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SOME FAMILY OF CENTER MANIFOLDS OF A FIXED INDETERMINATE POINT

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Abstract

In this article, we study the local dynamical structure of a rational mapping F of $\mathbf{P}^2$ at a fixed indeterminate point p. In the previous paper, using a sequence of points which is defined by blow-ups, we have constructed an invariant family of holomorphic curves at p. In this paper, using the same sequence of points, we approximate a set of points whose forward orbits stay in a neighborhood of p. Moreover, for a specific rational mapping we construct a family $\{W_{\mathbf{j}}\}_{\mathbf{j}\in\{1,2\}^{\mathbf{N}}}$ of center manifolds of p. The main result of this paper is to give the asymptotic expansion of the defining function of $W_{\mathbf{j}}$.

1. Introduction

Recently, several authors have researched rational maps on compact complex surfaces. Bedford-Kim [1], Mcmullen [6] and Uehara [12] construct many examples of automorphisms with positive entropy. Diller-Dujardin-Guedj [2], Dinh-Sibony [3], Diller-Farve [5] and others construct invariant currents for good birational maps. These results concern with *global* dynamics of rational maps.

In this paper, we study the *local* dynamical structure of a rational mapping F of the two-dimensional complex projective space $\mathbf{P}^2$ at an indeterminate point p. To say that p is a *fixed indeterminate point* means that F blows up p to a variety which contains p. It is remarked here that a fixed indeterminate point p is non-wandering, and we expect that there exists a local dynamical structure. Indeed, Yamagishi [13], [14] and Dinh-Dujardin-Sibony [4] showed that there exists a family $\{W_{\mathbf{j}}\}_{\mathbf{j}\in\{1,2\}^{\mathbf{N}}}$ of uncountably many currents or stable manifolds of p, which comprise what is called a *Cantor bouquet* of p.

On the other hand, we have constructed a Cantor bouquet by another method in [10]. By using a sequence of points $\{p_{j_1\cdots j_n}\}$ which is defined by blow-ups, we construct a family $\{W_{\mathbf{j}}\}_{\mathbf{j}\in J}$ of holomorphic curves at the point p, where

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J is a subset of a Cantor set $\{1,2\}^{\mathbf{N}}$. In [10], for the following rational mapping of $\mathbf{C}^2$:

$$F(x_1, x_2) = \left(ax_1, \frac{x_2(x_2 - x_1)}{x_1^2}\right) \quad \text{with } |a| > 4,$$

we showed that J is a proper subset of $\{1,2\}^{\mathbf{N}}$ and every $W_{\mathbf{j}}$ is an unstable manifold of p. Hence, our $\{W_{\mathbf{j}}\}_{\mathbf{j} \in J}$ is a generalization of a Cantor bouquet. Moreover, we construct an invariant surface for a rational mapping F of $\mathbf{P}^n$ which has a set I of indeterminate points with $\dim I = n - 2$ in [11].

In this paper, by using the blow-ups in the same way as in [10], we approximate a set of points whose all forward orbits stay in a neighborhood of a fixed indeterminate point p (see Theorem 2.2). As a prototypical example, we consider the following rational mapping F of $\mathbf{C}^2$:

$$F(x_1, x_2) = \left(x_1 + ax_1^2, \frac{x_2(2x_2 - 1)}{x_1^2}\right) \quad \text{with } a \neq 0$$

and construct $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$, which is a family of center manifolds of p (see Theorem 2.3). In [10], by using the sequence of points $\{p_{j_1 \cdots j_n}\}$, for every symbol sequence $\mathbf{j} \in \{1,2\}^{\mathbf{N}}$ we define a formal power series $\varphi_{\mathbf{j}}$ and we show that if a family $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$ of holomorphic curves is locally invariant at p, then every $\varphi_{\mathbf{j}}$ is a convergent power series and $W_{\mathbf{j}}$ is given by the graph of $\varphi_{\mathbf{j}}$. In general, it is known that the defining function $\psi_{\mathbf{j}}$ of a center manifold $W_{\mathbf{j}}$ is not always analytic. The main result of this paper is to show that the formal power series $\varphi_{\mathbf{j}}$ is the asymptotic expansion of $\psi_{\mathbf{j}}$ whether $\varphi_{\mathbf{j}}$ is a convergent power series or not (see Theorem 2.4).

This paper is organized as follows. In Section 2, we state some preliminary facts and our main theorems. Section 3 is devoted to the proof of Theorem 2.2. In the final section, Section 4, we construct the family $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$ of center manifolds of p for a given rational mapping F.

2. Preliminaries and main theorems

In this section, we fix the notation which will be used throughout this paper, and state our main theorems. Firstly, we fix once and for all a homogeneous coordinate system $[x_0 : x_1 : x_2]$ in $\mathbf{P}^2$; we shall often use the natural identification given by

$$\mathbf{C}^2 = \{[x_0 : x_1 : x_2] \in \mathbf{P}^2 \mid x_0 \neq 0\} \quad \text{and} \quad (x_1, x_2) = [1 : x_1 : x_2].$$

Consider the product space $\mathbf{C}^2 \times \mathbf{P}^1$ and define the subvariety $X \subset \mathbf{C}^2 \times \mathbf{P}^1$ as the following:

$$X := \{(x_1, x_2) \times [l_1 : l_2] \in \mathbf{C}^2 \times \mathbf{P}^1 \mid x_1 l_2 = (x_2 - \alpha) l_1\}$$

for the point $p = (0, \alpha) \in \mathbf{C}^2$.

Definition 2.1. The mapping $\pi : X \to \mathbf{C}^2$ defined by restricting the first projection $\mathbf{C}^2 \times \mathbf{P}^1 \to \mathbf{C}^2$ to X is called the blow-up of $\mathbf{C}^2$ centered at p.

It follows from the definition that $\pi^{-1}(p) = \{p\} \times \mathbf{P}^1$ and that

$$\pi : X \backslash \pi^{-1}(p) \to \mathbf{C}^2 \backslash \{p\} \text{ is biholomorphic.}$$

Put $E := \pi^{-1}(p)$; E is called the *exceptional curve*. Let us describe the structure of X. Define

$$U^i := \{(x_1, x_2) \times [l_1 : l_2] \in X \mid l_i \neq 0\} \quad \text{for } i = 1, 2.$$

Then, U^i is biholomorphic to the affine space $\mathbf{C}^2$ by the following maps:

$$\mu^1 : U^1 \ni (x_1, x_2) \times [l_1 : l_2] \mapsto (x_1, l_2/l_1) \in \mathbf{C}^2,$$
$$\mu^2 : U^2 \ni (x_1, x_2) \times [l_1 : l_2] \mapsto (l_1/l_2, x_2) \in \mathbf{C}^2.$$

Hence, $\{(U^i, \mu^i)\}_{i=1,2}$ gives a local chart of X. Let $(x_1, \tilde{x}_2)$ and $(\tilde{x}_1, x_2)$ be local coordinates on U^1 and U^2, respectively.

Proposition 2.1. *We have the following*:
(1) $\pi|_{U^1} : U^1 \ni (x_1, \tilde{x}_2) \mapsto (x_1, x_1\tilde{x}_2 + \alpha) \in \mathbf{C}^2$.
(2) $\pi|_{U^2} : U^2 \ni (\tilde{x}_1, x_2) \mapsto (\tilde{x}_1(x_2 - \alpha), x_2) \in \mathbf{C}^2$.
(3) $X \backslash U^1 = \{(\tilde{x}_1, x_2) \in U^2 \mid \tilde{x}_1 = 0\}$.
(4) $E \cap U^1 = \{(x_1, \tilde{x}_2) \in U^1 \mid x_1 = 0\}$, $E \cap U^2 = \{(\tilde{x}_1, x_2) \in U^2 \mid x_2 = \alpha\}$.
(5) $E \cap (U^2 \backslash U^1) = \{(\tilde{x}_1, x_2) = (0, \alpha) \in U^2\}$.

