A. GEZER AND M. ALTUNBAS
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ON THE GEOMETRY OF THE RESCALED RIEMANNIAN METRIC ON TENSOR BUNDLES OF ARBITRARY TYPE

AYDIN GEZER AND MURAT ALTUNBAS

Abstract

Let (M, g) be an n-dimensional Riemannian manifold and $T_1^1(M)$ be its $(1,1)$-tensor bundle equipped with the rescaled Sasaki type metric ${}^S g_f$ which rescale the horizontal part by a non-zero differentiable function f. In the present paper, we discuss curvature properties of the Levi-Civita connection and another metric connection of $T_1^1(M)$. We construct almost product Riemannian structures on $T_1^1(M)$ and investigate conditions for these structures to be locally decomposable. Also, some applications concerning with these almost product Riemannian structures on $T_1^1(M)$ are presented. Finally we introduce the rescaled Sasaki type metric ${}^S g_f$ on the (p,q)-tensor bundle and characterize the geodesics on the (p,q)-tensor bundle with respect to the Levi-Civita connection and another metric connection of ${}^S g_f$.

1. Introduction

Geometric structures on bundles have been object of much study since the middle of the last century. The natural lifts of the metric g, from a Riemannian manifold (M, g) to its tangent or cotangent bundles, induce new (pseudo) Riemannian structures, with interesting geometric properties. Maybe the best known Riemannian metric ${}^S g$ on the tangent bundle over Riemannian manifold (M, g) is that introduced by S. Sasaki in 1958 (see [32]), but in most cases the study of some geometric properties of the tangent bundle endowed with this metric led to the flatness of the base manifold. The metric ${}^S g$ is called the Sasaki metric in literature (for the recent survey on the Sasaki metric, see [16]). The Sasaki metric ${}^S g$ has been extensively studied by several authors and in many different contexts. In [39] (see also [40, 41]), B. V. Zayatuev introduced a Riemannian metric ${}^S g_f$ on the tangent bundle TM given by

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$$
\begin{aligned}
&{}^{S}g_f({}^{H}X, {}^{H}Y) = fg(X, Y),\\
&{}^{S}g_f({}^{H}X, {}^{V}Y) = {}^{S}g_f({}^{V}X, {}^{H}Y) = 0,\\
&{}^{S}g_f({}^{V}X, {}^{V}Y) = g(X, Y),
\end{aligned}
$$

for all vector fields X and Y on M, where $f > 0$, $f \in C^{\infty}(M)$. For $f = 1$, it follows that ${}^{S}g_f = {}^{S}g$, i.e. the metric ${}^{S}g_f$ is a generalization of the Sasaki metric ${}^{S}g$. In [35], J. Wang and Y. Wang called this metric the rescaled Sasaki metric and studied geodesics and some curvature properties for the rescaled Sasaki metric. Also, the authors studied the rescaled Sasaki type metric on the cotangent bundle $T^{*}M$ over a Riemannian manifold (M, g) (see [11]).

Let M be an n-dimensional differentiable manifold endowed with an almost product structure F and a Riemannian metric g such that $g(FX, Y) = g(X, FY)$, i.e. g is pure with respect to F for arbitrary vector fields X and Y on M. Then the triple (M, F, g) is called an almost product Riemannian manifold. Almost product Riemannian structures was firstly introduced by K. Yano in [38]. The classification of almost product Riemannian structures with respect to their covariant derivatives is described by A. M. Naveira in [27]. This is the analogue of the classification of almost Hermitian structures by A. Gray and L. Hervella in [15]. Later, M. Staikova and K. Gribachev obtained a classification of the almost product Riemannian structures, for which the trace vanishes (see [33]). Almost product structures on the tangent, cotangent and tensor bundles of a manifold M were considered by some authors (e.g. see [5, 6, 7, 11, 21, 25, 28, 31]).

Fibre bundles play an important role in every aspect of modern geometry and topology. Prime examples of fiber bundles are tensor bundles of arbitrary type over differentiable manifolds. The tangent bundle TM and cotangent bundle $T^{*}M$ are the special cases of a more general tensor bundle. The Sasaki type metric is defined on (p, q)-tensor bundles over Riemannian manifolds (see, [30]). In [28], the Levi-Civita connection of the Sasaki type metric on the $(1, 1)$-tensor bundle and all types of its curvature tensors are calculated and also investigated interesting relations between the geometric properties of the base manifold and its $(1, 1)$-tensor bundle with the Sasaki type metric. In addition, it is presented examples of almost para-Norden and para-Kähler-Norden metrics on the $(1, 1)$-tensor bundle with the Sasaki type metric.

Motivated by the above studies, our aim is to define the rescaled Sasaki type metric on tensor bundles of arbitrary type and study its some properties. The paper is structured as follows. In section 2, we review some introductory materials concerning with the tensor bundle $T_1^1(M)$ over an n-dimensional differentiable manifold M. In section 3, we get the conditions under which the tensor bundle $T_1^1(M)$ endowed with some almost product structures and the rescaled Sasaki type metric ${}^{S}g_f$ is a locally decomposable Riemannian manifold. Moreover, we give some applications related to the almost product structures on the tensor bundle $T_1^1(M)$. Section 4 and section 5 discuss curvature properties of the Levi-Civita connection and another metric connection of $T_1^1(M)$ with

${}^{S}g_f$. Section 6 deals with detailed descriptions of geodesics on the (p,q)-tensor bundles with respect to the Levi-Civita connection and another metric connection of ${}^{S}g_f$.

All manifolds, tensor fields and connections in the present paper are always assumed to be differentiable of class C^∞ (i.e. smooth). Also, we denote by $\Im^p_q(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im^p_q(T^p_q(M))$ the corresponding set on the (p,q)-tensor bundle $T^p_q(M)$. The Einstein summation convention is used, the range of the indices i, j, s being always $\{1,2,\dots,n\}$.

2. Preliminaries

2.1. The $(1,1)$-tensor bundle. Let M be a differentiable manifold of class C^∞ and finite dimension n. Then the set $T^1_1(M) = \bigcup_{P\in M} T^1_1(P)$ is, by definition, the tensor bundle of type $(1,1)$ over M, where $\cup$ denotes the disjoint union of the tensor spaces $T^1_1(P)$ for all $P \in M$. For any point $\tilde{P}$ of $T^1_1(M)$ such that $\tilde{P} \in T^1_1(P)$, the surjective correspondence $\tilde{P} \to P$ determines the natural projection $\pi : T^1_1(M) \to M$. The projection π defines the natural differentiable manifold structure of $T^1_1(M)$, that is, $T^1_1(M)$ is a C^∞-manifold of dimension $n+n^2$. If x^j are local coordinates in a neighborhood U of $P \in M$, then a tensor t at P which is an element of $T^1_1(M)$ is expressible in the form (x^j, t^i_j), where t^i_j are components of t with respect to the natural base. We may consider $(x^j, t^i_j) = (x^j, x^{\bar{j}}) = (x^J)$, $j = 1,\dots,n$, $\bar{j} = n+1,\dots,n+n^2$, $J = 1,\dots,n+n^2$ as local coordinates in a neighborhood $\pi^{-1}(U)$.

Let $X = X^i \dfrac{\partial}{\partial x^i}$ and $A = A^i_j \dfrac{\partial}{\partial x^i} \otimes dx^j$ be the local expressions in U of a vector field X and a $(1,1)$ tensor field A on M, respectively. Then the vertical lift ${}^{V}A$ of A and the horizontal lift ${}^{H}X$ of X are given, with respect to the induced coordinates, by

$$ {}^{V}A = \begin{pmatrix} {}^{V}A^j \\ {}^{V}A^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^i_j \end{pmatrix}, \tag{2.1} $$

and

$$ {}^{H}X = \begin{pmatrix} {}^{H}X^j \\ {}^{H}X^{\bar{j}} \end{pmatrix} = \begin{pmatrix} X^j \\ X^s(\Gamma^m_{sj} t^i_m - \Gamma^i_{sm} t^m_j) \end{pmatrix}, \tag{2.2} $$

where Γ^h_{ij} are the coefficients of the connection ∇ on M.

Let $\varphi \in \Im^1_1(M)$, which are locally represented by $\varphi = \varphi^i_j \dfrac{\partial}{\partial x^i} \otimes dx^j$. The vector fields $\gamma\varphi$ and $\tilde{\gamma}\varphi \in \Im^1_0(T^1_1(M))$ are respectively defined by

$$ \gamma\varphi = \begin{pmatrix} 0 \\ t^m_j \varphi^i_m \end{pmatrix}, $$

$$ \tilde{\gamma}\varphi = \begin{pmatrix} 0 \\ t^i_m \varphi^m_j \end{pmatrix} $$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_1^1(M)$. From (2.1) we easily see that the vector fields $\gamma\varphi$ and $\tilde{\gamma}\varphi$ determine respectively global vector fields on $T_1^1(M)$.

The Lie bracket operation of vertical and horizontal vector fields on $T_1^1(M)$ is given by the formulas

$$\begin{cases} [{}^H X, {}^H Y] = {}^H [X, Y] + (\tilde{\gamma} - \gamma) R(X, Y), \\ [{}^H X, {}^V A] = {}^V (\nabla_X A), \\ [{}^V A, {}^V B] = 0 \end{cases} \tag{2.3}$$

for any $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_1^1(M)$, where R is the curvature tensor field of the connection ∇ defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ and $(\tilde{\gamma} - \gamma) R(X, Y) = \begin{pmatrix} 0 \\ t_m^i R_{klj}^m X^k Y^l - t_j^m R_{klm}^i X^k Y^l \end{pmatrix}$ (for details, see [3, 28, 30]).

2.2. Expressions in the adapted frame. We insert the adapted frame which allows the tensor calculus to be efficiently done in $T_1^1(M)$. With the connection ∇ on M, we can introduce adapted frames on each induced coordinate neighborhood $\pi^{-1}(U)$ of $T_1^1(M)$. In each local chart $U \subset M$, we write $X_{(j)} = \partial_j = \delta_j^h \partial_h \in \Im_0^1(M)$, $A^{(\bar{j})} = \partial_i \otimes dx^j = \delta_i^k \delta_h^j \partial_k \otimes dx^h \in \Im_1^1(M)$, $j = 1, \dots, n$, $\bar{j} = n+1, \dots, n+n^2$. Then from (2.1) and (2.2), we see that these vector fields have respectively local expressions

$$\begin{aligned} {}^H X_{(j)} &= \delta_j^h \partial_h + (-t_h^s \Gamma_{js}^k + t_s^k \Gamma_{jh}^s) \partial_{\bar{h}} \\ {}^V A^{(\bar{j})} &= \delta_i^k \delta_h^j \partial_{\bar{h}} \end{aligned}$$

with respect to the natural frame $\{\partial_h, \partial_{\bar{h}}\}$ in $T_1^1(M)$, where $\partial_h = \dfrac{\partial}{\partial x^h}$, $\partial_{\bar{h}} = \dfrac{\partial}{\partial x^{\bar{h}}}$, $x^{\bar{h}} = t_h^k$ and δ_j^i is the Kronecker's. These $n + n^2$ vector fields are linearly independent and they generate the horizontal distribution of the connection ∇ and the vertical distribution of $T_1^1(M)$, respectively. The set $\{{}^H X_{(j)}, {}^V A^{(\bar{j})}\}$ is called the frame adapted to the connection ∇ in $\pi^{-1}(U) \subset T_1^1(M)$. By denoting

$$\begin{aligned} E_j &= {}^H X_{(j)}, \\ E_{\bar{j}} &= {}^V A^{(\bar{j})}, \end{aligned} \tag{2.4}$$

we can write the adapted frame as $\{E_\alpha\} = \{E_j, E_{\bar{j}}\}$. The indices $\alpha, \beta, \gamma, \dots = 1, \dots, n + n^2$ indicate the indices with respect to the adapted frame.

Using (2.1), (2.2) and (2.4), we have

$$ {}^V A = \begin{pmatrix} 0 \\ A_j^i \end{pmatrix}, \tag{2.5}$$

and

$$ {}^H X = \begin{pmatrix} X^j \\ 0 \end{pmatrix} \tag{2.6}$$

with respect to the adapted frame $\{E_\alpha\}$ (for details, see [28]). By the straightforward calculations, we have the lemma below.

LEMMA 1. *The Lie brackets of the adapted frame of* $T_1^1(M)$ *satisfy the following identities*:

$$[E_l, E_j] = (t_s^v R_{ljr}^s - t_r^s R_{ljs}^v) E_{\bar{r}},$$
$$[E_l, E_{\bar{j}}] = (\delta_r^j \Gamma_{li}^v - \delta_i^v \Gamma_{lr}^j) E_{\bar{r}},$$
$$[E_{\bar{l}}, E_{\bar{j}}] = 0,$$

where R_{ijl}^s *denote the components of the curvature tensor of the connection* ∇ *on* M.

