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A NOTE ON SERRIN'S OVERDETERMINED PROBLEM

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Abstract

We consider the solution of the torsion problem

$$-\Delta u = N \ \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial\Omega,$$

where Ω is a bounded domain in $\mathbf{R}^N$.

Serrin's celebrated symmetry theorem states that, if the normal derivative u_ν is constant on $\partial\Omega$, then Ω must be a ball. In [6], it has been conjectured that Serrin's theorem may be obtained *by stability* in the following way: first, for the solution u of the torsion problem prove the estimate

$$r_e - r_i \le C_t \left(\max_{\Gamma_t} u - \min_{\Gamma_t} u \right)$$

for some constant C_t depending on t, where r_e and r_i are the radii of an annulus containing $\partial\Omega$ and Γ_t is a surface parallel to $\partial\Omega$ at distance t and sufficiently close to $\partial\Omega$; secondly, if in addition u_ν is constant on $\partial\Omega$, show that

$$\max_{\Gamma_t} u - \min_{\Gamma_t} u = o(C_t) \quad \text{as} \ t \to 0^+.$$

The estimate constructed in [6] is not sharp enough to achieve this goal. In this paper, we analyse a simple case study and show that the scheme is successful if the admissible domains Ω are ellipses.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ and let u be the solution of the torsion problem

$$-\Delta u = N \ \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial\Omega. \tag{1.1}$$

Serrin's celebrated symmetry theorem [11] states that, if there exists a solution of (1.1) whose (exterior) normal derivative u_ν is constant on $\partial\Omega$, that is such that

$$u_\nu = c \quad \text{on} \ \partial\Omega, \tag{1.2}$$

then Ω is a ball and u is radially symmetric.

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As is well-known, the proof of Serrin makes use of the *method of moving planes* (see [11, 7]), a refinement of Alexandrov's reflection principle [2].

The aim of this note is to probe the feasibility of a new proof of Serrin's symmetry theorem based on a comparison with another overdetermined problem for (1.1). In fact, it has been noticed that, under certain sufficient conditions on $\partial\Omega$, if the solution of (1.1) is constant on a surface *parallel* to $\partial\Omega$, that is, if for some small $t > 0$

$$u = k \text{ on } \Gamma_t, \quad \text{where } \Gamma_t = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = t\}, \tag{1.3}$$

then Ω must be a ball (see [9, 10, 6] and [12]).

Condition (1.3) was first studied in [9] (see also [10] and [5] for further developments), motivated by an investigation on *time-invariant level surfaces* of a nonlinear non-degenerate *fast diffusion* equation (tailored upon the heat equation), and was used to extend to nonlinear equations the symmetry results obtained in [8] for the heat equation. The proof still hinges on the method of moving planes, that can be applied in a much simplified manner, since the overdetermination in (1.3) takes place inside Ω. Under slightly different assumptions and by a different proof—still based on the method of moving planes—a similar result was obtained in [12] independently.

The evident similarity between the two problems arouses a natural question: *is condition* (1.3) *weaker or stronger than* (1.2)?

As pointed out in [6], (1.3) seems to be weaker than (1.2), as explained by the following two observations: (i) as (1.3) does not imply (1.2), the latter can be seen as the limit of a sequence of conditions of type (1.3) with $k = k_n$ and $t = t_n$ and k_n and t_n vanishing as $n \to \infty$; (ii) as (1.2) does not imply (1.3) either, if u satisfies (1.1)–(1.2), then the oscillation of u on a surface parallel to the boundary becomes smaller than usual, the closer the surface is to $\partial\Omega$. More precisely, if $u \in C^1(\overline{\Omega})$, by a Taylor expansion argument, it is easy to verify that

$$\max_{\Gamma_t} u - \min_{\Gamma_t} u = o(t) \quad \text{as } t \to 0 \tag{1.4}$$

—that becomes a $O(t^2)$ as $t \to \infty$ when $u \in C^2(\overline{\Omega})$.