Proof. For $(x_1, \tilde{x}_2) \in U^1 \cong \mathbf{C}^2$,

$$\pi \circ (\mu^1)^{-1}(x_1, \tilde{x}_2) = \pi((x_1, x_1\tilde{x}_2 + \alpha) \times [1 : \tilde{x}_2]) = (x_1, x_1\tilde{x}_2 + \alpha).$$

By a similar discussion, we have the other claims. □

By pasting $\mathbf{C}^2 = \{[x_0 : x_1 : x_2] \in \mathbf{P}^2 \mid x_0 \neq 0\}$ on the other charts of $\mathbf{P}^2$, we obtain the blow-up of $\mathbf{P}^2$ centered at $[1 : 0 : \alpha]$. To simplify our notation, we denote this also by $\pi : X \to \mathbf{P}^2$. In this paper, let $F : \mathbf{P}^2 \to \mathbf{P}^2$ be a rational mapping with an indetermianate point $p = [1 : 0 : 0]$ and concentrate our attention on the dynamics of F in the chart

$$\mathbf{C}^2 = \{[x_0 : x_1 : x_2] \in \mathbf{P}^2 \mid x_0 \neq 0\}.$$

Remark that $p = (0, 0)$ is our indeterminate point on $\mathbf{C}^2$. Put the space of symbol sequences

$$\{1, 2\}^{\mathbf{N}} := \{\mathbf{j} = (j_1, j_2, \ldots) \mid j_n = 1 \text{ or } 2, n \in \mathbf{N}\}.$$

Let us define a rational mapping

$$\tilde{F} : X \to \mathbf{P}^2 \quad by\ \tilde{F} := F \circ \pi,$$

where π is the blow-up centered at $p = (0,0)$. *In this paper, we assume that* $\tilde{F}$ *satisfies the following condition* (A.0), *see Figure* 1:

$$(A.0)\begin{cases} (1)\ \textit{For any point } q \in E, \textit{ there exists an open neighborhood } N \textit{ of } q \\ \textit{such that } \tilde{F} \textit{ is holomorphic on } N \\ (2)\ \tilde{F}^{-1}(p) \cap E \textit{ consists of two points } p_{j_1}\ (j_1 = 1,2) \textit{ and} \\ (3)\ \textit{there exists an open neighborhood } N_{j_1} \textit{ of } p_{j_1}\ (j_1 = 1,2) \\ \textit{such that } \tilde{F} \textit{ is biholomorphic on } N_{j_1}. \end{cases}$$

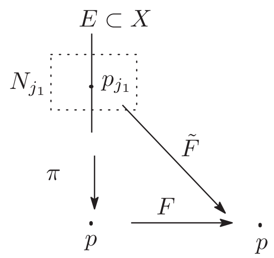

FIGURE 1

Remark 2.1. (2) of condition $(A.0)$ implies that p is a fixed indeterminate point of F. If $\tilde{F}^{-1}(p) \cap E$ consists of finite points p_{j_1} $(j_1 = 1, 2, \dots k)$, then we can show a similar result, for the space of symbol sequences $\{1, 2, \dots, k\}^{\mathbf{N}}$, in exactly the same way.

Remark 2.2. In [13], Yamagishi showed that if F satisfies $(A.0)$ and contracts some open neighborhood N_p of p in some direction, then there exists a family $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$ of uncountably many local stable manifolds of p. $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$ is called a Cantor bouquet of p.

By (4) and (5) of Proposition 2.1, if $p_{j_1} \in E \cap U^1$, then we can put $p_{j_1} = (0, \alpha_{j_1}) \in U^1$ for some α_{j_1}. If $p_{j_1} \in E \backslash U^1$, then we have $p_{j_1} = (0,0) \in U^2$. In either case, we can put $p_{j_1} = (0, \alpha_{j_1})$ in some chart U^k for $k = 1$ or 2. Together with the identification $U^k \cong \mathbf{C}^2$, for $p_{j_1} \in U^k$, we define the subvariety

$$X_{j_1} := \{(z_1, z_2) \times [l_1 : l_2] \in U^k \times \mathbf{P}^1 \mid z_1 l_2 = (z_2 - \alpha_{j_1}) l_1\}$$

with the local chart $\{(U^i_{j_1}, \mu^i_{j_1})\}_{i=1,2}$ of X_{j_1}, the blow-up $\pi_{j_1} : X_{j_1} \to U^k$ centered at p_{j_1}, and the exceptional curve $E_{j_1} := \pi_{j_1}^{-1}(p_{j_1})$ analogous to the definitions for X, $\{(U^i, \mu^i)\}_{i=1,2}$, π and E. Moreover, by pasting the chart U^k which contains p_{j_1} on the other charts of X, we obtain the blow-up $\pi_{j_1} : X_{j_1} \to X$. In [10], we have shown that there exists a sequence of infinitely many blow-ups for rational mappings $F : \mathbf{P}^2 \to \mathbf{P}^2$ satisfying $(A.0)$. To state our main theorems, we introduce the construction of blow-ups (see Figure 2).

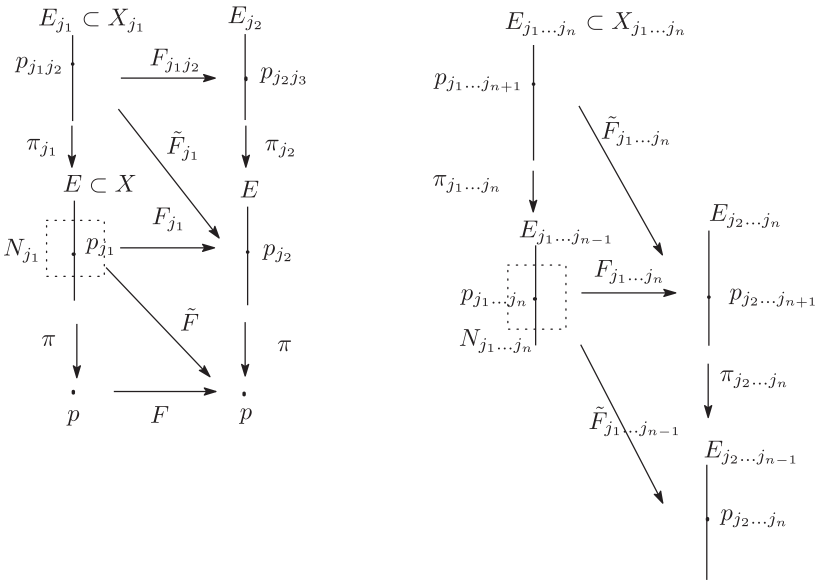

FIGURE 2

Step 1. (1) Define a rational mapping

$$F_{j_1} := \pi^{-1} \circ \tilde{F} : N_{j_1} \to X.$$

Then, from the definition of π, the point p_{j_1} is an indeterminate point of F_{j_1}.

(2) Define a rational mapping

$$\tilde{F}_{j_1} := F_{j_1} \circ \pi_{j_1} : \pi_{j_1}^{-1}(N_{j_1}) \to X,$$

where $\pi_{j_1} : X_{j_1} \to X$ is the blow-up of X centered at p_{j_1}.

(3) It follows that $\tilde{F}_{j_1}|_{E_{j_1}} : E_{j_1} \to E$ is bijective and put $p_{j_1 j_2} := \tilde{F}_{j_1}^{-1}(p_{j_2}) \in E_{j_1}$.

Then, there exists an open neighborhood $N_{j_1 j_2}$ of $p_{j_1 j_2}$ such that $\tilde{F}_{j_1}|_{N_{j_1 j_2}}$ is biholomorphic.

Repeat this process inductively and define the following (see Figure 2):

Step n. For $n \in \mathbf{N}$ with $n \geq 2$,

(1) define a rational mapping

$$F_{j_1 \cdots j_n} := \pi_{j_2 \cdots j_n}^{-1} \circ \tilde{F}_{j_1 \cdots j_{n-1}} : N_{j_1 \cdots j_n} \to X_{j_2 \cdots j_n},$$

(2) define a rational mapping

$$\tilde{F}_{j_1 \cdots j_n} := F_{j_1 \cdots j_n} \circ \pi_{j_1 \cdots j_n} : \pi_{j_1 \cdots j_n}^{-1}(N_{j_1 \cdots j_n}) \to X_{j_2 \cdots j_n},$$

where $\pi_{j_1 \cdots j_n} : X_{j_1 \cdots j_n} \to X_{j_1 \cdots j_{n-1}}$ is the blow-up centered at $p_{j_1 \cdots j_n}$ and $E_{j_1 \cdots j_n}$ is the exceptional curve of $X_{j_1 \cdots j_n}$. Then, we have the following theorem.