3. Almost product Riemannian structures on the $(1,1)$-tensor bundle

Let $T_1^1(M)$ be the $(1,1)$-tensor bundle over a Riemannian manifold (M,g). For each $P \in M$, the extension of scalar product g (marked by G) is defined on the tensor space $\pi^{-1}(P) = T_1^1(P)$ by $G(A,B) = g_{it} g^{jl} A_j^i B_l^t$ for all $A, B \in T_1^1(P)$. The rescaled Sasaki type metric ${}^S g_f$ is defined on $T_1^1(M)$ by the following three equations

$$ {}^S g_f({}^V A, {}^V B) = {}^V(G(A,B)), \tag{3.1}$$

$$ {}^S g_f({}^V A, {}^H Y) = 0, \tag{3.2}$$

$$ {}^S g_f({}^H X, {}^H Y) = {}^V(f g(X,Y)) \tag{3.3}$$

for any $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_1^1(M)$, where $f > 0$, $f \in C^\infty(M)$ (for $f = 1$, see [28]). From the equations (3.1)–(3.3), by virtue of (2.5) and (2.6), the rescaled Sasaki type metric ${}^S g_f$ and its inverse have components with respect to the adapted frame $\{E_\alpha\}$:

$$ ({}^S g_f)_{\beta\gamma} = \begin{pmatrix} ({}^S g_f)_{jl} & ({}^S g_f)_{j\bar{l}} \\ ({}^S g_f)_{\bar{j}l} & ({}^S g_f)_{\bar{j}\bar{l}} \end{pmatrix} = \begin{pmatrix} f g_{jl} & 0 \\ 0 & g_{it} g^{jl} \end{pmatrix}, \quad x^{\bar{l}} = t_l^t \tag{3.4}$$

and

$$ ({}^S g_f)^{\beta\gamma} = \begin{pmatrix} ({}^S g_f)^{jl} & ({}^S g_f)^{j\bar{l}} \\ ({}^S g_f)^{\bar{j}l} & ({}^S g_f)^{\bar{j}\bar{l}} \end{pmatrix} = \begin{pmatrix} \frac{1}{f} g^{jl} & 0 \\ 0 & g^{it} g_{jl} \end{pmatrix}, \quad x^{\bar{j}} = t_j^i. \tag{3.5}$$

For the Levi-Civita connection of the rescaled Sasaki type metric ${}^S g_f$ we give the next theorem.

THEOREM 1. *Let* (M,g) *be a Riemannian manifold and equip its tensor bundle* $T_1^1(M)$ *with the rescaled Sasaki type metric* ${}^S g_f$. *Then the corresponding Levi-Civita connection* $\tilde{\nabla}$ *satisfies the followings*:

$$(3.6)\quad \begin{cases} \text{i) } \tilde{\nabla}_{E_l}E_j = \left\{\Gamma^r_{lj} + \frac{1}{2f}{}^fA^r_{lj}\right\}E_r + \left\{\frac{1}{2}R^s_{ljr}t^v_s - \frac{1}{2}R^v_{ljs}t^s_r\right\}E_{\bar{r}}, \\ \text{ii) } \tilde{\nabla}_{E_l}E_{\bar{j}} = \left\{\frac{1}{2f}g_{ia}R^{s\ j\ r}_{\ .\ .\ l}t^a_s - \frac{1}{2f}g^{jb}R^r_{isl}t^s_b\right\}E_r + \{\Gamma^v_{li}\delta^j_r - \Gamma^j_{lr}\delta^v_i\}E_{\bar{r}}, \\ \text{iii) } \tilde{\nabla}_{E_{\bar{l}}}E_j = \left\{\frac{1}{2f}g_{ta}R^{s\ l\ r}_{\ .\ .\ j}t^a_s - \frac{1}{2f}g^{lb}R^r_{tsj}t^s_b\right\}E_r, \\ \text{iv) } \tilde{\nabla}_{E_{\bar{l}}}E_{\bar{j}} = 0 \end{cases}$$

with respect to the adapted frame, where ${}^fA^h_{ji}$ *is a tensor field of type* $(1,2)$ *defined by* ${}^fA^h_{ji} = (f_j\delta^h_i + f_i\delta^h_j - f^h_{.}g_{ji})$ *and* $f_i = \partial_i f$, $R^{s\ j\ r}_{\ .\ .\ l} = g^{as}g^{bj}R^r_{abl}$.

Proof. The connection $\tilde{\nabla}$ is characterized by the Koszul formula:

$$2^Sg_f(\tilde{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}) = \tilde{X}(^Sg_f(\tilde{Y},\tilde{Z})) + \tilde{Y}(^Sg_f(\tilde{Z},\tilde{X})) - \tilde{Z}(^Sg_f(\tilde{X},\tilde{Y}))$$
$$-{}^Sg_f(\tilde{X},[\tilde{Y},\tilde{Z}]) + {}^Sg_f(\tilde{Y},[\tilde{Z},\tilde{X}]) + {}^Sg_f(\tilde{Z},[\tilde{X},\tilde{Y}])$$

for all vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ on $T^1_1(M)$. One can verify the Koszul formula for pairs $\tilde{X} = E_l, E_{\bar{l}}$ and $\tilde{Y} = E_j, E_{\bar{j}}$ and $\tilde{Z} = E_k, E_{\bar{k}}$. In calculations, the formulas (2.4), Lemma 1 and the first Bianchi identity for R should be applied. We omit standart calculations. □

Let $\tilde{X}, \tilde{Y} \in \Im^1_0(T^1_1(M))$. Then the covariant derivative $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ has components

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y}^\alpha = \tilde{X}^\gamma E_\gamma \tilde{Y}^\alpha + \tilde{\Gamma}^\alpha_{\gamma\beta}\tilde{X}^\beta\tilde{Y}^\gamma$$

with respect to the adapted frame $\{E_\alpha\}$. Using (2.4), (2.5), (2.6) and (3.6), we have the following proposition.

Proposition 1. *Let* (M,g) *be a Riemannian manifold and* $\tilde{\nabla}$ *be the Levi-Civita connection of the tensor bundle* $T^1_1(M)$ *equipped with the rescaled Sasaki type metric* ^Sg_f. *Then the corresponding Levi-Civita connection satisfies the following relations*:

i) $\tilde{\nabla}_{^HX}{}^HY = {}^H\left(\nabla_XY + \frac{1}{2f}{}^fA(X,Y)\right) + \frac{1}{2}(\tilde{\gamma} - \gamma)R(X,Y)$,

ii) $\tilde{\nabla}_{^HX}{}^VB = \frac{1}{2f}{}^H(g^{bj}R(t_b,B_j)X + g_{ai}(t^a(g^{-1}\circ R(\quad,X)\tilde{B}^i) + {}^V(\nabla_XB)$,

iii) $\tilde{\nabla}_{^VC}{}^HY = \frac{1}{2f}{}^H(g^{bl}R(t_b,C_l)Y + g_{at}(t^a(g^{-1}\circ R(\quad,Y)\tilde{C}^t))$,

iv) $\tilde{\nabla}_{^VC}{}^VB = 0$

for all $X, Y \in \Im^1_0(M)$ *and* $B, C \in \Im^1_1(M)$, *where* $C_l = (C^i_l)$, $\tilde{C}^t = (g^{bl}C^t_l) = (C^{bt}_{\ .})$, $t_l = (t^a_l)$, $t^a = (t^a_b)$, $R(\quad,X)Y \in \Im^1_1(M)$, $g^{-1}\circ R(\quad,X)Y \in \Im^1_0(M)$ *and* ${}^fA(X,Y) = X(f)Y + Y(f)X - g(X,Y)\circ(df)^*$ *(for* $f = 1$, *see* [28]).

An almost product Riemannian manifold (M, F, g) is an n-dimensional differentiable manifold M endowed with a positive definite Riemannian metric g and a non-trivial tensor field F of type $(1,1)$ such that

$$F^2 = I$$

and

$$g(FX, Y) = g(X, FY) \tag{3.7}$$

for all $X, Y \in \Im_0^1(M)$. Such a metric also is referred as pure metric with respect to F. A locally decomposable Riemannian manifold can be defined as a triple (M, F, g) which consist of a differentiable manifold M equipped with an almost product structure F and a pure metric g such that $\nabla F = 0$, where ∇ is the Levi-Civita connection of g. It is well known that $\nabla F = 0$ is equivalent to decomposability of the pure metric g [31], i.e. $\phi_F g = 0$, where ϕ_F is the Tachibana operator [34, 36]: $(\phi_F g)(X, Y, Z) = (FX)(g(Y,Z)) - X(g(FY,Z)) + g((L_Y F)X, Z) + g(Y, (L_Z F)X)$.

Let us define an almost product structure on $T_1^1(M)$ as follows:

$$J({}^H X) = -{}^H X \tag{3.8}$$
$$J({}^V A) = {}^V A$$

for any $X \in \Im_0^1(M)$ and $A \in \Im_1^1(M)$. One can easily check that the rescaled Sasaki type metric ${}^S g_f$ is pure with respect to the almost product structure J. Hence we state the following theorem.

Theorem 2. *Let (M, g) be a Riemannian manifold and $T_1^1(M)$ be its tensor bundle equipped with the rescaled Sasaki type metric ${}^S g_f$ and the almost product structure J. The triple $(T_1^1(M), J, {}^S g_f)$ is an almost product Riemannian manifold.*

We now give conditions for the rescaled Sasaki type metric ${}^S g_f$ to be decomposable with respect to the almost product structure J. Using the definition of the rescaled Sasaki type metric ${}^S g_f$ and the almost product structure J and by using the fact that ${}^V A\,{}^V(G(B,C)) = 0$, ${}^V A\,{}^V(fg(Y,Z)) = 0$ and ${}^H X\,{}^V(fg(Y,Z)) = {}^V(X(fg(Y,Z)))$ we calculate

$$(\phi_J {}^S g_f)(\tilde{X}, \tilde{Y}, \tilde{Z}) = (J\tilde{X})({}^S g_f(\tilde{Y}, \tilde{Z})) - \tilde{X}({}^S g_f(J\tilde{Y}, \tilde{Z})) + {}^S g_f((L_{\tilde{Y}} J)\tilde{X}, \tilde{Z}) + {}^S g_f(\tilde{Y}, (L_{\tilde{Z}} J)\tilde{X})$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Im_0^1(T_1^1(M))$. For pairs $\tilde{X} = {}^H X, {}^V A$, $\tilde{Y} = {}^H Y, {}^V B$ and $\tilde{Z} = {}^H Z, {}^V C$, then we get

$$(\phi_J {}^S g_f)({}^H X, {}^V B, {}^H Z) = 2\,{}^S g_f({}^V B, (\tilde{\gamma} - \gamma) R(X, Z)) \tag{3.9}$$
$$(\phi_J {}^S g_f)({}^H X, {}^H Y, {}^V C) = 2\,{}^S g_f((\tilde{\gamma} - \gamma) R(X, Y), {}^V C).$$

for all $X, Y, Z \in \Im_0^1(M)$ and $A, B, C \in \Im_1^1(M)$, and the others are zero. Since $\phi_J {}^S g_f = 0$ is equivalent to $\tilde{\nabla} J = 0$, we have the following theorem.

Theorem 3. *Let (M, g) be a Riemannian manifold and let $T_1^1(M)$ be its tensor bundle equipped with the rescaled Sasaki type metric ${}^S g_f$ and the almost product structure J. The triple $(T_1^1(M), J, {}^S g_f)$ is a locally decomposable Riemannian manifold if and only if M is flat.*

Remark 1. Let (M, g) be a Riemannian manifold and let $T_1^1(M)$ be its tensor bundle equipped with the rescaled Sasaki type metric ${}^S g_f$. The diagonal lift ${}^D\gamma$ of $\gamma \in \Im_1^1(M)$ to $T_1^1(M)$ is defined by the formulas

$$ {}^D\gamma {}^H X = {}^H(\gamma(X)) $$

$$ {}^D\gamma {}^V A = -{}^V(\gamma(A)) $$

for any $X \in \Im_0^1(M)$ and $A \in \Im_1^1(M)$ [13]. The diagonal lift ${}^D I$ of the identity tensor field $I \in \Im_1^1(M)$ has the following properties

$$ {}^D I {}^H X = {}^H X $$

$$ {}^D I {}^V A = -{}^V A $$

and satisfies $({}^D I)^2 = I_{T_1^1(M)}$. Thus, ${}^D I$ is an almost product structure. Also, the rescaled Sasaki type metric ${}^S g_f$ is pure with respect to ${}^D I$, i.e. the triple $(T_1^1(M), {}^D I, {}^S g_f)$ is an almost product Riemannian manifold. Finally, by using ϕ-operator, we can say that the rescaled Sasaki type metric ${}^S g_f$ is decomposable with respect to ${}^D I$ if and only if M is flat.

