tau)^2}{\frac{1}{c} + 2\alpha(\tau)}\right)}\, d\tau. \tag{23}$$

If $b = 0$, we define $\hat{f}_h(t,x)$ simply by omitting $\alpha(t)$, $\varepsilon(t)$ and $\bar{\eta} \star \varphi(t,x)$ in the expressions of $\hat{\theta}_h(t)$ and $\hat{f}_h(t,x)$ above.

By (12), we have $\rho(t)^2 - 2\varepsilon(t)^2 > 0$. This implies that $\hat{\theta}_h : I \to \mathbf{R}$ is well defined, smooth and increasing.

Note that $\langle \hat{f}_h(t,x) \hat{f}_h(t,x)\rangle = -\left(\frac{1}{c} + 2\alpha(t)\right) + |\tilde{\eta}(t)|^2 - |\tilde{\eta}(t)|^2 + 2\alpha(t) = -\frac{1}{c}$. Thus the image of $\hat{f}_h$ is contained in $\mathbf{H}^{8n-7-5a}_a(-c)$. By a standard computation,

$$\begin{aligned}
(\hat{f}_h)^*(\langle\,,\rangle) &= \left(-\frac{\alpha'(t)^2}{\frac{1}{c} + 2\alpha(t)} + \left(\frac{1}{c} + 2\alpha(t)\right)\hat{\theta}_h'(t)^2 + |\tilde{\eta}'(t)|^2\right) dt^2 \\
&\quad + (\tilde{\eta} \star h)^*(\langle\,,\rangle) + (\bar{\eta} \star \hat{\varphi})^*(\langle\,,\rangle) \\
&= \rho(t)^2\, dt^2 + \eta_1(t)^2\, dx_1^2 + \cdots + \eta_{a+b}(t)^2\, dx_{a+b}^2.
\end{aligned}$$

Thus $\hat{f}_h : M^n \to \mathbf{H}^{8n-7-5a}_a(-c)$ is an isometric immersion.

CLAIM 3.4. *The immersion $\hat{f}_h$ is injective.*

In fact, assume that $\hat{f}_h(t^1, x^1) = \hat{f}_h(t^2, x^2)$. Using (22), we have

$$\sqrt{\frac{1}{c} + 2\alpha(t^1)}\, h(\hat{\theta}_h(t^1)) = \sqrt{\frac{1}{c} + 2\alpha(t^2)}\, h(\hat{\theta}_h(t^2)).$$

Since $\langle h(u), h(u)\rangle = -1$, for all $u \in \mathbf{R}$, we obtain $\frac{1}{c} + 2\alpha(t^1) = \frac{1}{c} + 2\alpha(t^2)$, hence $h(\hat{\theta}_h(t^1)) = h(\hat{\theta}_h(t^2))$. This implies that $t^1 = t^2$, since the function $\sinh(\hat{\theta}_h(t))$, with $t \in I$, is increasing. The argument to show that $x^1 = x^2$ is similar the one as given in Claim 3.2. Claim 3.4 is proved.

CLAIM 3.5. *$\hat{f}_h : M^n \to \mathbf{H}^{8n-7-5a}_a(-c)$ is an isometric embedding.*

In fact, we just need to prove that the inverse map $(\hat{f}_h)^{-1} : \hat{f}_h(I \times \mathbf{R}^{n-1}) \to I \times \mathbf{R}^{n-1}$ is continuous. Let $y_m = \hat{f}_h(t_m, x^m)$ be a sequence that converges to a point $y_\infty = \hat{f}_h(t_\infty, x^\infty)$. Using (22), we obtain

$$\lim \sqrt{\frac{1}{c} + 2\alpha(t_m)}h(\hat{\theta}_h(t_m)) = \sqrt{\frac{1}{c} + 2\alpha(t_\infty)}h(\hat{\theta}_h(t_\infty)).$$

Using $\langle h(u), h(u)\rangle = -1$, we obtain $\lim\left(\frac{1}{c} + 2\alpha(t_m)\right) = \frac{1}{c} + 2\alpha(t_\infty) > 0$, hence $\lim h(\hat{\theta}_h(t_m)) = h(\hat{\theta}_h(t_\infty))$. This implies that $\lim \sinh(T_1(\hat{\theta}_h(t_m))) = \sinh(T_1(\hat{\theta}_h(t_\infty)))$. Using that $\sinh(T_1(\hat{\theta}_h(t)))$ is a diffeomorphism of I onto its image, it follows that $\lim t_m = t_\infty$. The argument to show that $\lim x^m = x^\infty$ is also similar to Claim 3.3. Thus Claim 3.5 is proved.