Theorem 2.1 ([10, Theorem 2.2]). *Assume that a rational mapping F with the indeterminate point p satisfies the condition* $(A.0)$. *Then, for every $n \in \mathbf{N}$, $j_n = 1, 2$, there exists a sequence of points*

$$p_{j_1 \cdots j_{n+1}} := \tilde{F}^{-1}_{j_1 \cdots j_n}(p_{j_2 \cdots j_{n+1}}) \in E_{j_1 \cdots j_n}.$$

Moreover, there exist open neighborhoods $N_{j_1 \cdots j_{n+1}}$ of $p_{j_1 \cdots j_{n+1}}$ and $\tilde{N}_{j_2 \cdots j_{n+1}}$ of $p_{j_2 \cdots j_{n+1}}$ such that $\tilde{F}_{j_1 \cdots j_n}|_{N_{j_1 \cdots j_{n+1}}} : N_{j_1 \cdots j_{n+1}} \to \tilde{N}_{j_2 \cdots j_{n+1}}$ is biholomorphic.

For any open neighborhood N_p of p, define

$$\Lambda := \bigcap_{k \geq 1}^{\infty} (\pi \circ \tilde{F}^{-1})^k (N_p) \cap N_p.$$

It is clear from the definition that Λ is the set of points whose all forward orbits stay in N_p.

Proposition 2.2. *For any point $q \in \Lambda \backslash \bigcup_{k \geq 0}^{\infty} F^{-k}(p)$, $F^k(q) \in N_p$ for all $k \geq 1$.*

In [13] and [14], Yamagishi showed that if F satisfies a stability condition at p then there exists a Cantor bouquet $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$ which consists of local stable manifolds $W_{\mathbf{j}}$ of p. It follows from the definition of local stable manifolds that

$$\bigcup_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}} W_{\mathbf{j}} \subset \Lambda \quad \textit{for some open neighborhood } N_p \textit{ of } p.$$

Hence, Λ is a generalization of a Cantor bouquet for a fixed indeterminate point p and the main purpose of this paper is to describe the shape of Λ. To do this, we need the following condition:

$$(A.1) \begin{cases} p_{j_1} \in U^1 \cap E \ \textit{and} \ p_{j_1 \cdots j_{n+1}} \in U^1_{j_1 \cdots j_n} \cap E_{j_1 \cdots j_n} \\ \textit{for any } n \in \mathbf{N}, \ j_n = 1, 2 \end{cases}$$

where $U^1_{j_1 \cdots j_n}$ is the local chart of $X_{j_1 \cdots j_n}$ which is defined by

$$U^1_{j_1 \cdots j_n} := \{(z_1, z_2) \times [l_1 : l_2] \in X_{j_1 \cdots j_n} \mid l_1 \neq 0\}.$$

By using this chart, for any symbol sequence $\mathbf{j} = (j_1, \ldots j_n, \ldots) \in \{1,2\}^{\mathbf{N}}$, there exists a sequence of complex numbers $\alpha_{j_1 \cdots j_{n+1}} \in \mathbf{C}$ such that $p_{j_1 \cdots j_{n+1}} = (0, \alpha_{j_1 \cdots j_{n+1}}) \in U^1_{j_1 \cdots j_n}$ for any n. By using this sequence $\{\alpha_{j_1 \cdots j_n}\}$, for any $n \in \mathbf{N}$, $j_n = 1, 2$ define a polynomial

$$\varphi_{j_1 \cdots j_n}(x_1) := \alpha_{j_1} x_1 + \alpha_{j_1 j_2} x_1^2 + \cdots + \alpha_{j_1 \cdots j_n} x_1^n,$$

and a polydisk of radius ε with center $p_{j_1 \cdots j_{n+1}}$

$$\Delta^2_{j_1 \cdots j_{n+1}}(\varepsilon) := \{(z_1, z_2) \in U^1_{j_1 \cdots j_n} \mid |z_1| < \varepsilon, |z_2 - \alpha_{j_1 \cdots j_{n+1}}| < \varepsilon\},$$

for some positive constant ε. Then, it follows from the definition of a blow-up $\pi_{j_1 \cdots j_n}|_{U^1_{j_1 \cdots j_n}}$ in (1) of Proposition 2.1 that

$$(*1) \quad \pi \circ \pi_{j_1} \circ \cdots \circ \pi_{j_1 \cdots j_n}(\Delta^2_{j_1 \cdots j_{n+1}}(\varepsilon))$$
$$= \{(x_1, x_2) \in \mathbf{C}^2 \mid |x_1| < \varepsilon, |x_2 - \varphi_{j_1 \cdots j_{n+1}}(x_1)| < \varepsilon |x_1|^{n+1}\}.$$

Put $\Lambda_{j_1 \cdots j_{n+1}}(\varepsilon)$ equal to the right-hand side of $(*1)$. Then, we have the following theorem.

THEOREM 2.2. *Let F be a rational mapping satisfying the conditions* $(A.0)$ *and* $(A.1)$. *For any $n \in \mathbf{N}$ and for any sufficiently small open neighborhood of N_p of p, there exists a constant $\varepsilon > 0$ such that*

$$p \in \Lambda \subset \bigcup_{j_1 \cdots j_{n+1} = 1, 2} \Lambda_{j_1 \cdots j_{n+1}}(\varepsilon).$$

Remark 2.3. For every $\mathbf{j} \in \{1, 2\}^{\mathbf{N}}$, put the formal power series $\varphi_{\mathbf{j}}(x_1) := \sum \alpha_{j_1 \cdots j_n} x_1^n$. In [10], we show that if a family $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$ of holomorphic curves is locally invariant at p, then every $\varphi_{\mathbf{j}}$ is a convergent power series and every holomorphic curve $W_{\mathbf{j}}$ has the following form:

$$W_{\mathbf{j}} = \{(x_1, x_2) \in \mathbf{C}^2 \mid |x_1| < \delta_{\mathbf{j}}, x_2 = \varphi_{\mathbf{j}}(x_1)\},$$

where $\delta_{\mathbf{j}}$ is a radius of the domain of definition of $\varphi_{\mathbf{j}}$. On the other hand, in Theorem 2.2, we approximate Λ by the set $\Lambda_{j_1 \cdots j_{n+1}}(\varepsilon)$ whether $\varphi_{\mathbf{j}}$ is a convergent power series and Λ consists of holomorphic curves or not.

As a prototypical example, consider the following rational mapping of $\mathbf{C}^2$:

$$(*2) \quad F(x_1, x_2) = \left(x_1 + a x_1^2, \frac{x_2(2x_2 - x_1)}{x_1^2} \right) \quad \text{with } a \neq 0.$$

Our F satisfies conditions (A.0) and (A.1); therefore, Theorems 2.1 and 2.2 can be applied for F. In particular, $\tilde{F}$ is locally biholomorphic at p_{j_1}, and we put G_{j_1} equal to the inverse branch of $\tilde{F}$ with $G_{j_1}(p) = p_{j_1}$. Then, define a graph transformation Γ_{j_1} $(j_1 = 1, 2)$ on some appropriate function space. By the contraction mapping principle, we have the following theorems.

THEOREM 2.3. *Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be the rational mapping as in* $(*2)$. *For every symbol sequence $\mathbf{j} \in \{1, 2\}^{\mathbf{N}}$, there exists a continuous function $x_2 = \psi_{\mathbf{j}}(x_1)$ on some disk $\Delta(\delta) := \{x_1 \in \mathbf{C} \mid |x_1| < \delta\}$ satisfies the following conditions*:

Put

$$W_{\mathbf{j}} := \{(x_1, x_2) \in \Delta(\delta) \times \mathbf{C} \mid x_2 = \psi_{\mathbf{j}}(x_1), x_1 \in \Delta(\delta)\}.$$

The family $\{W_{\mathbf{j}}\}_{\mathbf{j}\in\{1,2\}^{\mathbf{N}}}$ *is invariant with respect to* F *at* p. *Here, to say* $\{W_{\mathbf{j}}\}_{\mathbf{j}\in\{1,2\}^{\mathbf{N}}}$ *is invariant with respect to* F *at* p *means that for any symbol sequence* $\mathbf{j}\in\{1,2\}^{\mathbf{N}}$, *there exists some open neighborhood* $N_{\mathbf{j}}$ *of* p *such that*

$$p \in \pi \circ G_{j_1}(W_{\sigma(\mathbf{j})}) \cap N_{\mathbf{j}} \subset W_{\mathbf{j}},$$

where $\sigma : \{1,2\}^{\mathbf{N}} \to \{1,2\}^{\mathbf{N}}$ *is the shift operator.*