This remark suggests the possibility that Serrin's symmetry result may be obtained *by stability* in the following way: first, for the solution u of the torsion problem (1.1) prove the estimate

$$r_e - r_i \le C_t \left(\max_{\Gamma_t} u - \min_{\Gamma_t} u \right) \tag{1.5}$$

for some constant C_t depending on t, where r_e and r_i are the radii of an annulus containing $\partial\Omega$; secondly, if in addition u_ν is constant on $\partial\Omega$, show that

$$\max_{\Gamma_t} u - \min_{\Gamma_t} u = o(C_t) \quad \text{as } t \to 0^+.$$

In the same spirit of (1.5), based on [1], in [6] we proved an estimate that quantifies the radial symmetry of Ω in terms of the following quantity:

$$[u]_{\Gamma_t} = \sup_{\substack{z, w \in \Gamma_t \\ z \neq w}} \frac{|u(z) - u(w)|}{|z - w|}. \tag{1.6}$$

In fact, it was proved that there exist two constants $\varepsilon, C_t > 0$ such that, if $[u]_{\Gamma_t} \leq \varepsilon$, then there are two concentric balls B_{r_i} and B_{r_e} such that

$$B_{r_i} \subset \Omega \subset B_{r_e} \quad \text{and} \quad r_e - r_i \leq C_t [u]_{\Gamma_t}. \tag{1.7}$$

The constant C_t only depends on t, N, the regularity of $\partial\Omega$ and the diameter of Ω.

The calculations in [6] imply that C_t blows-up exponentially as t tends to 0, which is too fast for our purposes, since $[u]_{\Gamma_t}$ cannot vanish faster than t^2, when (1.2) holds. The exponential dependence of C_t on t is due to the method of proof we employed, which is based on the idea of refining the method of moving planes from a quantitative point of view. As that method is based on the maximum (or comparison) principle, its quantitative counterpart is based on *Harnack's inequality* and some quantitative versions of *Hopf's boundary lemma*. The exponential dependence of the constant involved in Harnack's inequality leads to that of C_t. Recent (unpublished) calculations, based on more refined versions of Harnack's inequality, show that the growth rate of C_t can be improved, but they are still inadequate to achieve our goal. Approaches to stability based on the ideas contained in [3] and [4] do not seem to work for problem (1.1)–(1.3).

In this note, we shall show that our scheme (i)–(ii) is successful, at least in the case $N = 2$ and if the admissible domains are ellipses: in this case, the deviation from radial symmetry can be exactly computed in terms of the oscillation of u on Γ_t. We obtain (1.5) with $C_t = O(t^{-1})$ as $t \to 0^+$; thus, formula (1.4) yields the desired symmetry.[1]

2. Section 2

We begin by defining the three quantities that we shall exactly compute later on. Let $\Gamma \subset \mathbf{R}^2$ be a C^1-regular closed simple curve and let $z(s)$, $s \in [0, |\Gamma|)$ be its parameterization by arc-lenght. For a function $u : \Gamma \to \mathbf{R}$, we will consider the seminorms

$$|u|_\Gamma = \sup_{\substack{0 \leq s, s' \leq |\Gamma| \\ s \neq s'}} \frac{|u(z(s)) - u(z(s'))|}{\min(|s - s'|, |\Gamma| - |s - s'|)}, \quad [u]_\Gamma = \sup_{\substack{z, w \in \Gamma \\ z \neq w}} \frac{|u(z) - u(w)|}{|z - w|}, \tag{2.1}$$

and the *oscillation*

$$\operatorname*{osc}_\Gamma u = \max_\Gamma u - \min_\Gamma u. \tag{2.2}$$

[1] Of course, in this very special case, there is a trivial proof of symmetry, but this is not the point.