Now let $\hat{f}_s : I \times \mathbf{R}^{n-1} \to \mathbf{R}_a^{8n-4-6a} = \mathbf{R}^4 \times (\mathbf{R}_1^2)^a \times \mathbf{R}^{8b}$ be the map

$$\hat{f}_s(t,x) = \left(\sqrt{\frac{1}{2c} + \frac{|\tilde{\eta}(t)|^2}{2} - \alpha(t)}\mathscr{C}(\hat{\theta}_s(t)), \tilde{\eta} \star h(t,x), \bar{\eta} \star \hat{\varphi}(t,x)\right), \tag{24}$$

where $\mathscr{C}(u) = (\cos(T_1(u)), \sin(T_1(u)), \cos(T_2(u)), \sin(T_2(u)))$, with $u \in \mathbf{R}$. Further, the function $\alpha : I \to [0, \infty)$ is given as in (13), and $\hat{\theta}_s : I \to \mathbf{R}$ is defined by

$$\hat{\theta}_s(t) = \int_{t_0}^t \sqrt{\frac{G(\tau)}{\frac{1}{2c} + \frac{|\tilde{\eta}(\tau)|^2}{2} - \alpha(\tau)}}\, d\tau,$$

where $G : I \to \mathbf{R}$ is the function

$$G(t) = \rho(t)^2 + |\tilde{\eta}'(t)|^2 - 2\varepsilon(t)^2 - 2\frac{\left(\left(-\alpha(t) + \frac{1}{2}|\tilde{\eta}(t)|^2\right)'\right)^2}{4\left(\frac{1}{2c} + \frac{|\tilde{\eta}(t)|^2}{2} - \alpha(t)\right)}.$$

If $b = 0$, we define $\hat{f}_s$ by simply omitting $\alpha(t)$ and $\bar{\eta} \star \hat{\varphi}(t,x)$ in the definitions of $\hat{\theta}_s(t)$ and $\hat{f}_s(t,x)$ above.

We claim that we can choose the step functions S_1 and S_2 sufficiently large so that $\hat{\theta}_s$ is well defined and smooth. By (14), we already have $\frac{1}{2c} - \alpha(t) > 0$. Furthermore, by a simple computation,

$$\begin{aligned} G(t) &= \rho(t)^2 + |\tilde{\eta}'(t)|^2 - \frac{\langle \tilde{\eta}'(t), \tilde{\eta}(t)\rangle^2}{\frac{1}{c} + |\tilde{\eta}(t)|^2 - 2\alpha(t)} - \Delta(t) \\ &\geq \rho(t)^2 + |\tilde{\eta}'(t)|^2\left[1 - \frac{|\tilde{\eta}(t)|^2}{\frac{1}{c} + |\tilde{\eta}(t)|^2 - 2\alpha(t)}\right] - \Delta(t), \end{aligned}$$

where $\Delta(t) = 2\varepsilon(t)^2 + \dfrac{\alpha'(t)^2 - 2\alpha'(t)\langle\tilde{\eta}'(t), \tilde{\eta}(t)\rangle}{\frac{1}{c} + |\tilde{\eta}(t)|^2 - 2\alpha(t)}$. Note that we can take $\varepsilon(t)$, $\alpha(t)$ and $\alpha'(t)$ as smaller as we want if $S_1(t)$ and $S_2(t)$ become larger. Thus, we can choose the step functions S_1 and S_2 sufficiently large so that $\Delta(t) < \rho(t)^2$. This implies that $\hat{\theta}_s(t)$ is well defined, smooth and increasing.

Note that $\langle \hat{f}_s(t,x), \hat{f}_s(t,x)\rangle = \frac{1}{c}$ since $|\mathscr{C}(u)|^2 = 2$, $|\bar{\eta} \star \hat{\varphi}(t,x)|^2 = 2\alpha(t)$ and $\langle \tilde{\eta} \star h(t,x), \tilde{\eta} \star h(t,x)\rangle = -|\tilde{\eta}(t)|^2$. Thus the image $\hat{f}_s(I \times \mathbf{R}^{n-1}) \subset \mathbf{S}_a^{8n-5-6a}(c)$.