Now we shall give some applications related to almost product Riemannian structures on the $(1, 1)$-tensor bundle.

3.1. Let us consider the almost product structure J defined by (3.8) and the Levi-Civita connection $\tilde{\nabla}$ given by Proposition 1. We define a $(1, 2)$ tensor field on $T_1^1(M)$ by

$$ \tilde{S}(\tilde{X}, \tilde{Y}) = \frac{1}{2}\{(\tilde{\nabla}_{J\tilde{Y}} J)\tilde{X} + J((\tilde{\nabla}_{\tilde{Y}} J)\tilde{X}) - J((\tilde{\nabla}_{\tilde{X}} J)\tilde{Y})\} $$

for all $\tilde{X}, \tilde{Y} \in \Im_0^1(T_1^1(M))$. Then the linear connection

$$ {}^{(P)}\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \tilde{\nabla}_{\tilde{X}} \tilde{Y} - \tilde{S}(\tilde{X}, \tilde{Y}) $$

is an almost product connection on $T_1^1(M)$ (for almost product connection, see [22]).

Theorem 4. *Let (M, g) be a Riemannian manifold and let $T_1^1(M)$ be its tensor bundle equipped with the rescaled Sasaki type metric ${}^S g_f$ and the almost product structure J. Then the almost product connection ${}^{(P)}\tilde{\nabla}$ constructed by the Levi-Civita connection $\tilde{\nabla}$ of ${}^S g_f$ and the almost product structure J is as follows*:

$$(3.10)\quad \begin{cases} \text{i) } {}^{(P)}\tilde{\nabla}_{{}^{H}X}{}^{H}Y = {}^{H}\left(\nabla_X Y + \frac{1}{2f}{}^{f}A(X,Y)\right), \\ \text{ii) } {}^{(P)}\tilde{\nabla}_{{}^{H}X}{}^{V}B = {}^{V}(\nabla_X B), \\ \text{iii) } {}^{(P)}\tilde{\nabla}_{{}^{V}C}{}^{H}Y = \frac{3}{2f}{}^{H}(g^{bl}R(t_b, C_l)Y + g_{at}(t^a(g^{-1}\circ R(\quad, Y)\tilde{C}^t)), \\ \text{iv) } {}^{(P)}\tilde{\nabla}_{{}^{V}C}{}^{V}B = 0. \end{cases}$$

Similarly, by means of the almost product structure ${}^{D}I$ and the Levi-Civita connection $\tilde{\nabla}$ of the rescaled Sasaki type metric ${}^{S}g_f$, another almost product connection can be constructed. Denoting by ${}^{(P)\tilde{\nabla}}T$, the torsion tensor of ${}^{(P)}\tilde{\nabla}$, we have from (2.3) and (3.10)

$$\begin{aligned} &{}^{(P)\tilde{\nabla}}T({}^{V}C, {}^{V}B) = 0, \\ &{}^{(P)\tilde{\nabla}}T({}^{V}C, {}^{H}Y) = \frac{3}{2f}{}^{H}(g^{bl}R(t_b, C_l)Y + g_{at}(t^a(g^{-1}\circ R(\quad, Y)\tilde{C}^t)), \\ &{}^{(P)\tilde{\nabla}}T({}^{H}X, {}^{H}Y) = -(\tilde{\gamma} - \gamma)R(X, Y). \end{aligned}$$

Hence we have the theorem below.

THEOREM 5. *Let (M, g) be a Riemannian manifold and let $T_1^1(M)$ be its tensor bundle. The almost product connection ${}^{(P)}\tilde{\nabla}$ constructed by the Levi-Civita connection $\tilde{\nabla}$ of the rescaled Sasaki type metric ${}^{S}g_f$ and the almost product structure J is symmetric if and only if M is flat.*

Note that if there exists a symmetric almost product connection on M, then the almost product structure is integrable [22]. The converse is also true [10]. It is known that the integrability of an almost product structure F is equivalent to the vanishing of the Nijenhuis tensor N_F given by

$$N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + [X, Y].$$

In view of Theorem 5, we can say that the almost product structure J is integrable if and only if M is flat.

3.2. O. Gil-Medrano and A. M. Naveira proved that both distributions of the almost product structure on the almost product Riemannian manifold (M, F, g) are totally geodesic if and only if $\sigma_{X,Y,Z}\,g((\nabla_X F)Y, Z) = 0$ for any $X, Y, Z \in \Im_0^1(M)$, where σ is the cyclic sum by three arguments [14]. In [29], the authors proved that $\sigma_{X,Y,Z}\,g((\nabla_X F)Y, Z) = 0$ is equivalent to $(\phi_F g)(X, Y, Z) + (\phi_F g)(Y, Z, X) + (\phi_F g)(Z, X, Y) = 0$. We compute

$$A(\tilde{X}, \tilde{Y}, \tilde{Z}) = (\phi_J{}^{S}g_f)(\tilde{X}, \tilde{Y}, \tilde{Z}) + (\phi_J{}^{S}g_f)(\tilde{Y}, \tilde{Z}, \tilde{X}) + (\phi_J{}^{S}g_f)(\tilde{Z}, \tilde{X}, \tilde{Y})$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Im_0^1(T_1^1(M))$. By means of (3.9), we have $A(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$ for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Im_0^1(T_1^1(M))$. Hence we state the following theorem.

Theorem 6. *Let (M, g) be a Riemannian manifold and $T_1^1(M)$ be its tensor bundle equipped with the rescaled Sasaki type metric ${}^S g_f$ and the almost product structure J defined by* (3.8). *Both distributions of the almost product Riemannian manifold $(T_1^1(M), J, {}^S g_f)$ are totally geodesic.*

3.3. The famous golden section $\eta = \dfrac{1+\sqrt{5}}{2} \approx 1,61803398874989\ldots$ being the root of the equation $x^2 - x - 1 = 0$ is an irrational number which has many applications in mathematics, computational science, biology, art, architecture, nature, etc. In the last few years, the golden proportion has played an increasing role in modern physical research and it has a unique significant role in atomic physics [20]. The golden proportion has also interesting properties in topology of four-manifolds, in conformal field theory, in mathematical probability theory and in Cantorian spacetime [23, 24]. Inspired by golden ratio, a new structure on a Riemannian manifold was constructed by M. Crasmareanu and C. Hretcanu [4, 18, 19]. Also, they called this structure the golden structure. Let ψ be a $(1,1)$ tensor field on a manifold M. If the polynomial $X^2 - X - 1$ is the minimal polynomial for a structure ψ satisfying $\psi^2 - \psi - I = 0$, then ψ is a golden structure on M and (M, ψ) is a golden manifold. Let (M, g) be a Riemannian manifold endowed with the golden structure ψ such that

$$g(\psi X, Y) = g(X, \psi Y),$$

for all $X, Y \in \Im_0^1(M)$. The triple (M, ψ, g) is named a golden Riemannian manifold.

If ψ is a golden structure on M, then

$$F = \frac{1}{\sqrt{5}}(2\psi - I) \tag{3.11}$$

is an almost product structure on M. Conversely,

$$\psi = \frac{1}{2}(I + \sqrt{5}F) \tag{3.12}$$

is a golden structure on M. If a Riemannian metric g is pure with respect to an almost product structure F, then the Riemannian metric g is pure with respect to the corresponding golden structure ψ. A simple computation, using the expression of the corresponding almost product structure via (3.11) gives:

$$\phi_F g = \frac{2}{\sqrt{5}} \phi_\psi g. \tag{3.13}$$

Let (M, ψ, g) be a golden Riemannian manifold and F its corresponding almost product structure. In [12], the first author and collaborators have proved that 1) The golden structure ψ is integrable if $\phi_\psi g = 0$ (or equivalently $\phi_F g = 0$)

and 2) The manifold M is a locally decomposable golden Riemannian manifold if and only if $\phi_F g = 0$ (or equivalently $\phi_\psi g = 0$).

By means of the almost product structure J, from (3.12) we can construct a golden structure on $T_1^1(M)$ defined by the formulas

$$\begin{cases} \tilde{\psi}({}^H X) = \left(\dfrac{1-\sqrt{5}}{2}\right) {}^H X, \\ \tilde{\psi}({}^V A) = \left(\dfrac{1+\sqrt{5}}{2}\right) {}^V A. \end{cases} \tag{3.14}$$

for any $X \in \Im_0^1(M)$ and $A \in \Im_1^1(M)$. Also the following hold

$${}^S g_f(\tilde{\psi}\tilde{X}, \tilde{Y}) = {}^S g_f(\tilde{X}, \tilde{\psi}\tilde{Y})$$

for any $\tilde{X}, \tilde{Y} \in \Im_0^1(T_1^1(M))$, i.e. ${}^S g_f$ is pure with respect to $\tilde{\psi}$. In view of Theorem 3, by (3.13), we have the following result.

Corollary 1. *Let (M, g) be a Riemannian manifold and let $T_1^1(M)$ be its tensor bundle equipped with the rescaled Sasaki type metric ${}^S g_f$ and the golden structure $\tilde{\psi}$ associated with the almost product structure J. The triple $(T_1^1(M), \tilde{\psi}, {}^S g_f)$ is a locally decomposable golden Riemannian manifold if and only if M is flat.*

Remark 2. Another golden structure associated with the almost product structure ${}^D I$ is as follows:

$$\begin{cases} \bar{\psi}({}^H X) = \left(\dfrac{1+\sqrt{5}}{2}\right) {}^H X, \\ \bar{\psi}({}^V A) = \left(\dfrac{1-\sqrt{5}}{2}\right) {}^V A. \end{cases}$$

Similarly, we say that the triple $(T_1^1(M), \bar{\psi}, {}^S g_f)$ is a locally decomposable golden Riemannian manifold if and only if M is flat.

4. Curvature properties of the rescaled Sasaki type metric ${}^S g_f$ on the $(1,1)$-tensor bundle

The Riemannian curvature tensor $\tilde{R}$ of $T_1^1(M)$ with the rescaled Sasaki type metric ${}^S g_f$ is obtained from the well-known formula

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]}\tilde{Z}$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Im_0^1(T_1^1(M))$. Then from Lemma 1 and Theorem 1, we get the following proposition.

PROPOSITION 2. *The components of the curvature tensor $\tilde{R}$ of the tensor bundle $T_1^1(M)$ with the rescaled Sasaki type metric ${}^S g_f$ are given as follows*:

$$
\begin{aligned}
(4.1)\quad \tilde{R}(E_m, E_l)E_j = \Big\{ & R_{mlj}^r + \frac{1}{4f}(g_{ka}R^{s\,h\,r}_{\cdot\,\cdot\,m}R^p_{ljh} - g_{ka}R^{s\,h\,r}_{\cdot\,\cdot\,l}R^p_{mjh} - 2g_{ka}R^{s\,h\,r}_{\cdot\,\cdot\,j}R^p_{mlh})t_s^a t_p^k \\
& + \frac{1}{4f}(g_{ka}R^{s\,h\,r}_{\cdot\,\cdot\,l}R^k_{mjp} - g_{ka}R^{s\,h\,r}_{\cdot\,\cdot\,m}R^k_{ljp} + 2g_{ka}R^{s\,h\,r}_{\cdot\,\cdot\,j}R^k_{mlp})t_s^a t_h^p \\
& + \frac{1}{4f}(g^{hb}R^r_{kpl}R^s_{mjh} - g^{hb}R^r_{kpm}R^s_{ljh} + 2g^{hb}R^r_{kpj}R^s_{mlh})t_b^p t_s^k \\
& + \frac{1}{4f}(g^{hb}R^r_{ksm}R^k_{ljp} - g^{hb}R^r_{ksl}R^k_{mjp} - 2g^{hb}R^r_{ksj}R^k_{mlp})t_b^s t_h^p \\
& + \nabla_m\left(\frac{1}{2f}A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f}A^r_{mj}\right) + \frac{1}{4f^2}A^r_{ms}A^s_{lj} - \frac{1}{4f^2}A^r_{ls}A^s_{mj}\Big\} E_r \\
+ \Big\{ & \frac{1}{2}(\nabla_m R^s_{ljr} - \nabla_l R^s_{mjr})t_s^v + \frac{1}{2}(\nabla_l R^v_{mjs} - \nabla_m R^v_{ljs})t_r^s \\
& + \frac{1}{4f}((R^s_{mhr}t_s^v - R^v_{mhs}t_r^s)A^h_{lj} - (R^s_{lhr}t_s^v - R^v_{lhs}t_r^s)A^h_{mj})\Big\} E_{\bar{r}},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}(E_{\bar{m}}, E_l)E_j = \Big\{ & -\frac{1}{2f}g_{na}(\nabla_l R^{s\,m\,r}_{\cdot\,\cdot\,j})t_s^a + \frac{1}{2f}g^{mb}(\nabla_l R^r_{nsj})t_b^s \\
& + \frac{1}{4f^2}(g_{na}R^{s\,m\,r}_{\cdot\,\cdot\,h}A^h_{lj}t_s^a - g^{mb}R^r_{nsh}A^h_{lj}t_b^s + g^{mb}R^h_{nsj}A^r_{lh}t_b^s \\
& - g_{na}R^{s\,m\,h}_{\cdot\,\cdot\,j}A^r_{lh}t_s^a + 2f_l g_{na}R^{s\,m\,r}_{\cdot\,\cdot\,j}t_s^a - 2f_l g^{mb}R^r_{nsj}t_b^s)\Big\} E_r \\
+ \Big\{ & \frac{1}{2}R^m_{ljr}\delta^v_n - \frac{1}{2}R^v_{ljn}\delta^m_r - \frac{1}{4f}(R^s_{lhr}g_{na}R^{p\,m\,h}_{\cdot\,\cdot\,j})t_s^v t_p^a \\
& + \frac{1}{4f}(R^s_{lhr}g^{mb}R^h_{npj})t_s^v t_b^p + \frac{1}{4f}(R^v_{lhs}g_{na}R^{p\,m\,h}_{\cdot\,\cdot\,j})t_r^s t_p^a \\
& - \frac{1}{4f}(R^v_{lhs}g^{mb}R^h_{npj})t_r^s t_b^p\Big\} E_{\bar{r}},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}(E_m, E_{\bar{l}})E_j = \Big\{ & \frac{1}{2f}g_{ta}(\nabla_m R^{s\,l\,r}_{\cdot\,\cdot\,j})t_s^a - \frac{1}{2f}g^{lb}(\nabla_m R^r_{tsj})t_b^s \\
& + \frac{1}{4f^2}(g_{ta}R^{s\,l\,h}_{\cdot\,\cdot\,j}A^r_{mh}t_s^a - g^{lb}R^h_{tsj}A^r_{mh}t_b^s + g^{lb}R^r_{tsh}A^h_{mj}t_b^s \\
& - g_{ta}R^{s\,l\,r}_{\cdot\,\cdot\,h}A^h_{mj}t_s^a - 2f_m g_{ta}R^{s\,l\,h}_{\cdot\,\cdot\,j}t_s^a + 2f_m g^{lb}R^r_{tsj}t_b^s)\Big\} E_r
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\{-\frac{1}{2}R^{l}_{mjr}\delta^{v}_{t}+\frac{1}{2}R^{v}_{mjt}\delta^{l}_{r}+\frac{1}{4f}(R^{s}_{mhr}g_{ta}R^{p\,l\,h}_{\cdot\,\cdot\,j})t^{v}_{s}t^{a}_{p}\\
&\quad-\frac{1}{4f}(R^{s}_{mhr}g^{lb}R^{h}_{tpj})t^{v}_{s}t^{p}_{b}-\frac{1}{4f}(R^{v}_{mhp}g_{ta}R^{s\,l\,h}_{\cdot\,\cdot\,j})t^{p}_{r}t^{a}_{s}\\
&\quad+\frac{1}{4f}(R^{v}_{mhs}g^{lb}R^{h}_{tpj})t^{s}_{r}t^{p}_{b}\Big\}E_{\bar r},
\end{aligned}
$$

$$
\begin{aligned}
\tilde R(E_{\bar m},E_{\bar l})E_{j}=&\Big\{\frac{1}{f}g_{tn}R^{m\,l\,r}_{\cdot\,\cdot\,j}-\frac{1}{f}g^{lm}R^{r}_{tnj}\\
&+\frac{1}{4f^{2}}(g_{na}R^{s\,m\,r}_{\cdot\,\cdot\,h}g_{tb}R^{p\,l\,h}_{\cdot\,\cdot\,j}-g_{ta}R^{s\,l\,r}_{\cdot\,\cdot\,h}g_{nb}R^{p\,m\,h}_{\cdot\,\cdot\,j})t^{a}_{s}t^{b}_{p}\\
&+\frac{1}{4f^{2}}(g_{ta}R^{s\,l\,r}_{\cdot\,\cdot\,h}g^{mb}R^{h}_{npj}-g_{na}R^{s\,m\,r}_{\cdot\,\cdot\,h}g^{lb}R^{h}_{tpj})t^{a}_{s}t^{p}_{b}\\
&+\frac{1}{4f^{2}}(g^{lb}R^{r}_{tph}g_{na}R^{s\,m\,h}_{\cdot\,\cdot\,j}-g^{mb}R^{r}_{nph}g_{ta}R^{s\,l\,h}_{\cdot\,\cdot\,j})t^{p}_{b}t^{a}_{s}\\
&+\frac{1}{4f^{2}}(g^{ma}R^{r}_{nsh}g^{lb}R^{h}_{tsj}-g^{la}R^{r}_{tsh}g^{mb}R^{h}_{npj})t^{s}_{a}t^{p}_{b}\Big\}E_{r},
\end{aligned}
$$

$$
\begin{aligned}
\tilde R(E_{m},E_{l})E_{\bar j}=&\Big\{\frac{1}{2f}g_{ia}(\nabla_{m}R^{s\,j\,r}_{\cdot\,\cdot\,l}-\nabla_{l}R^{s\,j\,r}_{\cdot\,\cdot\,m})t^{a}_{s}+\frac{1}{2f}g^{jb}(\nabla_{l}R^{r}_{ism}-\nabla_{m}R^{r}_{isl})t^{s}_{b}\\
&+\frac{1}{4f^{2}}(g_{ia}R^{s\,j\,h}_{\cdot\,\cdot\,l}A^{r}_{mh}t^{a}_{s}-g^{jb}R^{h}_{isl}A^{r}_{mh}t^{s}_{b}+g^{jb}R^{h}_{ism}A^{r}_{lh}t^{s}_{b}-g_{ia}R^{s\,j\,h}_{\cdot\,\cdot\,m}A^{r}_{lh}t^{a}_{s}\\
&-2(f_{m}g_{ia}R^{s\,j\,h}_{\cdot\,\cdot\,l}t^{a}_{s}+f_{m}g^{jb}R^{r}_{isl}t^{s}_{b}+f_{l}g_{ia}R^{s\,j\,r}_{\cdot\,\cdot\,m}t^{a}_{s}-f_{l}g^{jb}R^{r}_{ism}t^{s}_{b})\Big\}E_{r}\\
&+\Big\{R^{v}_{mli}\delta^{j}_{r}-R^{j}_{mlr}\delta^{v}_{i}+\frac{1}{4f}(R^{s}_{mhr}g_{ia}R^{p\,j\,h}_{\cdot\,\cdot\,l}-R^{s}_{lhr}g_{ia}R^{p\,j\,h}_{\cdot\,\cdot\,m})t^{v}_{s}t^{a}_{p}\\
&\quad+\frac{1}{4f}(R^{s}_{lhr}g^{jb}R^{h}_{ipm}-R^{s}_{mhr}g^{jb}R^{h}_{ipl})t^{v}_{s}t^{p}_{b}\\
&\quad+\frac{1}{4f}(R^{v}_{lhp}g_{ia}R^{s\,j\,h}_{\cdot\,\cdot\,m}-R^{v}_{mhp}g_{ia}R^{s\,j\,h}_{\cdot\,\cdot\,l})t^{p}_{r}t^{a}_{s}\\
&\quad+\frac{1}{4f}(R^{v}_{mhs}g^{jb}R^{h}_{ipl}-R^{v}_{lhs}g^{jb}R^{h}_{ipm})t^{s}_{r}t^{p}_{b}\Big\}E_{\bar r},
\end{aligned}
$$

$$
\begin{aligned}
\tilde R(E_{m},E_{\bar l})E_{\bar j}=&\Big\{-\frac{1}{2f}g_{it}R^{l\,j\,r}_{\cdot\,\cdot\,m}+\frac{1}{2f}g^{jl}R^{r}_{itm}-\frac{1}{4f^{2}}(g_{ta}R^{s\,l\,r}_{\cdot\,\cdot\,h}g_{ib}R^{p\,j\,h}_{\cdot\,\cdot\,m})t^{a}_{s}t^{b}_{p}\\
&+\frac{1}{4f^{2}}(g_{ta}R^{s\,l\,r}_{\cdot\,\cdot\,h}g^{jb}R^{h}_{ipm})t^{a}_{s}t^{p}_{b}+\frac{1}{4f^{2}}(g^{lb}R^{r}_{tph}g_{ia}R^{s\,j\,h}_{\cdot\,\cdot\,m})t^{p}_{b}t^{a}_{s}\\
&-\frac{1}{4f^{2}}(g^{la}R^{r}_{tsh}g^{jb}R^{h}_{ipm})t^{s}_{a}t^{p}_{b}\Big\}E_{r},
\end{aligned}
$$

$$\tilde{R}(E_{\bar{m}}, E_l)E_{\bar{j}} = \left\{ \frac{1}{2f} g_{in} R^{m\ j\ r}_{.\ .\ l} - \frac{1}{2f} g^{jm} R^r_{inl} + \frac{1}{4f^2} (g_{na} R^{s\ m\ r}_{.\ .\ h} g_{ib} R^{p\ j\ h}_{.\ .\ l}) t^a_s t^b_p \right.$$

$$- \frac{1}{4f^2} (g_{na} R^{s\ m\ r}_{.\ .\ h} g^{jb} R^h_{ipl}) t^a_s t^p_b - \frac{1}{4f^2} (g^{mb} R^r_{nph} g_{ia} R^{s\ j\ h}_{.\ .\ m}) t^p_b t^a_s$$

$$\left. + \frac{1}{4f^2} (g^{ma} R^r_{nsh} g^{jb} R^h_{ipl}) t^s_a t^p_b \right\} E_r$$

$$\tilde{R}(E_{\bar{m}}, E_{\bar{l}})E_{\bar{j}} = 0$$

with respect to the adapted frame $\{E_\alpha\}$ (*for* $f = 1$, *see* [28]).

We now compare the geometries of the Riemannian manifold (M, g) and its tensor bundle $T^1_1(M)$ with the rescaled Sasaki type metric ${}^S g_f$.

THEOREM 7. *Let* (M, g) *be a Riemannian manifold and* $T^1_1(M)$ *be its tensor bundle with the rescaled Sasaki type metric* ${}^S g_f$. *Then* $T^1_1(M)$ *is locally flat if and only if M is locally flat and* $\nabla_m\left(\frac{1}{2f} A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f} A^r_{mj}\right) + \frac{1}{4f^2} A^r_{ms} A^s_{lj} - \frac{1}{4f^2} A^r_{ls} A^s_{mj} = 0$.

Proof. If

$$\nabla_m\left(\frac{1}{2f} A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f} A^r_{mj}\right) + \frac{1}{4f^2} A^r_{ms} A^s_{lj} - \frac{1}{4f^2} A^r_{ls} A^s_{mj} = 0$$

in the equations (4.1), then $R \equiv 0$ implies $\tilde{R} \equiv 0$. Conversely, if we assume $\tilde{R} \equiv 0$, then from the first equation in (4.1) in the point $(x^i, t^j_i) = (x^i, 0) \in T^1_1(M)$, we get

$$(\tilde{R}(E_m, E_l)E_j)_{(x^i, 0)} = 0$$

$$\left[\left\{ R^r_{mlj} + \frac{1}{4f} (g_{ka} R^{s\ h\ r}_{.\ .\ m} R^p_{ljh} - g_{ka} R^{s\ h\ r}_{.\ .\ l} R^p_{mjh} - 2 g_{ka} R^{s\ h\ r}_{.\ .\ j} R^p_{mlh}) t^a_s t^k_p \right.\right.$$

$$+ \frac{1}{4f} (g_{ka} R^{s\ h\ r}_{.\ .\ l} R^k_{mjp} - g_{ka} R^{s\ h\ r}_{.\ .\ m} R^k_{ljp} + 2 g_{ka} R^{s\ h\ r}_{.\ .\ j} R^k_{mlp}) t^a_s t^p_h$$

$$+ \frac{1}{4f} (g^{hb} R^r_{kpl} R^s_{mjh} - g^{hb} R^r_{kpm} R^s_{ljh} + 2 g^{hb} R^r_{kpj} R^s_{mlh}) t^p_b t^k_s$$

$$+ \frac{1}{4f} (g^{hb} R^r_{ksm} R^k_{ljp} - g^{hb} R^r_{ksl} R^k_{mjp} - 2 g^{hb} R^r_{ksj} R^k_{mlp}) t^s_b t^p_h$$

$$\left. + \nabla_m\left(\frac{1}{2f} A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f} A^r_{mj}\right) + \frac{1}{4f^2} A^r_{ms} A^s_{lj} - \frac{1}{4f^2} A^r_{ls} A^s_{mj} \right\} E_r$$

$$+\left\{\frac{1}{2}(\nabla_m R^s_{ljr} - \nabla_l R^s_{mjr})t^v_s + \frac{1}{2}(\nabla_l R^v_{mjs} - \nabla_m R^v_{ljs})t^s_r \right.$$

$$\left. + \frac{1}{4f}((R^s_{mhr}t^v_s - R^v_{mhs}t^s_r)A^h_{lj} - (R^s_{lhr}t^v_s - R^v_{lhs}t^s_r)A^h_{mj})\right\}E_{\bar{r}}\Big]_{(x^i,0)} = 0$$

$$R^r_{mlj} + \nabla_m\left(\frac{1}{2f}A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f}A^r_{mj}\right) + \frac{1}{4f^2}A^r_{ms}A^s_{lj} - \frac{1}{4f^2}A^r_{ls}A^s_{mj} = 0.$$