THEOREM 2.4. *Let* $F : \mathbf{C}^2 \to \mathbf{C}^2$ *be the rational mapping as in* $(*2)$. *For every symbol sequence* $\mathbf{j}\in\{1,2\}^{\mathbf{N}}$, *the formal power series* $\varphi_{\mathbf{j}}(x_1) = \sum \alpha_{j_1\cdots j_n} x_1^n$ *is an asymptotic expansion of the continuous function* $\psi_{\mathbf{j}}(x_1)$ *in Theorem* 2.3. *Here, to say* $\varphi_{\mathbf{j}}(x_1)$ *is an asymptotic expansion of* $\psi_{\mathbf{j}}(x_1)$ *means that for any* $n\in\mathbf{N}$, *there exist some constants* $\delta_n > 0$ *and* $M_n > 0$ *such that for any* $x_1 \in \Delta(\delta_n)$,

$$|\psi_{\mathbf{j}}(x_1) - \alpha_{j_1}x_1 - \cdots - \alpha_{j_1\cdots j_{n-1}}x_1^{n-1}| \le M_n|x_1|^n.$$

Remark 2.4. Since the first component of F is $q(x_1) := x_1 + ax_1^2$, $q(x_1)$ has attracting and repelling regions on the x_1 plane whose boundary contains 0 (for details, see [8]). Therefore, $W_{\mathbf{j}}$ contains not only a local stable set but also a local unstable set of p. Hence, our $\{W_{\mathbf{j}}\}_{\mathbf{j}\in\{1,2\}^{\mathbf{N}}}$ is a generalization of a Cantor bouquet.

3. Proof of Theorem 2.2

To prove Theorem 2.2, we proceed by induction on n (see Figure 3).

By Theorem 2.1, $\tilde{F}_{j_1\cdots j_n}$ is biholomorphic at an open neighborhood of $p_{j_1\cdots j_{n+1}}$. Together with the fact $\tilde{F}_{j_1\cdots j_n}(p_{j_1\cdots j_{n+1}}) = p_{j_2\cdots j_{n+1}}$, one can choose a sequence of open neighborhoods $N_{j_1\cdots j_n}$ of $p_{j_1\cdots j_n}$ and $\tilde{N}_{j_2\cdots j_{n+1}}$ of $p_{j_2\cdots j_{n+1}}$ such that $\tilde{F}_{j_1\cdots j_n}(N_{j_1\cdots j_{n+1}}) = \tilde{N}_{j_2\cdots j_{n+1}}$. Hence, for any $n\in\mathbf{N}$ and for any sufficiently small open neighborhood N_p of p, there exists an open neighborhood $\tilde{N}_{j_1\cdots j_{n+1}}$ of $p_{j_1\cdots j_{n+1}}$ such that

$$\tilde{F}\circ\cdots\circ\tilde{F}_{j_2\cdots j_n}\circ\tilde{F}_{j_1\cdots j_n}(\tilde{N}_{j_1\cdots j_{n+1}}) = N_p,$$
$$(\pi\circ\tilde{F}^{-1})^{n+1}(N_p)\cap N_p = \bigcup_{j_1\cdots j_{n+1}=1,2} \pi\circ\cdots\circ\pi_{j_1\cdots j_{n+1}}(\tilde{N}_{j_1\cdots j_{n+1}})\cap N_p.$$

Then, it is easy to see from the definition that

$$\bigcap_{k\ge 1}^{\infty}(\pi\circ\tilde{F}^{-1})^k(N_p)\cap N_p \subset (\pi\circ\tilde{F}^{-1})^{n+1}(N_p)\cap N_p,$$

$$\Lambda = \bigcap_{k\ge 1}^{\infty}(\pi\circ\tilde{F}^{-1})^k(N_p)\cap N_p \subset \bigcup_{j_1\cdots j_{n+1}=1,2}\pi\circ\cdots\circ\pi_{j_1\cdots j_n}(\tilde{N}_{j_1\cdots j_{n+1}})\cap N_p.$$

Take a positive constant $\varepsilon > 0$ satisfying $\tilde{N}_{j_1\cdots j_{n+1}} \subset \Delta^2_{j_1\cdots j_{n+1}}(\varepsilon)$ for any $j_1,\ldots,j_{n+1} = 1,2$. Then, Theorem 2.2 is proved.

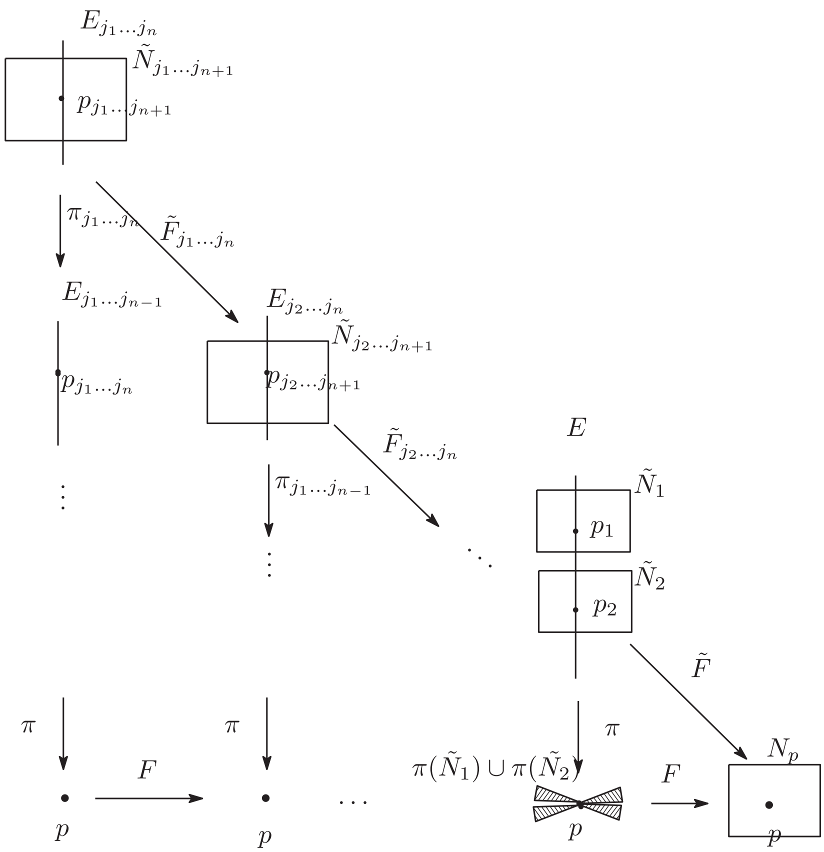

Figure 3

4. Proof of Theorems 2.3 and 2.4

In this section, as an application consider the following rational map of $\mathbf{C}^2$:

$$F(x_1, x_2) = \left(x_1 + a x_1^2, \frac{x_2(2x_2 - x_1)}{x_1^2} \right)$$

with $a \neq 0$. Now, let us start the proof of Theorem 2.3. In the following part, we shall give a proof which is based on an argument by Hadamard–Perron Theorem in [7, Theorem 6.2.8] and the construction of the Cantor bouquet in [13].

From some easy calculations, one can check that our F satisfies the conditions $(A.0)$ and $(A.1)$. Hence, Theorems 2.1 and 2.2 can be applied for F, and for any infinite symbol sequence $\mathbf{j} = (j_1, j_2, \dots) \in \{1, 2\}^{\mathbf{N}}$, there exists the sequence of points $\{p_{j_1\cdots j_n}\}$ such that $p_{j_1\cdots j_n} = (0, \alpha_{j_1\cdots j_n}) \in U^1_{j_1\cdots j_{n-1}}$. *In the rest of this paper, we identify U^1 which is the local chart of X with $\mathbf{C}^2$.*

Since the condition $(A.0)$ holds, $\tilde{F}$ is a locally biholomorphic mapping on some neighborhoods of p_{j_2}, and there are positive constants r, r', and inverse branches $G_{j_2} : \Delta^2(r) \to \Delta^2_{j_2}(r')$ of $\tilde{F}$, where $\Delta^2(r) := \Delta(r) \times \Delta(r)$. Let $\rho : \mathbf{C}^2 \to [0,1]$ be a C^1-function such that

$$\rho(z_1, z_2) = \begin{cases} 1 & \text{on } \Delta^2_{j_1}(r) \\ 0 & \text{on } \mathbf{C}^2 \backslash \Delta^2_{j_1}(2r). \end{cases}$$