We now consider an ellipse

$$E = \left\{ z = (x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\},$$

with semi-axes a and b normalized by $a^{-2} + b^{-2} = 1$, and let

$$\Gamma_t = \{ z \in E : \operatorname{dist}(z, \partial E) = t \} \tag{2.3}$$

be the curve parallel to ∂E at distance t; Γ_t is still regular and simple if t is smaller than the minimal radius of curvature of ∂E, that is for

$$0 \le t < \frac{\min(a^3, b^3)}{2a^2 b^2}. \tag{2.4}$$

When $\Omega = E$, the solution u of (1.1) is clearly given by

$$u(x, y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}. \tag{2.5}$$

LEMMA 2.1. *Let u be given by* (2.5) *and let t satisfy* (2.4). *Then, we have*:

(i) $|u|_{\Gamma_t} = |a - b| \dfrac{a + b}{a^2 b^2} t$;

(ii) $[u]_{\Gamma_t} = |u|_{\Gamma_t}$;

(iii) $\operatorname*{osc}_{\Gamma_t} u = |a - b| \dfrac{a + b}{a^2 b^2} \left(\dfrac{2ab}{a + b} - t \right) t.$

Proof. The standard parametrization of ∂E is

$$\gamma(\theta) = (a \cos \theta, b \sin \theta), \quad \theta \in [0, 2\pi];$$

thus,

$$\Gamma_t = \left\{ \gamma(\theta) - tJ \frac{\gamma'(\theta)}{|\gamma'(\theta)|} : \theta \in [0, 2\pi) \right\},$$

where J is the rotation matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that the outward unit normal is

$$\nu(\theta) = J \frac{\gamma'(\theta)}{|\gamma'(\theta)|}.$$

(i) The mean value theorem then tells us that

$$\frac{|u(z(s)) - u(z(s'))|}{\min(|s - s'|, |\Gamma_t| - |s - s'|)} = |\langle Du(z(\sigma)), z'(\sigma) \rangle|, \tag{2.6}$$

for some $\sigma \in [0, |\Gamma_t|]$. Since Γ_t is parallel to ∂E, we have

$$z'(\sigma) = \frac{\gamma'(\theta(\sigma))}{|\gamma'(\theta(\sigma))|},$$

where $\theta(\sigma)$ is such that

$$z(\sigma) = \gamma(\theta(\sigma)) - t\nu(\theta).$$

By (2.5), we have that

$$|\langle Du(z(\sigma)), z'(\sigma)\rangle| = 2|\langle Az(\sigma), z'(\sigma)\rangle| \quad \text{with } A = \begin{pmatrix} a^{-2} & 0 \\ 0 & b^{-2} \end{pmatrix},$$

and hence

$$|\langle Du(z(\sigma)), z'(\sigma)\rangle| = 2\left|\frac{\langle A\gamma(\theta), \gamma'(\theta)\rangle}{|\gamma'(\theta)|} - t\frac{\langle AJ\gamma'(\theta), \gamma'(\theta)\rangle}{|\gamma'(\theta)|^2}\right|,$$

with $\theta = \theta(\sigma)$.

Straightforward computations give:

$$\gamma'(\theta) = (-a\sin\theta, b\cos\theta), \quad |\gamma'(\theta)| = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta},$$

$$\langle A\gamma(\theta), \gamma'(\theta)\rangle = 0, \quad \langle AJ\gamma'(\theta), \gamma'(\theta)\rangle = \frac{|a^2 - b^2|}{ab}\sin\theta\cos\theta.$$

Therefore,

$$|\langle Du(z(\sigma)) \cdot z'(\sigma)\rangle| = \frac{|a^2 - b^2|}{ab}\frac{2|\tan\theta|}{a^2\tan^2\theta + b^2}t;$$

this expression achieves its maximum if $|\tan\theta| = b/a$, that gives:

$$\max_{0\le\sigma\le|\Gamma_t|} |\langle Du(z(\sigma)), z'(\sigma)\rangle| = |a^{-2} - b^{-2}|t.$$

From (2.6) we conclude.

(ii) By a symmetry argument, we can always assume that $[u]_{\Gamma_t}$ is attained for points z and w (that may possibly coincide) in the first quadrant of the cartesian plane.