From the last equation, we can say that $R \equiv 0$ and $\nabla_m\left(\frac{1}{2f}A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f}A^r_{mj}\right) + \frac{1}{4f^2}A^r_{ms}A^s_{lj} - \frac{1}{4f^2}A^r_{ls}A^s_{mj} = 0$, which completes the proof. □

Corollary 2. *Let (M, g) be a Riemannian manifold and $T^1_1(M)$ be its tensor bundle with the rescaled Sasaki type metric ${}^Sg_f$. Suppose that $f = C(const.)$, then $(T^1_1(M), {}^Sg_f)$ is locally flat if and only if M is locally flat.*

Proof. Let $f = C(const.)$, then ${}^fA^h_{ji} = (f_j\delta^h_i + f_i\delta^h_j - f^h_{.}g_{ji}) = 0$ from which $\nabla_m\left(\frac{1}{2f}A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f}A^r_{mj}\right) + \frac{1}{4f^2}A^r_{ms}A^s_{lj} - \frac{1}{4f^2}A^r_{ls}A^s_{mj} = 0$. In the case, on following the proof of Theorem 7, the result is directly obtained. □

We now turn our attention to the Ricci tensor and scalar curvature of the rescaled Sasaki type metric ${}^Sg_f$. Let $\tilde{R}_{\alpha\beta} = \tilde{R}^\sigma_{\sigma\alpha\beta}$ and $\tilde{r} = ({}^Sg_f)^{\alpha\beta}\tilde{R}_{\alpha\beta}$ denote the Ricci tensor and scalar curvature of the rescaled Sasaki type metric ${}^Sg_f$, respectively. From (4.1), the components of the Ricci tensor $\tilde{R}_{\alpha\beta}$ are characterized by

$$\begin{aligned}
(4.2)\qquad \tilde{R}_{\bar{l}\bar{j}} &= -\frac{1}{4f^2}(g_{ta}R^{s\,l\,r}_{\,.\,.\,h}g_{ib}R^{p\,j\,h}_{\,.\,.\,r})t^a_s t^b_p + \frac{1}{4f^2}(g_{ta}R^{s\,l\,r}_{\,.\,.\,h}g^{jb}R^h_{ipr})t^a_s t^p_b \\
&\quad + \frac{1}{4f^2}(g^{lb}R^r_{tph}g_{ia}R^{s\,j\,h}_{\,.\,.\,r})t^p_b t^a_s - \frac{1}{4f^2}(g^{lb}R^r_{tsh}g^{ja}R^h_{ipr})t^s_b t^p_a, \\
\tilde{R}_{\bar{l}j} &= \frac{1}{2f}g_{ta}(\nabla_r R^{s\,l\,r}_{\,.\,.\,j})t^a_s - \frac{1}{2f}g^{lb}(\nabla_r R^r_{tsj})t^s_b \\
&\quad + \frac{1}{4f^2}(g_{ta}R^{s\,l\,h}_{\,.\,.\,j}A^r_{rh}t^a_s - g^{lb}R^h_{tsj}A^r_{rh}t^s_b + g^{lb}R^r_{tsh}A^h_{rj}t^s_b \\
&\quad - g_{ta}R^{s\,l\,r}_{\,.\,.\,h}A^h_{rj}t^a_s - 2f_r g_{ta}R^{s\,l\,h}_{\,.\,.\,j}t^a_s + 2f_r g^{lb}R^r_{tsj}t^s_b), \\
\tilde{R}_{l\bar{j}} &= \frac{1}{2f}g_{ia}(\nabla_r R^{s\,j\,r}_{\,.\,.\,l})t^a_s - \frac{1}{2f}g^{jb}(\nabla_r R^r_{isl})t^s_b \\
&\quad + \frac{1}{4f^2}(g_{ia}R^{s\,j\,h}_{\,.\,.\,l}A^r_{rh}t^a_s - g^{jb}R^h_{isl}A^r_{rh}t^s_b + g^{jb}R^h_{isr}A^r_{lh}t^s_b \\
&\quad - g_{ia}R^{s\,j\,h}_{\,.\,.\,r}A^r_{lh}t^a_s - 2f_r g_{ia}R^{s\,j\,h}_{\,.\,.\,l}t^a_s + 2f_r g^{jb}R^r_{isl}t^s_b),
\end{aligned}$$

$$\begin{aligned}\tilde{R}_{lj} = R_{lj} &- \frac{1}{4f}(g_{ka}R^{s\,h\,r}_{\cdot\,\cdot\,l}R^{p}_{rjh})t^a_s t^k_p - \frac{1}{2f}(g_{ka}R^{s\,h\,r}_{\cdot\,\cdot\,j}R^{p}_{rlh})t^a_s t^k_p \\ &- \frac{1}{4f}(R^s_{lhr}g_{va}R^{p\,r\,h}_{\cdot\,\cdot\,j})t^v_s t^a_p - \frac{1}{4f}(g^{hb}R^r_{ksl}R^k_{rjp})t^s_b t^p_h \\ &- \frac{1}{2f}(g^{hb}R^r_{ksj}R^k_{rlp})t^s_b t^p_h - \frac{1}{4f}(R^v_{lhs}g^{rb}R^h_{vpj})t^s_r t^p_b \\ &+ \frac{1}{2f}(g_{ka}R^{s\,h\,r}_{\cdot\,\cdot\,j}R^k_{rlp})t^a_s t^p_h + \frac{1}{2f}(g^{hb}R^r_{kpj}R^s_{rlh})t^p_b t^k_s \\ &+ \nabla_r\left(\frac{1}{2f}A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f}A^r_{rj}\right) + \frac{1}{4f^2}A^r_{rs}A^s_{lj} - \frac{1}{4f^2}A^r_{ls}A^s_{rj}.\end{aligned}$$

with respect to the adapted frame. From (3.5) and (4.2), the scalar curvature of the rescaled Sasaki type metric ${}^Sg_f$ is given by

$$\begin{aligned}\tilde{r} = \frac{1}{f}r &- \frac{1}{4f^2}g^{ab}g^{hk}g^{lj}g^{ti}R_{slhv}R_{pjkr}t^s_a t^p_b \\ &- \frac{1}{4f^2}g_{cd}g^{lj}g^{hk}g^{rv}R^s_{rlh}R^p_{vjk}t^c_s t^d_p + \frac{1}{2f^2}R^h_{cpr}R^{r\,b\,s}_{h\,\cdot\,\cdot}t^c_s t^p_b \\ &+ \frac{1}{f}g^{lj}\left(\nabla_r\left(\frac{1}{2f}A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f}A^r_{rj}\right) + \frac{1}{4f^2}A^r_{rs}A^s_{lj} - \frac{1}{4f^2}A^r_{ls}A^s_{rj}\right).\end{aligned}$$

Thus we have the result as follows.

THEOREM 8. *Let (M,g) be a Riemannian manifold and $T^1_1(M)$ be its tensor bundle with the rescaled Sasaki type metric ${}^Sg_f$. Let r be the scalar curvature of g and $\tilde{r}$ be the scalar curvature of ${}^Sg_f$. Then the following equation holds*:

$$\begin{aligned}\tilde{r} = \frac{1}{f}r &- \frac{1}{4f^2}g^{ab}g^{hk}g^{lj}g^{ti}R_{slhv}R_{pjkr}t^s_a t^p_b \\ &- \frac{1}{4f^2}g_{cd}g^{lj}g^{hk}g^{rv}R^s_{rlh}R^p_{vjk}t^c_s t^d_p + \frac{1}{2f^2}R^h_{cpr}R^{r\,b\,s}_{h\,\cdot\,\cdot}t^c_s t^p_b + {}^fL,\end{aligned}$$

where

$${}^fL = \frac{1}{f}g^{lj}\left(\nabla_r\left(\frac{1}{2f}A^r_{lj}\right) - \nabla_l\left(\frac{1}{2f}A^r_{rj}\right) + \frac{1}{4f^2}A^r_{rs}A^s_{lj} - \frac{1}{4f^2}A^r_{ls}A^s_{rj}\right).$$

Let now (M,g), $n > 2$ be a Riemannian manifold of constant curvature κ, i.e.

$$R^s_{kmj} = \kappa(\delta^s_k g_{mj} - \delta^s_m g_{kj})$$

and

$$r = n(n-1)\kappa.$$

Then, from Theorem 8 we have

$$\begin{aligned}
\tilde{r} &= \frac{1}{f} r - \frac{1}{4f^2} g^{ab} g^{hk} g^{vr} g_{lj} R^l_{hvs} R^j_{krp} t^s_a t^p_b \\
&\quad - \frac{1}{4f^2} g_{cd} g^{lj} g^{hk} g^{rv} R^s_{rlh} R^p_{vjk} t^c_s t^d_p + \frac{1}{2f^2} g^{re} g^{bz} R^h_{cpr} R^s_{hez} t^c_s t^p_b + {}^f L \\
&= \frac{1}{f} r - \frac{1}{4f^2} g^{ab} g^{hk} g^{vr} g_{lj} (\kappa(\delta^j_h g_{vs} - \delta^j_v g_{hs}) \kappa(\delta^l_k g_{rp} - \delta^l_r g_{kp}) t^s_a t^p_b \\
&\quad - \frac{1}{4f^2} g_{cd} g^{lj} g^{hk} g^{rv} (\kappa(\delta^s_r g_{lh} - \delta^s_l g_{rh}) \kappa(\delta^p_v g_{jk} - \delta^p_j g_{vk}) t^c_s t^d_p \\
&\quad + \frac{1}{2f^2} g^{re} g^{bz} (\kappa(\delta^h_c g_{pr} - \delta^h_p g_{cr}) \kappa(\delta^s_h g_{ez} - \delta^s_e g_{hz})) t^c_s t^p_b + {}^f L \\
&= \frac{1}{f} n(n-1)\kappa - \frac{1}{4f^2} \kappa^2 n g^{ab} g_{sp} t^s_a t^p_b + \frac{1}{4f^2} \kappa^2 g^{ab} g_{sp} t^s_a t^p_b + \frac{1}{4f^2} \kappa^2 g^{ab} g_{sp} t^s_a t^p_b \\
&\quad - \frac{1}{4f^2} \kappa^2 n g^{ab} g_{sp} t^s_a t^p_b - \frac{1}{4f^2} \kappa^2 n g_{cd} g^{rv} t^c_r t^d_v + \frac{1}{4f^2} \kappa^2 g_{cd} g^{rj} t^c_r t^d_j \\
&\quad + \frac{1}{4f^2} \kappa^2 g_{cd} g^{lv} t^c_l t^d_v - \frac{1}{4f^2} \kappa^2 n g_{cd} g^{lj} t^c_l t^d_j + \frac{1}{2f^2} \kappa^2 \delta^s_c \delta^b_p t^c_s t^p_b \\
&\quad - \frac{1}{2f^2} \kappa^2 \delta^b_c \delta^s_p t^c_s t^p_b - \frac{1}{2f^2} \kappa^2 \delta^b_c \delta^s_p t^c_s t^p_b + \frac{1}{2f^2} \kappa^2 \delta^s_c \delta^b_p t^c_s t^p_b + {}^f L \\
&= \frac{1}{f} n(n-1)\kappa - \frac{1}{2f^2} \kappa^2 \|t\|^2 (n-1) - \frac{1}{2f^2} \kappa^2 \|t\|^2 (n-1) \\
&\quad + \frac{1}{f^2} \kappa^2 t^c_c t^p_p - \frac{1}{f^2} \kappa^2 t^c_p t^p_c + {}^f L \\
&= \frac{1}{f} (n-1)\kappa \left\{ n - \frac{1}{f} \|t\|^2 \kappa + \frac{1}{f} \kappa^2 (trace\ t)^2 - (trace\ t^2) \right\} + {}^f L.
\end{aligned}$$

Thus we have

THEOREM 9. *Let (M, g), $n > 2$ be a Riemannian manifold of constant curvature κ. Then the scalar curvature $\tilde{r}$ of $(T^1_1(M), {}^S g_f)$ is*

$$\tilde{r} = \frac{1}{f}(n-1)\kappa \left(n - \frac{1}{f} \|t\|^2 \kappa \right) + \frac{1}{f} \kappa^2 ((trace\ t)^2 - (trace\ t^2)) + {}^f L,$$

where $\|t\|^2 = g_{kl} g^{ij} t^i_k t^j_l$.