By a direct computation, we show that

$$\begin{aligned}
\hat{f}_s^*(\langle\, , \rangle) &= \left(\frac{\left(\left(-\alpha(t) + \frac{1}{2}|\tilde{\eta}(t)|^2\right)\right)'^2}{\frac{1}{c} + |\tilde{\eta}(t)|^2 - 2\alpha(t)} + \left(\frac{1}{2c} + \frac{|\tilde{\eta}(t)|^2}{2} - \alpha(t)\right)\hat{\theta}_s'(t)^2 \right) dt^2 \\
&\quad + (\tilde{\eta} \star h)^*(\langle\, , \rangle) + (\bar{\eta} \star \hat{\varphi})^*(\langle\, , \rangle) \\
&= \rho(t)^2\, dt^2 + \eta_1(t)^2\, dx_1^2 + \cdots + \eta_{a+b}(t)^2\, dx_{a+b}^2.
\end{aligned}$$

This implies that $\hat{f}_s : M^n \to \mathbf{S}_a^{8n-5-6a}(c)$ is an isometric immersion.

Claim 3.6. *The immersion $\hat{f}_s$ is injective.*

In fact, assume that $\hat{f}_s(t^1, x^1) = \hat{f}_s(t^2, x^2)$, for some $t^1, t^2 \in I$ and $x^1, x^2 \in \mathbf{R}^{n-1}$. Using (24),

$$\sqrt{\frac{1}{c} + |\tilde{\eta}(t^1)|^2 - 2\alpha(t^1)}\,\mathscr{C}(\hat{\theta}_s(t^1)) = \sqrt{\frac{1}{c} + |\tilde{\eta}(t^2)|^2 - 2\alpha(t^2)}\,\mathscr{C}(\hat{\theta}_s(t^2)).$$

Since $|\mathscr{C}(\hat{\theta}_s(t))|^2 = 2$, for all $t \in I$, we obtain $|\tilde{\eta}(t^1)|^2 - 2\alpha(t^1) = |\tilde{\eta}(t^2)|^2 - 2\alpha(t^2)$, hence $\mathscr{C}(\hat{\theta}_s(t^1)) = \mathscr{C}(\hat{\theta}_s(t^2))$. This implies that $t^1 = t^2$, since $\sin(T_1(\hat{\theta}_s(t)))$, with $t \in I$, is increasing. The argument to show that $x^1 = x^2$ is similar to that one given in Claim 3.2.

Claim 3.7. *$\hat{f}_s : M^n \to \mathbf{S}_a^{8n-5-6a}(c)$ is an isometric embedding.*

In fact, we just need to prove that the inverse map $(\hat{f}_s)^{-1} : \hat{f}_s(I \times \mathbf{R}^{n-1}) \to I \times \mathbf{R}^{n-1}$ is continuous. Let $y_m = \hat{f}_s(t_m, x^m)$ be a sequence that converges to a point $y_\infty = f(t_\infty, x^\infty)$. Using (24),

$$\lim \sqrt{\frac{1}{c} + |\tilde{\eta}(t_m)|^2 - 2\alpha(t_m)}\,\mathscr{C}(\hat{\theta}_s(t_m)) = \sqrt{\frac{1}{c} + |\tilde{\eta}(t_\infty)|^2 - 2\alpha(t_\infty)}\,\mathscr{C}(\hat{\theta}_s(t_\infty)).$$

Since $|\mathscr{C}(\hat{\theta}_s(t))|^2 = 2$, for all $t \in I$, we have $\lim(|\tilde{\eta}(t_m)|^2 - 2\alpha(t_m)) = |\tilde{\eta}(t_\infty)|^2 - 2\alpha(t_\infty) > 0$, hence $\lim \mathscr{C}(\hat{\theta}_s(t_m)) = \mathscr{C}(\hat{\theta}_s(t_\infty))$. This implies that $\lim \sin(T_1(\hat{\theta}_s(t_m))) = \sin(T_1(\hat{\theta}_s(t_\infty)))$. Using that $\sin(T_1(\hat{\theta}_s(t)))$ is a diffeomorphism of I onto its image, it follows that $\lim t_m = t_\infty$. The argument to show that $\lim x^m = x^\infty$ is similar to Claim 3.3. Thus, Claim 3.7 is proved. Theorem 1.1 is proved.

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