Now, suppose that the value $[u]_{\Gamma_t}$ is attained for two points $z, w \in \Gamma_t$ with $z \neq w$. Let $s \to z(s) \in \Gamma_t$ be a parametrization by arclength of Γ_t such that $z(0) = z$ and let $\omega = z'(0)$ be the tangent unit vector to Γ_t at z. The function defined by

$$f(s) = \frac{u(z(s)) - u(w)}{|z(s) - w|}$$

has a relative maximum at $s = 0$ and hence $f'(0) = 0$; thus,

$$\frac{\langle Du(z), \omega\rangle}{|z - w|} = \frac{u(z) - u(w)}{|z - w|}\frac{\langle z - w, \omega\rangle}{|z - w|^2}.$$

Therefore, since $\langle z - w, \omega \rangle \neq 0$, we have that

$$[u]_{\Gamma_t} = \frac{\langle Du(z), \omega \rangle}{\langle z - w, \omega \rangle} |z - w|,$$

that gives a contradiction, since the right-hand side increases with z if the angle between $z - w$ and ω decreases.

As a consequence, we infer that

$$[u]_{\Gamma_t} = \lim_{n \to \infty} \frac{u(z_n) - u(w_n)}{|z_n - w_n|} \quad \text{where } z_n, w_n \in \Gamma_t \text{ and } |z_n - w_n| \to 0.$$

Thus, by compactness, we can find a point $z \in \Gamma_t$ such that

$$[u]_{\Gamma_t} = \langle Du(z), \omega \rangle,$$

where ω is the tangent unit vector to Γ_t at z.

It is clear now that $[u]_{\Gamma_t} = |u|_{\Gamma_t}$.

(iii) If (2.4) holds, the maximum and minimum of u on Γ_t are attained at the points on Γ_t whose projections on ∂E respectively maximize and minimize $|Du|$ on ∂E. Thus, (iii) follows at once.

In fact, for a point $z = \gamma(\theta) - t\nu(\theta)$ on Γ_t, calculations give that

$$\begin{aligned} u(z) &= 1 - \langle A\gamma(\theta), \gamma(\theta) \rangle + 2t \langle A\gamma(\theta), \nu(\theta) \rangle - t^2 \langle A\nu(\theta), \nu(\theta) \rangle \\ &= 2t \langle A\gamma(\theta), \nu(\theta) \rangle - t^2 \langle A\nu(\theta), \nu(\theta) \rangle, \end{aligned}$$

where

$$\begin{aligned} \langle A\gamma(\theta), \nu(\theta) \rangle &= \frac{1}{ab} \sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}; \\ \langle A\nu(\theta), \nu(\theta) \rangle &= \frac{1}{a^2 b^2} \frac{b^4 \cos^2 \theta + a^4 \sin^2 \theta}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}, \end{aligned}$$

so that, by the substitution $\xi = \sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}$, we obtain that

$$u(z) = \frac{2t}{ab} \xi + \frac{t^2}{\xi^2} - (a^{-2} + b^{-2}) t^2.$$

Since (2.4) holds, this function is respectively maximal or minimal when $\xi = \min(a, b)$ or $\max(a, b)$. $\square$

Therefore, for an ellipse E, [6, Theorem 1.1] can be stated as follows, together with two analogues.

THEOREM 2.2. *Let u be the solution of* (1.1) *in an ellipse E of semi-axes a and b. Let Γ_t be the curve* (2.3) *parallel to ∂E at distance t satisfying* (2.4).

Then, there are two concentric balls B_{r_i} and B_{r_e} such that $B_{r_i} \subset E \subset B_{r_e}$ and

$$r_e - r_i = \frac{1}{t}\frac{a^2 b^2}{a+b}|u|_{\Gamma_t}; \quad r_e - r_i = \frac{1}{t}\frac{a^2 b^2}{a+b}[u]_{\Gamma_t};$$

$$r_e - r_i = \frac{1}{t}\frac{a^2 b^2}{2ab - (a+b)t} \operatorname*{osc}_{\Gamma_t} u.$$

Proof. The largest ball contained in E and the smallest ball containing E are centered at the origin and have radii $\min(a,b)$ and $\max(a,b)$, respectively; hence, $r_e - r_i = |a-b|$ and the desired formulas follow from Lemma 2.1. □

Now, we turn to Serrin problem (1.1)–(1.2). The following lemma holds for quite general domains in general dimension.