5. Other metric connections of the rescaled Sasaki type metric ${}^Sg_f$ on the $(1,1)$-tensor bundle

Let ∇ be a linear connection on a manifold M. The connection ∇ is symmetric if its torsion tensor vanishes, otherwise it is non-symmetric. If there is a Riemannian metric g on M such that $\nabla g = 0$, then the connection ∇ is a metric connection, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In section 3, we have considered the Levi-Civita connection $\tilde\nabla$ of the rescaled Sasaki type metric ${}^Sg_f$ on the tensor bundle $T_1^1(M)$ over (M,g). The connection is the unique connection which satisfies $\tilde\nabla_\alpha({}^Sg_f)_{\beta\gamma} = 0$ and has a zero torsion. H. A. Hayden [17] introduced a metric connection with a non-zero torsion on a Riemanian manifold. Now we are interested in a metric connection ${}^{(M)}\tilde\nabla$ of the rescaled Sasaki type metric ${}^Sg_f$ whose torsion tensor ${}^{(M)\tilde\nabla}T^{\varepsilon}_{\gamma\beta}$ is skew-symmetric in the indices γ and β. We denote components of the connection ${}^{(M)}\tilde\nabla$ by ${}^{(M)}\tilde\Gamma$. The metric connection ${}^{(M)}\tilde\nabla$ satisfies

$$ {}^{(M)}\tilde\nabla_\alpha({}^Sg_f)_{\beta\gamma} = 0 \quad \text{and} \quad {}^{(M)}\tilde\Gamma^{\gamma}_{\alpha\beta} - {}^{(M)}\tilde\Gamma^{\gamma}_{\beta\alpha} = {}^{(M)\tilde\nabla}T^{\gamma}_{\alpha\beta}. \tag{5.1} $$

On the equation (5.1) is solved with respect to ${}^{(M)}\tilde\Gamma^{\gamma}_{\alpha\beta}$, one finds the following solution [17]

$$ {}^{(M)}\tilde\Gamma^{\gamma}_{\alpha\beta} = \tilde\Gamma^{\gamma}_{\alpha\beta} + \tilde U^{\gamma}_{\alpha\beta}, \tag{5.2} $$

where $\tilde\Gamma^{\gamma}_{\alpha\beta}$ is components of the Levi-Civita connection of the rescaled Sasaki type metric ${}^Sg_f$,

$$ \tilde U_{\alpha\beta\gamma} = \frac{1}{2}\left({}^{(M)\tilde\nabla}T_{\alpha\beta\gamma} + {}^{(M)\tilde\nabla}T_{\gamma\alpha\beta} + {}^{(M)\tilde\nabla}T_{\gamma\beta\alpha}\right) \tag{5.3} $$

and

$$ \tilde U_{\alpha\beta\gamma} = \tilde U^{\varepsilon}_{\alpha\beta}({}^Sg_f)_{\varepsilon\gamma}, \quad {}^{(M)\tilde\nabla}T_{\alpha\beta\gamma} = T^{\varepsilon}_{\alpha\beta}({}^Sg_f)_{\varepsilon\gamma}. $$

If we put

$$ {}^{(M)\tilde\nabla}T^{\bar r}_{lj} = t^m_r R^v_{ljm} - t^v_m R^m_{ljr} \tag{5.4} $$

all other ${}^{(M)\tilde\nabla}T^{\gamma}_{\alpha\beta}$ not related to ${}^{(M)\tilde\nabla}T^{\bar r}_{lj}$ being assumed to be zero. We choose this ${}^{(M)\tilde\nabla}T^{\gamma}_{\alpha\beta}$ in $T_1^1(M)$ which is skew-symmetric in the indices γ and β as torsion tensor and determine a metric connection on $T_1^1(M)$ with respect to the rescaled Sasaki type metric ${}^Sg_f$. By using (3.5), (5.3) and (5.4), we get non-zero components of $\tilde U^{\gamma}_{\alpha\beta}$ as follows:

$$ \tilde U^{\bar r}_{lj} = \frac{1}{2}(t^s_r R^v_{ljs} - t^v_s R^s_{ljr}), $$

$$ \tilde U^{r}_{l\bar j} = \frac{1}{2f}(g^{jb}R^r_{isl}t^s_b - g_{ia}R^{s\,j\,r}_{\cdot\,\cdot\,l}t^a_s), $$

$$ \tilde U^{r}_{\bar l j} = \frac{1}{2f}(g^{lb}R^r_{tsj}t^s_b - g_{ta}R^{s\,l\,r}_{\cdot\,\cdot\,j}t^a_s), $$

with respect to the adapted frame. From (5.2) and Theorem 1, we have

PROPOSITION 3. *Let (M, g) be a Riemannian manifold and $T_1^1(M)$ be its tensor bundle with the rescaled Sasaki type metric ${}^Sg_f$. The metric connection ${}^{(M)}\tilde{\nabla}$ with respect to ${}^Sg_f$ satisfy*

$$\begin{cases} \text{i) } {}^{(M)}\tilde{\nabla}_{E_l}E_j = \left\{\Gamma_{lj}^r + \dfrac{1}{2f}{}^fA_{lj}^r\right\}E_r, \\ \text{ii) } {}^{(M)}\tilde{\nabla}_{E_l}E_{\bar{j}} = \{\Gamma_{li}^v\delta_r^j - \Gamma_{lr}^j\delta_i^v\}E_{\bar{r}}, \\ \text{iii) } {}^{(M)}\tilde{\nabla}_{E_{\bar{l}}}E_j = 0, \\ \text{iv) } {}^{(M)}\tilde{\nabla}_{E_{\bar{l}}}E_{\bar{j}} = 0 \end{cases}$$

with respect to the adapted frame, where ${}^fA_{ji}^h$ is a tensor field of type $(1,2)$ defined by ${}^fA_{ji}^h = (f_j\delta_i^h + f_i\delta_j^h - f^h_{\cdot}g_{ji})$.

Remark 3. If $f = C(const.)$, the metric connection ${}^{(M)}\tilde{\nabla}$ on $T_1^1(M)$ of the rescaled Sasaki type metric ${}^Sg_f$ coincides with the metric connection ${}^H\nabla$ of the Sasaki type metric ${}^Sg$, where ${}^H\nabla$ is the horizontal lift of the Levi-Civita connection ∇ of g (for the metric connection ${}^H\nabla$, see [28]).

For the curvature tensor ${}^{(M)}\tilde{R}$ of the metric connection ${}^{(M)}\tilde{\nabla}$, we state the following result.

PROPOSITION 4. *Let (M, g) be a Riemannian manifold and $T_1^1(M)$ be its tensor bundle with the rescaled Sasaki type metric ${}^Sg_f$. The curvature tensor ${}^{(M)}\tilde{R}$ of the metric connection ${}^{(M)}\tilde{\nabla}$ satisfies the followings*:

$$\begin{cases} {}^{(M)}\tilde{R}(E_m, E_l)E_j = \left\{R_{mlj}^r + \nabla_m\left(\dfrac{1}{2f}A_{lj}^r\right) - \nabla_l\left(\dfrac{1}{2f}A_{mj}^r\right)\right. \\ \qquad\qquad\qquad\qquad \left. + \dfrac{1}{4f^2}A_{ms}^rA_{lj}^s - \dfrac{1}{4f^2}A_{ls}^rA_{mj}^s\right\}E_r, \\ {}^{(M)}\tilde{R}(E_m, E_l)E_{\bar{j}} = \{R_{mli}^v\delta_r^j - R_{mlr}^j\delta_i^v\}E_{\bar{r}}, \quad \textit{otherwise} = 0 \end{cases}$$

with respect to the adapted frame.

The non-zero component of the contracted curvature tensor field (Ricci tensor field) ${}^{(M)}\tilde{R}_{\gamma\beta} = {}^{(M)}\tilde{R}_{\alpha\gamma\beta}^{\alpha}$ of the metric connection ${}^{(M)}\tilde{\nabla}$ is as follows:

$${}^{(M)}\tilde{R}_{lj} = R_{lj} + \nabla_r\left(\frac{1}{2f}A_{lj}^r\right) - \nabla_l\left(\frac{1}{2f}A_{rj}^r\right) + \frac{1}{4f^2}A_{rs}^rA_{lj}^s - \frac{1}{4f^2}A_{ls}^rA_{rj}^s.$$

For the scalar curvature ${}^{(M)}\tilde{r}$ of the metric connection ${}^{(M)}\tilde{\nabla}$ with respect to ${}^{S}g_{f}$, we obtain

$$
\begin{aligned}
{}^{(M)}\tilde{r} &= ({}^{S}g_{f})^{\gamma\beta}(M)\tilde{R}_{\gamma\beta} \\
&= \frac{1}{f}r + {}^{f}L,
\end{aligned} \tag{5.5}
$$

where ${}^{f}L = \frac{1}{f}g^{lj}\left\{\nabla_{r}\left(\frac{1}{2f}A_{lj}^{r}\right) - \nabla_{l}\left(\frac{1}{2f}A_{rj}^{r}\right) + \frac{1}{4f^{2}}A_{rs}^{r}A_{lj}^{s} - \frac{1}{4f^{2}}A_{ls}^{r}A_{rj}^{s}\right\}$ and r is the scalar curvature of ∇_{g}. If ${}^{f}L = 0$ in the equation (5.5), then (5.5) reduces to ${}^{(M)}\tilde{r} = \frac{1}{f}r$, which leads the following theorem.

Theorem 10. *Let (M, g) be a Riemannian manifold and the tensor bundle $T_{1}^{1}(M)$ be equipped with the rescaled Sasaki type metric ${}^{S}g_{f}$. Suppose that ${}^{f}L = 0$, then the tensor bundle $T_{1}^{1}(M)$ with the metric connection ${}^{(M)}\tilde{\nabla}$ has vanishing scalar curvature ${}^{(M)}\tilde{r}$ with respect to ${}^{S}g_{f}$ if and only if the scalar curvature r of ∇_{g} in M is zero.*

For a Riemannian metric g on M, there happen to be many ways to define metric connections associated with g. Now we shall give another class of metric connections on $T_{1}^{1}(M)$. Let F be an almost product structure and ∇ be a linear connection on a manifold M. The product conjugate connection ${}^{(F)}\nabla$ of ∇ is defined by

$$
{}^{(F)}\nabla_{X}Y = F(\nabla_{X}FY)
$$

for all $X, Y \in \Im_{0}^{1}(M)$. If (M, F, g) is an almost product Riemannian manifold, then $({}^{(F)}\nabla_{X}g)(FY, FZ) = (\nabla_{X}g)(Y, Z)$, i.e. ∇ is a metric connection with respect to g if and only if ${}^{(F)}\nabla$ is so. From this, we can say that if ∇ is the Levi-Civita connection of g, then ${}^{(F)}\nabla$ is a metric connection with respect to g [1].

By the almost product structure J defined by (3.8) and the Levi-Civita connection $\tilde{\nabla}$ given by Theorem 1, we write the product conjugate connection ${}^{(J)}\tilde{\nabla}$ of $\tilde{\nabla}$ as follows:

$$
{}^{(J)}\tilde{\nabla}_{\tilde{X}}\tilde{Y} = J(\tilde{\nabla}_{\tilde{X}}J\tilde{Y})
$$

for all $\tilde{X}, \tilde{Y} \in \Im_{0}^{1}(T_{1}^{1}(M))$. Also note that ${}^{(J)}\tilde{\nabla}$ is a metric connection of the rescaled Sasaki type metric ${}^{S}g_{f}$. The standart calculations give the following theorem.