Moreover, it follows from the (1) of Proposition 2.1 that π has the following form on the chart U^1

$$\pi(z_1, z_2) = (z_1, z_1 z_2).$$

Let $\pi_{p_{j_1}}$ be the Taylor expansion of π at $p_{j_1} = (0, \alpha_{j_1})$ and its has the following form:

$$\pi_{p_{j_1}}(z_1, z_2) = (z_1, \alpha_2 z_1 + z_1(z_2 - \alpha_2)).$$

By using ρ and $\pi_{p_{j_1}}$, define the C^1-mapping $g_{j_1 j_2} : \mathbf{C}^2 \to \mathbf{C}^2$

$$\begin{aligned} g_{j_1 j_2} &:= \rho \times (G_{j_2} \circ \pi_{p_{j_1}}) + (1 - \rho) \times \{(0, \alpha_{j_2}) + J(G_{j_2} \circ \pi_{p_{j_1}})\} \\ &= (0, \alpha_{j_2}) + J(G_{j_2} \circ \pi_{p_{j_1}}) + \rho \times \{G_{j_2} \circ \pi_{p_{j_1}} - (0, \alpha_{j_2}) - J(G_{j_2} \circ \pi_{p_{j_1}})\} \end{aligned} \tag{4.1}$$

where $J(G_{j_2} \circ \pi_{p_{j_1}})$ is the Jacobian matrix of $G_{j_2} \circ \pi_{p_{j_1}}$ at the point p_{j_1} (see Figure 4). Then, it follows from the definition that

$$g_{j_1 j_2} = \begin{cases} G_{j_2} \circ \pi_{p_{j_1}} & \text{on } \Delta^2_{j_1}(r) \\ (0, \alpha_{j_1}) + J(G_{j_2} \circ \pi_{p_{j_1}}) & \text{on } \mathbf{C}^2 \backslash \Delta^2_{j_1}(2r), \end{cases}$$

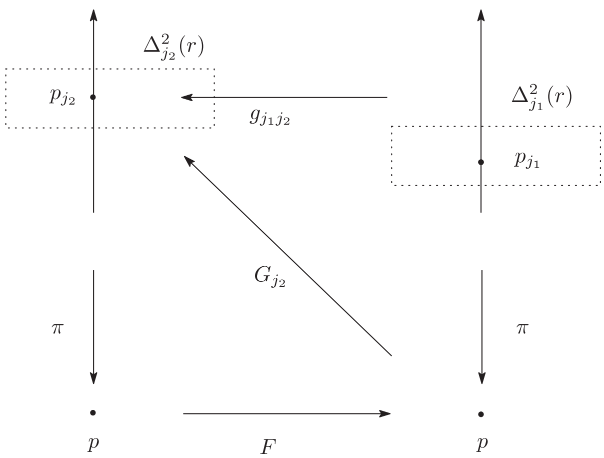

FIGURE 4

Lemma 4.1. *$g_{j_1 j_2}$ have the following form on $U^1 \cong \mathbf{C}^2$.*

$$g_{11}(z_1, z_2) = \left(z_1 + \rho \sum_{n \ge 2} a_n z_1^n, \rho \sum_{n \ge 1} b_n (z_1 z_2)^n \right) \tag{1}$$

$$g_{12}(z_1, z_2) = \left(z_1 + \rho \sum_{n \ge 2} a_n z_1^n, \alpha_2 + \rho \sum_{n \ge 1} b_n (z_1 z_2)^n \right) \tag{2}$$

$$\begin{aligned} g_{21}(z_1, z_2) = \Bigg(& z_1 + \rho \sum_{n \ge 2} a_n z_1^n, -\alpha_2 z_1 \\ & + \rho \left\{ -z_1(z_2 - \alpha_2) + \sum_{n \ge 2, n \ge k \ge 0} b_{nk} z_1^n (z_2 - \alpha_2)^k \right\} \Bigg) \end{aligned} \tag{3}$$

$$\begin{aligned} g_{22}(z_1, z_2) = \Bigg(& z_1 + \rho \sum_{n \ge 2} a_n z_1^n, \alpha_2 + \alpha_2 z_1 \\ & + \rho \left\{ z_1(z_2 - \alpha_2) + \sum_{n \ge 2, n \ge k \ge 0} b_{nk} z_1^n (z_2 - \alpha_2)^k \right\} \Bigg) \end{aligned} \tag{4}$$

Proof. On the chart U^1, $\tilde{F}$ can be written in the form

$$\tilde{F} := F \circ \pi(z_1, z_2) = (z_1 + a z_1^2, 2 z_2^2 - z_2).$$

Therefore, we see that $p_1 = (0, 0)$, $p_2 = (0, 1/2)$, $\alpha_1 = 0$, and $\alpha_2 = 1/2$. By direct calculation,

$$J\tilde{F}_{p_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J\tilde{F}_{p_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and, the Taylor expansion of G_{j_2} at $p = (0, 0)$

$$G_1(z_1, z_2) = \left(z_1 + \sum_{n \ge 2} a_n z_1^n, -z_2 + \sum_{k \ge 2} b_k z_2^k \right),$$

$$G_2(z_1, z_2) = (0, \alpha_2) + \left(z_1 + \sum_{n \ge 2} a_n z_1^n, z_2 + \sum_{k \ge 2} b_k z_2^k \right).$$

Then, it follows from the definitions,

$$G_1 \circ \pi_{p_1}(z_1, z_2) = \left(z_1 + \sum_{n \ge 2} a_n z_1^n, -z_1 z_2 + \sum_{n \ge 2} b_n (z_1 z_2)^n \right),$$

$$G_2 \circ \pi_{p_1}(z_1, z_2) = \left(z_1 + \sum_{n \ge 2} a_n z_1^n, \alpha_2 + z_1 z_2 + \sum_{n \ge 2} b_n (z_1 z_2)^n \right),$$

$$G_1 \circ \pi_{p_2}(z_1, z_2) = \left(z_1 + \sum_{n \geq 2} a_n z_1^n, -(\alpha_2 z_1 + z_1(z_2 - \alpha_2)) \right.$$
$$\left. + \sum_{n \geq 2} b_n (\alpha_2 z_1 + z_1(z_2 - \alpha_2))^n \right),$$
$$G_2 \circ \pi_{p_2}(z_1, z_2) = \left(z_1 + \sum_{n \geq 2} a_n z_1^n, \alpha_2 + (\alpha_2 z_1 + z_1(z_2 - \alpha_2)) \right.$$
$$\left. + \sum_{n \geq 2} b_n (\alpha_2 z_1 + z_1(z_2 - \alpha_2))^n \right).$$

Together with (4.1), we obtain (1) and (2) of Lemma 4.1. On the other hand, the second element of g_{21} has the following form:

$$\text{(4.2)} \quad -\alpha_2 z_1 + \rho \left\{ -z_1(z_1 - \alpha_2) + \sum_{n \geq 2} b_n (\alpha_2 z_1 + z_1(z_2 - \alpha_2))^n \right\}$$
$$= -\alpha_2 z_1 + \rho \left\{ -z_1(z_1 - \alpha_2) + \sum_{n \geq 2} b_n \sum_{k \geq 0}^{n} {}_nC_k (\alpha_2 z_1)^{n-k} (z_1(z_2 - \alpha_2))^k \right\}$$
$$= -\alpha_2 z_1 + \rho \left\{ -z_1(z_1 - \alpha_2) + \sum_{n \geq 2} b_n \sum_{k \geq 0}^{n} {}_nC_k \alpha_2^{n-k} z_1^n (z_2 - \alpha_2)^k \right\}$$

By changing coefficients of this power series,

$$(4.2) = -\alpha_2 z_1 + \rho \left\{ -z_1(z_1 - \alpha_2) + \sum_{n \geq 2, n \geq k \geq 0} b_{nk} z_1^n (z_2 - \alpha_2)^k \right\}.$$

Hence, the claim (3) holds. By similar calculation, we obtain the claim (4). We remark that $\rho = 0$ on $\mathbf{C}^2 \backslash \Delta_{j_1}^2(2r)$ and $g_{j_1 j_2}$ are well-defined on $\mathbf{C}^2$. □

Let γ be a positive constant satisfying $0 < \gamma < 1$ and $C_\gamma^{p_{j_1}}$ be the set of a function $\phi : \mathbf{C} \to \mathbf{C}$ which is Lipshitz continuous with Lipshitz constant $\mathrm{Lip}(\phi) \leq \gamma$ and $\phi(0) = \alpha_{j_1}$,

$$C_\gamma := C_\gamma^{p_1} \cup C_\gamma^{p_2},$$

and define a function $d : C_\gamma \times C_\gamma \to \mathbf{R}$ by

$$d(\phi, \psi) := \begin{cases} \sup_{z_1 \in \mathbf{C} \backslash \{0\}} \dfrac{|\phi(z_1) - \psi(z_1)|}{|z_1|} & \text{if } \phi, \psi \in C_\gamma^{p_k} \\ 3 \quad \text{if } \phi \in C_\gamma^{p_k} \text{ and } \psi \in C_\gamma^{p_l} \ (k \neq l). \end{cases}$$