Lemma 2.3. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with boundary of class C^2 and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of* (1.1) *satisfying* (1.2).
Then

$$\operatorname*{osc}_{\Gamma_t} u = o(t) \quad \text{as } t \to 0^+.$$

If, in addition, $u \in C^2(\overline{\Omega})$, then

$$[u]_{\Gamma_t} \quad \text{and} \quad |u|_{\Gamma_t} = o(t) \quad \text{as } t \to 0^+.$$

Proof. Let z and $z' \in \Gamma_t$ points at which u attains its maximum and minimum, respectively (for notational semplicity, we do not indicate their dependence on t). If t is sufficiently small, they have unique projections, say γ and γ', on $\partial\Omega$, so that we can write that $z = \gamma - t\nu(\gamma)$ and $z = \gamma' - t\nu(\gamma')$.
Since both u and u_ν are constant on $\partial\Omega$, Taylor's formula gives:

$$u(z) - u(z') = \int_0^t [\langle Du(\gamma' - \tau\nu(\gamma')), \nu(\gamma')\rangle - \langle Du(\gamma - \tau\nu(\gamma)), \nu(\gamma)\rangle]\, d\tau.$$

By the (uniform) continuity of the first derivatives of u (and the normals), the right-hand side of the last identity is a $o(t)$ as $t \to 0^+$.
We shall prove the second part of the theorem only for the semi-norm $[u]_{\Gamma_t}$, since that for $|u|_{\Gamma_t}$ runs similarly.
Let s and $s' \in [0, |\Gamma_t|]$ attain the first supremum in (2.1); we apply (2.6) and obtain that

$$\frac{|u(z(s)) - u(z(s'))|}{\min(|s-s'|, |\Gamma_t| - |s-s'|)} = |\langle Du(z(\sigma)), z'(\sigma)\rangle|,$$

for some $\sigma \in [0, |\Gamma_t|)$. Let $\gamma \in \partial\Omega$ be the projection of the point $z = z(\sigma)$ on $\partial\Omega$, that is $z = \gamma - t\nu(\gamma)$.

Since $\partial\Omega$ and Γ_t are parallel, the tangent unit vector $\tau(\gamma)$ to the curve $\sigma \mapsto \gamma(\sigma) \in \partial\Omega$ at γ equals the tangent unit vector $\tau(z)$ to the curve $\sigma \mapsto z(\sigma) \in \Gamma_t$ at z; the same occurs for the corresponding normal unit vectors $\nu(\gamma)$ and $\nu(z)$.

It is clear that $\langle Du(\gamma), \tau(\gamma)\rangle = 0$ and, since $u \in C^2(\overline{\Omega})$, by differentiating (1.2), we also have that $\langle D^2u(\gamma)\nu(\gamma), \tau(\gamma)\rangle = 0$; thus, by Taylor's formula, we obtain that

$$\langle Du(z(\sigma)), z'(\sigma)\rangle = \langle Du(\gamma), \tau(\gamma)\rangle - t\langle D^2u(\gamma)\nu(\gamma), \tau(\gamma)\rangle + R(s, s', t) = R(s, s', t).$$

Since the second derivatives of u are uniformly continuous on $\overline{\Omega}$, we conclude that the remainder term $R(s, s', t)$ is a $o(t)$ as $t \to 0^+$. □

THEOREM 2.4. *Let E be an ellipse of semi-axes a and b and assume that in E there exists a solution u of* (1.1) *satisfying* (1.2).

Then $a = b$, that is E is a ball and u is radially symmetric.

Proof. Theorem 2.2 and Lemma 2.3 in any case yield that

$$|a - b| = o(1) \quad \text{as } t \to 0^+,$$

which implies the assertion. □

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