Theorem 11. *Let (M, g) be a Riemannian manifold and let $T_{1}^{1}(M)$ be its tensor bundle equipped with the rescaled Sasaki type metric ${}^{S}g_{f}$ and the almost product structure J. Then the product conjugate connection* (*or metric connection*) *${}^{(J)}\tilde{\nabla}$ is as follows*:

$$\begin{cases} \text{i)} \ {}^{(J)}\tilde{\nabla}_{E_l}E_j = \left\{\Gamma_{lj}^r + \frac{1}{2f}{}^fA_{lj}^r\right\}E_r - \left\{\frac{1}{2}R_{ljr}^s t_s^v - \frac{1}{2}R_{ljs}^v t_r^s\right\}E_{\bar{r}}, \\ \text{ii)} \ {}^{(J)}\tilde{\nabla}_{E_l}E_{\bar{j}} = -\left\{\frac{1}{2f}g_{ia}R^{s\ j\ r}_{\ .\ .\ j}t_s^a - \frac{1}{2f}g^{jb}R_{isl}^r t_b^s\right\}E_r + \{\Gamma_{li}^v\delta_r^j - \Gamma_{lr}^j\delta_i^v\}E_{\bar{r}}, \\ \text{iii)} \ {}^{(J)}\tilde{\nabla}_{E_{\bar{l}}}E_j = \left\{\frac{1}{2f}g_{ta}R^{s\ l\ r}_{\ .\ .\ j}t_s^a - \frac{1}{2f}g^{lb}R_{tsj}^r t_b^s\right\}E_r, \\ \text{iv)} \ {}^{(J)}\tilde{\nabla}_{E_{\bar{l}}}E_{\bar{j}} = 0 \end{cases}$$

with respect to the adapted frame.

The relationship between curvature tensors R_∇ and $R_{{}^{(F)}\nabla}$ of the connections ∇ and ${}^{(F)}\nabla$ is follows: $R_{{}^{(F)}\nabla}(X, Y, Z) = F(R_\nabla(X, Y, FZ))$ for all $X, Y, Z \in \Im_0^1(M)$ [1]. By means of the almost product structure J defined by (3.8) and Proposition 2, from $\tilde{R}_{{}^{(J)}\tilde{\nabla}}(\tilde{X}, \tilde{Y}, \tilde{Z}) = J(\tilde{R}_{\tilde{\nabla}}(\tilde{X}, \tilde{Y}, J\tilde{Z}))$, components of the curvature tensor $\tilde{R}_{{}^{(J)}\tilde{\nabla}}$ of the product conjugate connection (or metric connection) ${}^{(J)}\tilde{\nabla}$ can easily be computed. Lastly, by using the almost product structure ${}^DI$, another metric connection of the rescaled Sasaki type metric ${}^Sg_f$ can be constructed.

6. The Rescaled Sasaki type metric on (p,q)-tensor bundles and its Geodesics

The set $T_q^p(M) = \bigcup_{P\in M} T_q^p(P)$ is (p,q) tensor bundles over M, where $T_q^p(P)$ is tensor spaces for all $P \in M$. For $\tilde{P} \in T_q^p(M)$, the surjective correspondence $\tilde{P} \to P$ determines the natural projection $\pi : T_q^p(M) \to M$. A system of local coordinates (U, x^j), $j = 1, \ldots, n$ in M induces on $T_q^p(M)$ a system of local coordinates $(\pi^{-1}(U),\ x^j,\ x^{\bar{j}} = t_{j_1\cdots j_q}^{i_1\cdots i_p})$, $\bar{j} = n+1, \ldots, n+n^{p+q}$, where $x^{\bar{j}} = t_{j_1\cdots j_q}^{i_1\cdots i_p}$ is the components of tensors t in each tensor space $T_q^p(M)_x$, $x \in U$ with respect to the natural base.

The vertical lift ${}^VA$ of $A \in \Im_q^p(M)$ and the horizontal lift ${}^HX$ of $X \in \Im_0^1(M)$ to $T_q^p(M)$ are given by

$${}^VA = \begin{pmatrix} {}^VA^j \\ {}^VA^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_{j_1\cdots j_q}^{i_1\cdots i_p} \end{pmatrix}$$

and

$${}^HX = \begin{pmatrix} {}^HX^j \\ {}^HX^{\bar{j}} \end{pmatrix} = \begin{pmatrix} X^j \\ X^s\left(\sum\limits_{\mu=1}^{q}\Gamma_{sj_\mu}^m t_{j_1\cdots m\cdots j_q}^{i_1\cdots i_p} - \sum\limits_{\lambda=1}^{p}\Gamma_{sm}^{i_\lambda} t_{j_1\cdots j_q}^{i_1\cdots m\cdots i_p}\right) \end{pmatrix},$$

where Γ_{ij}^h are the coefficients of the connection ∇ on M [3]. For $\varphi = \varphi_j^i \dfrac{\partial}{\partial x^i} \otimes dx^j \in \Im_1^1(M)$, the local expressions of the global vector fields $\gamma\varphi$ and $\tilde{\gamma}\varphi$ are as follows:

$$\gamma\varphi = \begin{pmatrix} 0 \\ \sum\limits_{\lambda=1}^{p} t_{j_1\cdots j_q}^{i_1\cdots m\cdots i_p}\varphi_m^{i_\lambda} \end{pmatrix} \quad \text{and} \quad \tilde{\gamma}\varphi = \begin{pmatrix} 0 \\ \sum\limits_{\mu=1}^{q} t_{j_1\cdots m\cdots j_q}^{i_1\cdots i_p}\varphi_{j_\mu}^{m} \end{pmatrix}.$$

Now, we define the adapted frame $\{E_\alpha\} = \{E_j, E_{\bar{j}}\}$ of $T_q^p(M)$ by

$$E_j = {}^H X_{(j)} = \delta_j^h \partial_h + \left(-\sum_{\lambda=1}^{p} \Gamma_{js}^{k_\lambda} t_{h_1\cdots h_q}^{k_1\cdots s\cdots k_p} + \sum_{\mu=1}^{q} \Gamma_{jh_\mu}^{s} t_{h_1\cdots s\cdots h_q}^{k_1\cdots k_p} \right) \partial_{\bar{h}},$$

$$E_{\bar{j}} = {}^V A^{(\bar{j})} = \delta_{i_1}^{k_1} \cdots \delta_{i_p}^{k_p} \delta_{h_1}^{j_1} \cdots \delta_{h_q}^{j_q} \partial_{\bar{h}}$$

with respect to the natural frame $\left\{ \frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^{\bar{h}}} \right\}$ in $T_q^p(M)$, where $X_{(j)} = \frac{\partial}{\partial x^j} = \delta_j^h \frac{\partial}{\partial x^h} \in \Im_0^1(M)$ and $A^{(\bar{j})} = \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q} = \delta_{i_1}^{k_1} \cdots \delta_{i_p}^{k_p} \delta_{h_1}^{j_1} \cdots \delta_{h_q}^{j_q} \partial_{k_1} \otimes \cdots \otimes \partial_{k_p} \otimes dx^{h_1} \otimes \cdots \otimes dx^{h_q} \in \Im_q^p(M)$.

With respect to the adapted frame $\{E_\alpha\}$, the vertical lift ${}^V A$ and the horizontal lift ${}^H X$ have respectively the components [30]

$${}^V A = \begin{pmatrix} 0 \\ A_{j_1\cdots j_q}^{i_1\cdots i_p} \end{pmatrix}$$

and

$${}^H X = \begin{pmatrix} X^j \\ 0 \end{pmatrix}.$$

The rescaled Sasaki type metric ${}^S g_f$ is defined on $T_q^p(M)$ by the three equations

$${}^S g_f({}^V A, {}^V B) = {}^V(G(A,B)),$$
$${}^S g_f({}^V A, {}^H Y) = 0,$$
$${}^S g_f({}^H X, {}^H Y) = {}^V(fg(X,Y))$$

for all $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_q^p(M)$, where

$$G(A,B) = g_{i_1 t_1} \cdots g_{i_p t_p} g^{j_1 l_1} \cdots g^{j_q l_q} A_{j_1\cdots j_q}^{i_1\cdots i_p} B_{l_1\cdots l_q}^{t_1\cdots t_p}.$$

The rescaled Sasaki type metric ${}^S g_f$ has components

$$({}^S g_f)_{\beta\gamma} = \begin{pmatrix} f g_{jl} & 0 \\ 0 & g_{i_1 t_1} \cdots g_{i_p t_p} g^{j_1 l_1} \cdots g^{j_q l_q} \end{pmatrix}, \quad x^{\bar{l}} = t_{l_1\cdots l_q}^{t_1\cdots t_p}$$

with respect to the adapted frame, g_{ij} and g^{ij} being local covariant and contravariant components g on M. By the Koszul formula and standart calculations give the following.

PROPOSITION 5. *The components of the Levi-Civita connection* $\hat{\nabla}$ *of the tensor bundle* $T_q^p(M)$ *with the rescaled Sasaki type metric* ${}^S g_f$ *are given as follows*:

$$\begin{cases} \hat{\Gamma}_{lj}^{\bar{r}} = \dfrac{1}{2}\sum_{\mu=1}^{q} R_{ljh_\mu}^{s} t_{h_1\cdots s\cdots h_q}^{k_1\cdots k_p} - \dfrac{1}{2}\sum_{\lambda=1}^{p} R_{ljs}^{k_\lambda} t_{h_1\cdots h_q}^{k_1\cdots s\cdots k_p}, \\ \hat{\Gamma}_{l\bar{j}}^{\bar{r}} = \sum_{\lambda=1}^{p} \Gamma_{ls}^{v_\lambda}\delta_{r_1}^{j_1}\cdots\delta_{r_q}^{j_q}\delta_{i_1}^{l_1}\cdots\delta_{i_\lambda}^{s}\cdots\delta_{i_p}^{l_p} - \sum_{\mu=1}^{q}\Gamma_{lr_\mu}^{s}\delta_{r_1}^{j_1}\cdots\delta_{s}^{j_\mu}\cdots\delta_{r_q}^{j_q}\delta_{i_1}^{l_1}\cdots\delta_{i_p}^{l_p}, \\ \hat{\Gamma}_{\bar{l}j}^{r} = \dfrac{1}{2f} g^{xr} g_{i_1t_1}\cdots g_{i_pt_p} g^{j_1h_1}\cdots g^{j_qh_q}\left(\sum_{\mu=1}^{q} R_{xjh_\mu}^{s} t_{h_1\cdots s\cdots h_q}^{k_1\cdots k_p} - \sum_{\lambda=1}^{p} R_{xjs}^{k_\lambda} t_{h_1\cdots h_q}^{k_1\cdots s\cdots k_p}\right), \\ \hat{\Gamma}_{l\bar{j}}^{r} = \dfrac{1}{2f} g^{xr} g_{t_1k_1}\cdots g_{t_pk_p} g^{l_1h_1}\cdots g^{l_qh_q}\left(\sum_{\mu=1}^{q} R_{xlh_\mu}^{s} t_{h_1\cdots s\cdots h_q}^{k_1\cdots k_p} - \sum_{\lambda=1}^{p} R_{xls}^{k_\lambda} t_{h_1\cdots h_q}^{k_1\cdots s\cdots k_p}\right), \\ \hat{\Gamma}_{lj}^{r} = \Gamma_{lj}^{r} + \dfrac{1}{2f}\,{}^f A_{lj}^{r}, \\ \hat{\Gamma}_{\bar{l}\bar{j}}^{\bar{r}} = 0, \quad \hat{\Gamma}_{\bar{l}\bar{j}}^{r} = 0, \quad \hat{\Gamma}_{\bar{l}j}^{\bar{r}} = 0, \end{cases}$$

with respect to the adapted frame, where ${}^f A_{ji}^h$ *is defined by* ${}^f A_{ji}^h = (f_j\delta_i^h + f_i\delta_j^h - f^h g_{ji})$ *and* $f_i = \partial_i f$.

An important geometric problem is to find the geodesics on the smooth manifolds with respect to the Riemannian metrics (see [2, 8, 9, 26, 30, 37]). In [37], K. Yano and S. Ishihara proved that the curves on the tangent bundles of Riemannian manifolds are geodesics with respect to certain lifts of the metric from the base manifold if and only if the curves are obtained as certain types of lifts of the geodesics from the base manifold. In this section, we shall characterize the geodesics on the (p,q)-tensor bundle with respect to the Levi-Civita connection and another metric connection of ${}^S g_f$.

Let $\tilde{\gamma} = \tilde{\gamma}(t)$ be a curve on $T_q^p(M)$ and suppose that $\tilde{\gamma}$ is locally expressed by $x^R = x^R(t)$, i.e. $x^r = x^r(t)$, $x^{\bar{r}} = t_{r_1\cdots r_q}^{v_1\cdots v_p}(t)$ with respect to the natural frame $\left\{\dfrac{\partial}{\partial x^I}\right\} = \left\{\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^{\bar{i}}}\right\}$, t being a parameter an arc length of $\tilde{\gamma}$. Then the curve $\gamma = \pi\circ\tilde{\gamma}$ on M is called the projection of the curve $\tilde{\gamma}$ and denoted by $\pi\tilde{\gamma}$ which is expressed locally by $x^r = x^r(t)$.