LEMMA 4.2. *C_γ is a complete metric space with respect to the metric d.*

Proof. Here, we only prove that d satisfies the triangle inequality. If $\phi, \psi \in C_\gamma^{p_1}$ and $\eta \in C_\gamma^{p_2}$, then $d(\phi, \eta) = d(\eta, \psi) = 3$. On the other hand, $d(\phi, \psi) < 2$. Indeed, it implies from the definition of C_γ that $|\phi(z_1) - \phi(0)| < \gamma|z_1|$ and $|\psi(z_1) - \psi(0)| < \gamma|z_1|$ and $\phi(0) = \psi(0) = \alpha_1$. Therefore,

$$d(\phi, \psi) \leq \sup_{z_1 \in \mathbf{C}\setminus\{0\}} \frac{|\phi(z_1) - \alpha_1| + |\psi(z_1) - \alpha_1|}{|z_1|} \leq 2\gamma < 2.$$

Hence, $d(\phi, \psi) \leq d(\phi, \eta) + d(\eta, \psi)$ holds. By similar arguments, the same inequality holds for the other cases. It is easy to check that C_γ is a complete metric space with this metric. □

Next, we define some graph transformation on C_γ. In the following part, we will go along the same line as in [7, Lemma 6.2.16].

Set

$$A(z_1, z_2) := \rho(z_1, z_2) \sum_{n \geq 2} a_n z_1^n,$$

$$A_1(z_1, z_2) := Re(A(z_1, z_2)), \quad A_2(z_1, z_2) := \mathrm{Im}(A(z_1, z_2)),$$

$$B(z_1, z_2) := \rho(z_1, z_2) \sum_{n \geq 1} b_n(z_1 z_2) \quad \text{and} \quad z_l = u_l + iv_l \ (l = 1, 2).$$

Then, we define a mapping

$$\tilde{A}_k : \mathbf{R}^4 \to \mathbf{R} \quad by \ \tilde{A}_k(u_1, v_1, u_2, v_2) := A_k(u_1 + iv_1, u_2 + iv_2)$$

and put $(\tilde{A}_k)_{u_l} := \partial \tilde{A}_k / \partial u_l$, $(\tilde{A}_k)_{v_l} := \partial \tilde{A}_k / \partial v_l$ $(k, l = 1, 2)$.

LEMMA 4.3. *There exist positive constants $r > 0$ and $\delta_0 > 0$ such that*

(1) $\sup_{(z_1, z_2) \in \mathbf{C}^2} |A(z_1, z_2)| < \delta_0 < 1$,

(2) $|A|_{C_1} := \sup_{(u_l, v_l) \in \mathbf{R}^2 k, l = 1, 2} \{|(\tilde{A}_k)_{u_l}|, |(\tilde{A}_k)_{v_l}|\} < \delta_0$,

(3) *For any $\phi \in C_\gamma$ and $z_1, z_1' \in \mathbf{C}$,*

(i) $|A(z_1, \phi(z_1)) - A(z_1', \phi(z_1'))| \leq 8\delta_0(1 + \gamma)|z_1 - z_1'|$ *and*

(ii) $|B(z_1, \phi(z_1)) - B(z_1', \phi(z_1'))| \leq 8\delta_0(1 + \gamma)|z_1 - z_1'|$,

(4) $0 < 8\delta_0(1 + \gamma) < 1$, $\dfrac{8\delta_0(1 + \gamma)}{1 - 8\delta_0(1 + \gamma)} < 1$.

Proof. Since $A(0, 0) = 0$, for any δ_0 with $0 < \delta_0 < 1$ there exists $r > 0$ such that

$$\sup_{(z_1, z_2) \in \mathbf{C}^2} |A(z_1, z_2)| = \sup_{(z_1, z_2) \in \Delta_j^2(2r)} \left| \rho \sum_{n \geq 2} a_n z_1^n \right| \leq \sup_{(z_1, z_2) \in \Delta_j^2(2r)} \left| \sum_{j \geq 2} a_j z_1^j \right| < \delta_0.$$

Then, (1) follows. Since $\sum_{j\geq 2} a_j z_1^j$ does not have linear terms and ρ is a C^1-function, similarly, one can prove (2). To prove (3), first we show that for any $(z_1, z_2), (z_1', z_2') \in \mathbf{C}^2$

$$|A(z_1, z_2) - A(z_1', z_2')| \leq 8|A|_{C_1}|(z_1 - z_1', z_2 - z_2')| \tag{4.3}$$

Indeed, it follows from the triangle inequality that

$$|A(z_1, z_2) - A(z_1', z_2')| \leq \sum_{k=1}^{2} |A_k(z_1, z_2) - A_k(z_1', z_2')| \tag{4.4}$$

Define a mapping $\tilde{\ell}(t) : [0, 1] \to \mathbf{R}^4$ by

$$t \mapsto (u_1' + t(u_1 - u_1'), v_1' + t(v_1 - v_1'), u_2' + t(u_2 - u_2'), v_2' + t(v_2 - v_2'))$$

and $A_{k\tilde{\ell}}(t) := \tilde{A}_k(\tilde{\ell}(t))$. Then, there exists $t_0 \in [0, 1]$ such that

$$\begin{aligned}|A_k(z_1, z_2) - A_k(z_1', z_2')| &= |A_{k\tilde{\ell}}(1) - A_{k\tilde{\ell}}(0)| = |(A_{k\tilde{\ell}})'(t_0)| \\ &\leq \sum_{l=1}^{2} |(\tilde{A}_k)_{u_l}(\tilde{\ell}(t_0))|\,|u_l - u_l'| + |(\tilde{A}_k)_{v_l}(\tilde{\ell}(t_0))|\,|v_l - v_l'|\end{aligned}$$

It follows from the inequalities for $l = 1, 2$

$$|u_l - u_l'|, |v_l - v_l'| \leq |(z_1 - z_1', z_2 - z_2')|$$

that

$$\text{the right-hand side of (4.4)} \leq 8|A|_{C_1}|(z_1 - z_1', z_2 - z_2')|,$$

and (4.3).

Put $z_2 = \phi(z_1)$. Together with the fact $\phi \in C_\gamma$, we prove (i) with respect to A. Similarly, we prove (ii) with respect to B.

From the proof of (1), by rechoosing $r > 0$, we assume that δ_0 satisfies (4). $\square$

For $\phi \in C_\gamma$, define the graph$(\phi) := \{(z_1, z_2) \in \mathbf{C}^2 \mid z_2 = \phi(z_1)\}$ and the map $K_\phi : \mathbf{C} \to \mathbf{C}$ by

$$K_\phi(z_1) := z_1 + \rho(z_1, \phi(z_1)) \sum_{n\geq 2} a_n z_1^n.$$

Then, we have the following lemma.

Lemma 4.4. *For any $\phi \in C_\gamma^{p_1}$, there exists $\psi \in C_\gamma^{p_1}$ such that*

$$g_{11}(\mathrm{graph}(\phi)) = \mathrm{graph}(\psi).$$

Proof. First, we will show that K_ϕ is a bijection. To do this, for any fixed $Z_1 \in \mathbf{C}$, we need to find a unique $z_1 \in \mathbf{C}$ such that $Z_1 = K_\phi(z_1)$, that is,

$$Z_1 = z_1 + \rho(z_1, \phi(z_1)) \sum_{n\geq 2} a_n z_1^n. \tag{4.5}$$

Define the map $\tilde{K}_\phi : \mathbf{C} \to \mathbf{C}$ by

$$\tilde{K}_\phi(z_1) := Z_1 - \rho(z_1, \phi(z_1)) \sum_{n\ge 2} a_n z_1^n.$$

Then, $\tilde{K}_\phi$ is a contracting map. Indeed, it follows from (3) of Lemma 4.3 that for any $z_1, z_1' \in \mathbf{C}$,

$$|\tilde{K}_\phi(z_1) - \tilde{K}_\phi(z_1')| \le |A(z_1, \phi(z_1)) - A(z_1', \phi(z_1'))| \le 8\delta_0(1+\gamma)|z_1 - z_1'|.$$

From (4) of Lemma 4.3, $\tilde{K}_\phi$ is contracting. Thus, by the contraction mapping principle, we see that equation (4.5) has a unique solution and K_ϕ is a bijection. Moreover, there exists a unique function ψ on $\mathbf{C}$ such that

$$g_{11}(\mathrm{graph}(\phi)) = \mathrm{graph}(\psi).$$

Next, we show that ψ is Lipschitz continuous with Lipschitz constant $\mathrm{Lip}(\psi) \le \gamma$. To do this, for any $(z_1, \phi(z_1)), (z_1', \phi(z_1')) \in \mathrm{graph}(\phi)$, set

$$(Z_1, Z_2) := g_{11}(z_1, \phi(z_1)), \quad (Z_1', Z_2') := g_{11}(z_1', \phi(z_1')).$$

It follows from Lemma 4.3 that

$$(4.6)\begin{cases} |Z_2' - Z_2| = |B(z_1', \phi(z_1')) - B(z_1, \phi(z_1))| \le 8\delta_0(1+\gamma)|z_1' - z_1|, \\ |Z_1' - Z_1| \ge |z_1' - z_1| - |A(z_1', \phi(z_1')) - A(z_1, \phi(z_1))| \\ \qquad = |z_1' - z_1| - 8\delta_0(1+\gamma)|z_1' - z_1| = (1 - 8\delta_0(1+\gamma))|z_1' - z_1|. \end{cases}$$

Along with (4.6), we have

$$|Z_2' - Z_2| \le \frac{8\delta_0(1+\gamma)}{1 - 8\delta_0(1+\gamma)}|Z_1' - Z_1|.$$

It follows from (4) of Lemma 4.3 that one can obtain $8\delta_0(1+\gamma)/\{1 - 8\delta_0(1+\gamma)\} < \gamma$. □

From an argument similar to the discussion of Lemma 4.3 and 4.4, we show the same claim for all $g_{j_1 j_2}$ $(j_1, j_2 = 1, 2)$ and define the graph transformation

$$\Gamma_{j_2} : C_\gamma \to C_\gamma^{p_{j_2}} \textit{ by } \phi \mapsto \psi = \Gamma_{j_2}(\phi) \quad \text{with} \quad g_{j_1 j_2}(\mathrm{graph}(\phi)) = \mathrm{graph}(\psi), \textit{ if } \phi \in C_\gamma^{p_{j_1}}.$$

LEMMA 4.5. *Γ_{j_2} is a contraction. Here, to say Γ_{j_2} is a contraction means that there is some constant $0 < \lambda < 1$ such that for any $\phi_1, \phi_2 \in C_\gamma$*

$$d(\Gamma_{j_2}(\phi_1), \Gamma_{j_2}(\phi_2)) \le \lambda d(\phi_1, \phi_2).$$

Proof. Here, only consider the case of Γ_1 and $\phi_i \in C_\gamma^{p_1}$. For any $\phi_i \in C_\gamma^{p_1}$, set $\psi_i := \Gamma_1(\phi_i) \in C_\gamma^{p_1}$ $(i = 1, 2)$. By using the previous estimates in the proof of (3) of Lemma 4.3, we have the following:

$$
\begin{aligned}
(4.7)\qquad & |\psi_1(K_{\phi_1}(z_1)) - \psi_2(K_{\phi_1}(z_1))| \\
& \le |\psi_1(K_{\phi_1}(z_1)) - \psi_2(K_{\phi_2}(z_1))| + |\psi_2(K_{\phi_2}(z_1)) - \psi_2(K_{\phi_1}(z_1))| \\
& \le |B(z_1, \phi_1(z_1)) - B(z_1, \phi_2(z_1))| + \gamma|K_{\phi_2}(z_1) - K_{\phi_1}(z_1)| \\
& \le 8\delta_0|\phi_1(z_1) - \phi_2(z_1)| + \gamma|A(z_1, \phi_2(z_1)) - A(z_1, \phi_1(z_1))| \\
& \le 8\delta_0|\phi_1(z_1) - \phi_2(z_1)| + 8\gamma\delta_0|\phi_2(z_1) - \phi_1(z_1)| \\
& = 8\delta_0(1+\gamma)|\phi_1(z_1) - \phi_2(z_1)|.
\end{aligned}
$$

Similarly, it follows from the fact $A(0, \phi(0)) = 0$ that

$$
\begin{aligned}
|K_{\phi_1}(z_1)| &= |z_1 + A(z_1, \phi_1(z_1))| \ge |z_1| - |A(z_1, \phi_1(z_1)) - A(0, \phi_1(0))| \\
&\ge |z_1| - 8\delta_0(1+\gamma)|z_1| = (1 - 8\delta_0(1+\gamma))|z_1|.
\end{aligned}
$$

From this together with (4.7), we show the following:

$$
\begin{aligned}
d(\psi_1, \psi_2) &= \sup_{z \in \mathbf{C}\setminus\{0\}} \frac{|\psi_1(K_{\phi_1}(z_1)) - \psi_2(K_{\phi_1}(z_1))|}{|K_{\phi_1(z_1)}|} \\
&\le \frac{8\delta_0(1+\gamma)}{1 - 8\delta_0(1+\gamma)} \sup_{z \in \mathbf{C}\setminus\{0\}} \frac{|\phi_1(z_1) - \phi_2(z_1)|}{|z_1|} \le \lambda d(\phi_1, \phi_2),
\end{aligned}
$$

where $\lambda := 8\delta_0(1+\gamma)/\{1 - 8\delta_0(1+\gamma)\}$. It follows from (4) of Lemma 4.3 that $0 < \lambda < 1$. □

Let S be the space of non-empty compact subsets of C_γ. Then, S is a complete metric space with respect to the Hausdorff metric. Defining a mapping

$$H : S \to S, \quad \text{by } \tilde{S} \mapsto H(\tilde{S}) := \Gamma_1(\tilde{S}) \cup \Gamma_2(\tilde{S}),$$

we see that H is a contraction on S, since Γ_j is a contraction mapping.

Thus, it follows from the contraction mapping principle that H has the unique fixed element $\tilde{s} \in S$, and $H^n(\tilde{S})$ converges to $\tilde{s}$ for any $\tilde{S} \in S$. Here, we choose a subset $\tilde{S}$ of S satisfying $\Gamma_{j_2}(\tilde{S}) \subset \tilde{S}$ for $j_2 = 1, 2$. Then

$$\bigcap_{n=0}^{\infty} H^n(\tilde{S}) = \tilde{s}.$$

Since $\Gamma_1(\tilde{S}) \cap \Gamma_2(\tilde{S}) = \emptyset$, for every symbol sequence $\mathbf{j} \in \{1,2\}^{\mathbf{N}}$, there exists a unique function $\tilde{\psi}_{\mathbf{j}} \in C_\gamma$ such that $\tilde{\psi}_{\mathbf{j}} = \bigcap_{n=1}^{\infty} \Gamma_{j_1} \circ \cdots \circ \Gamma_{j_n}(\tilde{S})$. By using $\tilde{\psi}_{\mathbf{j}}$, let us set

$$\tilde{W}_{\mathbf{j}} := \{(z_1, z_2) \in \mathbf{C}^2 \mid z_2 = \tilde{\psi}_{\mathbf{j}}(z_1)\}.$$

Then, it follows that $\Gamma_{j_1}(\tilde{\psi}_{\sigma(\mathbf{j})}) = \tilde{\psi}_{\mathbf{j}}$. Indeed,

$$\bigcap_{n=2}^{\infty} \Gamma_{j_2} \circ \cdots \circ \Gamma_{j_n}(\tilde{S}) = \tilde{\psi}_{\sigma(\mathbf{j})} \quad \textit{and} \quad \Gamma_{j_1}(\tilde{\psi}_{\sigma(\mathbf{j})}) \in \Gamma_{j_1} \circ \cdots \circ \Gamma_{j_n}(\tilde{S})$$

for every $n \in \mathbf{N}$. Hence, $\Gamma_{j_1}(\tilde{\psi}_{\sigma(\mathbf{j})}) \in \bigcap_{n=1}^{\infty} \Gamma_{j_1} \circ \cdots \circ \Gamma_{j_n}(\tilde{S})$. By the uniqueness of $\tilde{s}$, $\Gamma_{j_1}(\tilde{\psi}_{\sigma(\mathbf{j})}) = \tilde{\psi}_{\mathbf{j}}$. Take a small positive constant δ with $0 < \delta < r$, and put

$$\tilde{W}_{\mathbf{j}}^{\delta} := \tilde{W}_{\mathbf{j}} \cap (\Delta(\delta) \times \mathbf{C}) \quad \text{and} \quad W_{\mathbf{j}} := \pi(\tilde{W}_{\mathbf{j}}^{\delta}).$$

By (1) of Proposition 2.1, one can obtain that

$$W_{\mathbf{j}} = \{(x_1, x_2) \in \Delta(\delta) \times \mathbf{C} \mid x_2 = x_1 \tilde{\psi}_{\mathbf{j}}(x_1), x_1 \in \Delta(\delta)\}.$$

Put $\psi_{\mathbf{j}} := x_1 \tilde{\psi}_{\mathbf{j}}$. This is our required in Theorem 2.3. It is clear from $\Gamma_{j_1}(\tilde{\psi}_{\sigma(\mathbf{j})}) = \tilde{\psi}_{\mathbf{j}}$ that $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$ is invariant. Thus, the proof of Theorem 2.3 is complete.