A curve $\tilde{\gamma}$ is, by definition, a geodesic on $T_q^p(M)$ with respect to the Levi-Civita connection $\hat{\nabla}$ of ${}^S g_f$ if and only if it satisfies the following differential equations

$$\frac{d}{dt}\left(\frac{\omega^\alpha}{dt}\right) + \hat{\Gamma}_{\gamma\beta}^{\alpha}\frac{\omega^\gamma}{dt}\frac{\omega^\beta}{dt} = 0 \tag{6.1}$$

with respect to the adapted frame, where

$$\frac{\omega^r}{dt} = \frac{dx^r}{dt} \quad \text{and} \quad \frac{\omega^{\bar{r}}}{dt} = \frac{\delta t_{r_1\cdots r_q}^{v_1\cdots v_p}}{dt}$$

along a curve $\tilde{\gamma}$.

By means of Proposition 5, by (6.1), we have

$$\begin{aligned}(6.2)\quad &\frac{d}{dt}\left(\frac{\omega^r}{dt}\right)+\left(\Gamma^r_{lj}+\frac{1}{2f}{}^fA^r_{lj}\right)\frac{dx^l}{dt}\frac{dx^j}{dt}+\frac{1}{2f}g^{xr}g_{t_1k_1}\cdots g_{t_pk_p}g^{l_1h_1}\cdots g^{l_qh_q}\\ &\times\left(-\sum_{\lambda=1}^{p}R^{k_\lambda}_{xjs}t^{k_1\cdots s\cdots k_p}_{h_1\cdots h_q}+\sum_{\mu=1}^{q}R^{s}_{xjh_\mu}t^{k_1\cdots k_p}_{h_1\cdots s\cdots h_q}\right)\frac{\delta t^{t_1\cdots t_p}_{l_1\cdots l_q}}{dt}\frac{dx^j}{dt}\\ &+\frac{1}{2f}g^{xr}g_{i_1k_1}\cdots g_{i_pk_p}g^{j_1h_1}\cdots g^{j_qh_q}\\ &\times\left(-\sum_{\lambda=1}^{p}R^{k_\lambda}_{xls}t^{k_1\cdots s\cdots k_p}_{h_1\cdots h_q}+\sum_{\mu=1}^{q}R^{s}_{xlh_\mu}t^{k_1\cdots k_p}_{h_1\cdots s\cdots h_q}\right)\frac{dx^l}{dt}\frac{\delta t^{l_1\cdots l_p}_{j_1\cdots j_q}}{dt}\\ &=0,\end{aligned}$$

$$\begin{aligned}(6.3)\quad &\frac{d}{dt}\left(\frac{\delta t^{v_1\cdots v_p}_{r_1\cdots r_q}}{dt}\right)+\frac{1}{2}\left(\sum_{\mu=1}^{q}R^{s}_{ljh_\mu}t^{k_1\cdots k_p}_{h_1\cdots s\cdots h_q}-\sum_{\lambda=1}^{p}R^{k_\lambda}_{ljs}t^{k_1\cdots s\cdots k_p}_{h_1\cdots h_q}\right)\frac{dx^l}{dt}\frac{dx^j}{dt}\\ &+\left(\sum_{\lambda=1}^{p}\Gamma^{v_\lambda}_{ls}\delta^{j_1}_{r_1}\cdots\delta^{j_q}_{r_q}\delta^{l_1}_{i_1}\cdots\delta^{s}_{i_\lambda}\cdots\delta^{l_p}_{i_p}\right.\\ &\left.-\sum_{\mu=1}^{q}\Gamma^{s}_{lr_\mu}\delta^{j_1}_{r_1}\cdots\delta^{j_\mu}_{s}\cdots\delta^{j_q}_{r_q}\delta^{l_1}_{i_1}\cdots\delta^{l_p}_{i_p}\right)\frac{dx^l}{dt}\frac{\delta t^{i_1\cdots i_p}_{j_1\cdots j_q}}{dt}\\ &=0.\end{aligned}$$

Also, the equation (6.2) can be expressed as follows:

$$\begin{aligned}(6.4)\quad &\frac{\delta^2x^r}{dt^2}+\frac{1}{2f}{}^fA^r_{lj}\frac{dx^l}{dt}\frac{dx^j}{dt}+\frac{1}{f}g_{t_1k_1}\cdots g_{t_pk_p}g^{l_1h_1}\cdots g^{l_qh_q}\\ &\times\left(\sum_{\mu=1}^{q}R^{rs}_{.jh_\mu}t^{k_1\cdots k_p}_{h_1\cdots s\cdots h_q}-\sum_{\lambda=1}^{p}R^{rk_\lambda}_{.js}t^{k_1\cdots s\cdots k_p}_{h_1\cdots h_q}\right)\frac{\delta^{t_1\cdots t_p}_{l_1\cdots l_q}}{dt}\frac{dx^j}{dt}\\ &=0.\end{aligned}$$

Using the identity $\left(-\sum\limits_{\lambda=1}^{p}R^{k_\lambda}_{ljs}t^{k_1\cdots s\cdots k_p}_{h_1\cdots h_q}+\sum\limits_{\mu=1}^{q}R^{s}_{ljh_\mu}t^{k_1\cdots k_p}_{h_1\cdots s\cdots h_q}\right)\frac{dx^l}{dt}\frac{dx^j}{dt}=0$, from (6.3) we get the following relation

$$(6.5)\qquad \frac{\delta^2 t^{v_1\cdots v_p}_{r_1\cdots r_q}}{dt^2}=0.$$

From (6.4) and (6.5), we state the theorem below.

THEOREM 12. *Let $\tilde{\gamma}$ be a geodesic on $T_q^p(M)$ of the Levi-Civita connection $\hat{\nabla}$ of ${}^S g_f$. Then the tensor field $t_{r_1\cdots r_q}^{v_1\cdots v_p}(t)$ defined along γ satisfies the differential equations* (6.4) *and has vanishing second covariant derivative.*

Next, let γ be a curve on M expressed locally by $x^h = x^h(t)$ and $S_{j_1\cdots j_q}^{i_1\cdots i_p}(t)$ be a (p,q) tensor field along γ. Then, on the tensor bundle $T_q^p(M)$ over the Riemannian manifold M, we define a curve ${}^H\gamma$ by

$$\begin{cases} x^h = x^h(t), \\ x^{\bar{h}} = S_{h_1\cdots h_q}^{k_1\cdots k_p}(t). \end{cases}$$

If the curve ${}^H\gamma$ satisfies at all the points the relation

$$\frac{\delta S_{h_1\cdots h_q}^{k_1\cdots k_p}}{dt} = 0, \tag{6.6}$$

i.e. $S_{j_1\cdots j_q}^{i_1\cdots i_p}(t)$ is a parallel tensor field along γ, then the curve ${}^H\gamma$ is said to be a horizontal lift of γ. From (6.4) and (6.6), we obtain

$$\frac{\delta^2 x^r}{dt^2} + \frac{1}{2f}{}^f A_{lj}^r \frac{dx^l}{dt}\frac{dx^j}{dt} = 0.$$

If we take

$${}^f A_{lj}^r = (\partial_l f \delta_j^r + \partial_j f \delta_l^r - g^{rm}\partial_m f g_{lj}) = 0. \tag{6.7}$$

Contracting l and r in (6.7) it follows that $\partial_j f = 0$. Since this is true for any j, we can say $f = C(const.)$. Thus we have the following theorem.

THEOREM 13. *The horizontal lift of a geodesic on M is always geodesic on $T_q^p(M)$ with respect to the Levi-Civita connection $\hat{\nabla}$ of ${}^S g_f$ if and only if $f = C(const.)$.*

Following the same way in the section 5, by virtue of the Levi-Civita connection $\hat{\nabla}$ of ${}^S g_f$ on $T_q^p(M)$, we introduce a metric connection ${}^{(M)}\hat{\nabla}$ on $T_q^p(M)$ whose torsion tensor has components

$$\begin{cases} {}^{(M)\hat{\nabla}}T_{lj}^{\bar{r}} = \dfrac{1}{2}\displaystyle\sum_{\lambda=1}^{p} R_{ljs}^{k_\lambda} t_{h_1\cdots h_q}^{k_1\cdots s\cdots k_p} - \dfrac{1}{2}\sum_{\mu=1}^{q} R_{ljh_\mu}^{s} t_{h_1\cdots s\cdots h_q}^{k_1\cdots k_p}, \\ otherwise = 0 \end{cases}$$

with respect to ${}^S g_f$.

PROPOSITION 6. *Let (M,g) be a Riemannian manifold and $T_q^p(M)$ be its (p,q)-tensor bundle with the rescaled Sasaki type metric ${}^S g_f$. The components of the metric connection ${}^{(M)}\hat{\nabla}$ with respect to ${}^S g_f$ is given by*

$$\begin{cases} {}^{(M)}\hat{\Gamma}_{lj}^{\bar{r}} = 0, \quad {}^{(M)}\hat{\Gamma}_{l\bar{j}}^{r} = 0, \quad {}^{(M)}\hat{\Gamma}_{\bar{l}j}^{r} = 0, \quad {}^{(M)}\hat{\Gamma}_{\bar{l}\bar{j}}^{\bar{r}} = 0, \quad {}^{(M)}\hat{\Gamma}_{\bar{l}\bar{j}}^{r} = 0, \quad {}^{(M)}\hat{\Gamma}_{\bar{l}j}^{\bar{r}} = 0, \\ {}^{(M)}\hat{\Gamma}_{l\bar{j}}^{\bar{r}} = \sum_{\lambda=1}^{p} \Gamma_{ls}^{v_\lambda} \delta_{r_1}^{j_1} \cdots \delta_{r_q}^{j_q} \delta_{i_1}^{l_1} \cdots \delta_{i_\lambda}^{s} \cdots \delta_{i_p}^{l_p} - \sum_{\mu=1}^{q} \Gamma_{lr_\mu}^{s} \delta_{r_1}^{j_1} \cdots \delta_{s}^{j_\mu} \cdots \delta_{r_q}^{j_q} \delta_{i_1}^{l_1} \cdots \delta_{i_p}^{l_p}, \\ {}^{(M)}\hat{\Gamma}_{lj}^{r} = \Gamma_{lj}^{r} + \frac{1}{2f}\, {}^{f}\! A_{lj}^{r}, \end{cases}$$

with respect to the adapted frame.

Substituting the components ${}^{(M)}\hat{\Gamma}_{\gamma\beta}^{\alpha}$ of the metric connection ${}^{(M)}\hat{\nabla}$ into (6.1), we get

$$\begin{cases} \frac{d^2x^r}{dt^2} + \left(\Gamma_{lj}^{r} + \frac{1}{2f}\,{}^{f}\! A_{lj}^{r}\right) \frac{dx^l}{dt}\frac{dx^j}{dt} = 0, \\ \frac{d}{dt}\left(\frac{\delta t_{r_1\cdots r_q}^{v_1\cdots v_p}}{dt}\right) + \left(\sum_{\lambda=1}^{p} \Gamma_{ls}^{v_\lambda} \delta_{r_1}^{j_1} \cdots \delta_{r_q}^{j_q} \delta_{i_1}^{l_1} \cdots \delta_{i_\lambda}^{s} \cdots \delta_{i_p}^{l_p} \right. \\ \left. - \sum_{\mu=1}^{q} \Gamma_{lr_\mu}^{s} \delta_{r_1}^{j_1} \cdots \delta_{s}^{j_\mu} \cdots \delta_{r_q}^{j_q} \delta_{i_1}^{l_1} \cdots \delta_{i_p}^{l_p} \right) \frac{dx^l}{dt} \frac{\delta t_{j_1\cdots j_q}^{i_1\cdots i_p}}{dt} = 0. \end{cases} \tag{6.8}$$

Also, the second equation in (6.8) can be written the following

$$\frac{\delta^2 t_{r_1\cdots r_q}^{v_1\cdots v_p}}{dt^2} = 0. \tag{6.9}$$

Thus the first equation in (6.8) and (6.9) give the last result.

THEOREM 14. *Let $\tilde{\gamma}$ be a geodesic on $T_q^p(M)$ with respect to the metric connection ${}^{(M)}\hat{\nabla}$ of ${}^{S}g_f$. Then the projection γ of $\tilde{\gamma}$ is a geodesic with respect to the Levi-Civita connection ∇ on M and the tensor field $t_{r_1\cdots r_q}^{v_1\cdots v_p}(t)$ defined along γ has vanishing second covariant derivative if and only if $f = C(const.)$.*

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Aydin Gezer
Ataturk University
Faculty of Science
Department of Mathematics
25240, Erzurum
Turkey
E-mail: agezer@atauni.edu.tr

Murat Altunbas
Erzincan University
Faculty of Science and Art
Department of Mathematics
24030, Erzincan
Turkey
E-mail: maltunbas@erzincan.edu.tr