To prove Theorem 2.4, for any $\mathbf{j} \in \{1,2\}^{\mathbf{N}}$, put

$$W_{\mathbf{j}}^1 := \overline{\pi^{-1}(W_{\mathbf{j}} \backslash \{p\})}.$$

It is clear that $W_{\mathbf{j}}^1 \subset \tilde{W}_{\mathbf{j}}$. From the facts $\Gamma_{j_1}(\tilde{\psi}_{\sigma(\mathbf{j})}) = \tilde{\psi}_{\mathbf{j}}$ we have $p_{j_1} \in W_{\mathbf{j}}^1$. Put

$$W_{\mathbf{j}}^2 := \overline{\pi_{j_1}^{-1}(W_{\mathbf{j}}^1 \backslash \{p_{j_1}\})}.$$

Then, we have the following lemma (see Figure 5).

Lemma 4.6. $p_{j_1 j_2} \in E_{j_1} \cap W_{\mathbf{j}}^2$.

Proof. First, we remark that $p_{j_2} \in W_{\sigma(\mathbf{j})}^1$ such that $G_{j_1}(W_{\sigma(\mathbf{j})}) = W_{\mathbf{j}}^1$. It is clear from (1) of Proposition 2.1 that

$$W_{\mathbf{j}}^1 = \{(z_1, z_2) \in U^1 \mid z_2 = \tilde{\psi}_{\mathbf{j}}(z_1), z_1 \in \Delta(\delta)\} \quad \textit{and}$$
$$\pi_{j_1}^{-1}(W_{\mathbf{j}}^1 \backslash \{p_{j_1}\}) = \{(w_1, w_2) \in U_{j_1}^1 \mid w_1 w_2 = \tilde{\psi}_{\mathbf{j}}(w_1), w_1 \in \Delta(\delta)^*\},$$

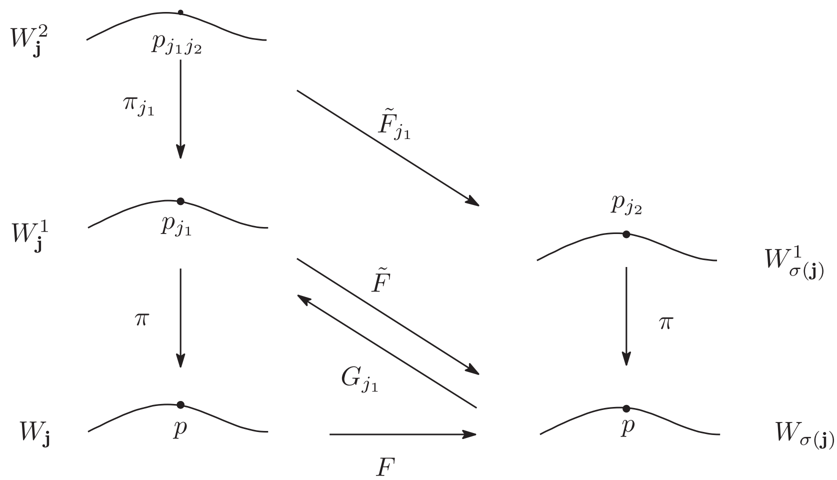

Figure 5

where $\Delta(\delta)^* := \Delta(\delta) \backslash \{0\}$. Define $\psi_{\mathbf{j}}^2(w_1) := \tilde{\psi}_{\mathbf{j}}(w_1)/w_1$. Then, $\psi_{\mathbf{j}}^2$ is continuous for $w_1 \in \Delta(\delta)^*$. Put $\Psi(w_1) := (w_1, \psi_{\mathbf{j}}^2(w_1))$. It follows that for $w_1 \in \Delta(\delta)^*$,

$$\tilde{F}_{j_1}(\Psi(w_1)) = \pi^{-1} \circ \tilde{F} \circ \pi_{j_1}(\Psi(w_1)) \textit{ is biholomorphic},$$
$$\tilde{F}_{j_1}(\Psi(w_1)) \in W^1_{\sigma(\mathbf{j})} \quad \textit{and} \quad \lim_{w_1 \to 0} \tilde{F}_{j_1}(\Psi(w_1)) = p_{j_2}.$$

By Theorem 2.1, there exists a point $p_{j_1 j_2} = (0, \alpha_{j_1 j_2}) \in E_{j_1} \cap U^1_{j_1}$ such that $p_{j_1 j_2} = \tilde{F}_{j_1}^{-1}(p_{j_2})$ and $\tilde{F}_{j_1}$ is biholomorphic at $p_{j_1 j_2}$, we know that $\lim_{w_1 \to 0} \Psi(w_1) = p_{j_1 j_2}$. By defining $\psi_{\mathbf{j}}^2(0) := \alpha_{j_1 j_2}$, we see that $\psi_{\mathbf{j}}^2$ is continuous at $w_1 = 0$. Hence, $p_{j_1 j_2} \in W_{\mathbf{j}}^2$. □

Moreover, by setting

$$W_{\mathbf{j}}^2 := \{(w_1, w_2) \in U^1_{j_1} \mid w_2 = \psi_{\mathbf{j}}^2(w_1), w_1 \in \Delta(\delta)\}$$

and repeating this process inductively, the sequence of points $p_{j_1 \cdots j_n}$ in Theorem 2.1 satisfies $p_{j_1 \cdots j_n} \in W_{\mathbf{j}}^n$ for any $n \geq 1$. Take a positive constant δ_n with $\delta_n > \delta > 0$ and $\Delta^2_{j_1 \cdots j_n}(\delta_n) \supset W_{\mathbf{j}}^n$. Then,

$$W_{\mathbf{j}} \subset \pi \circ \pi_{j_1} \circ \cdots \circ \pi_{j_1 \cdots j_{n-1}}(\Delta^2_{j_1 \cdots j_n}(\delta_n)).$$

From $(*1)$, we have

$$W_{\mathbf{j}} \subset \{(x_1, x_2) \in \mathbf{C}^2 \mid |x_1| < \delta_n, |x_2 - \varphi_{j_1 \cdots j_{n-1}}(x_1)| < \delta_n |x_1|^n\}$$

and we have proved Theorem 2.4.

Remark 4.1. Generally, $\tilde{\psi}_{\mathbf{j}}$ depends on the construction of an extension mapping $g_{j_1 j_2}$ and is not always unique (see [7]). Put

$$q(x_1) := x_1 + a x_1^2 \quad \text{and} \quad P := \{x_1 \in \mathbf{C} \mid q^n(x_1) \to 0\}.$$

It is known that P is non empty open set and $0 \in \partial P$ (see [8]). Then,

$$F^n(x_1, x_2) \to p \quad \text{as } n \to \infty \quad \textit{for any } (x_1, x_2) \in W_{\mathbf{j}} \cap \{P \times \mathbf{C}\} \textit{ with } x_1 \neq 0$$

and $\psi_{\mathbf{j}}(x_1)$ is determined uniquely for any $x_1 \in P$. By Theorem 2.2, it implies that for any fixed $n \in \mathbf{N}$ and any sufficiently small open neighborhood N_p of p there exists a constant $\varepsilon > 0$ such that

$$\bigcup_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}} W_{\mathbf{j}} \cap \{P \times \mathbf{C}\} \cap N_p \subset \Lambda \subset \bigcup_{j_1 \cdots j_{n+1} = 1, 2} \Lambda_{j_1 \cdots j_{n+1}}(\varepsilon).$$

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