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COLLAPSE OF THE MEAN CURVATURE FLOW FOR ISOPARAMETRIC SUBMANIFOLDS IN NON-COMPACT SYMMETRIC SPACES

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Abstract

It is known that principal orbits of Hermann actions on a symmetric space of non-compact type are curvature-adapted isoparametric submanifolds having no focal point of non-Euclidean type on the ideal boundary of the ambient symmetric space. In this paper, we investigate the mean curvature flows for such a curvature-adapted isoparametric submanifold and its focal submanifold. Concretely the investigation is performed by investigating the mean curvature flows for the lift of the submanifold to an infinite dimensional pseudo-Hilbert space through a pseudo-Riemannian submersion.

1. Introduction

Let f_t's ($t \in [0, T)$) be a one-parameter C^∞-family of immersions of a manifold M into a Riemannian manifold N, where T is a positive constant or $T = \infty$. Define a map $\widetilde{f} : M \times [0, T) \to N$ by $\widetilde{f}(x, t) = f_t(x)$ ($(x, t) \in M \times [0, T)$). If, for each $t \in [0, T)$, $\widetilde{f}_*\left(\left(\frac{\partial}{\partial t}\right)_{(x,t)}\right)$ is the mean curvature vector of $f_t : M \hookrightarrow N$, then f_t's ($t \in [0, T)$) is called a mean curvature flow. In particular, if f_t's are embeddings, then we call $M_t := f_t(M)$'s ($0 \in [0, T)$) rather than f_t's ($0 \in [0, T)$) a mean curvature flow. Liu-Terng [LT] investigated the mean curvature flows for an isoparametric submanifold in a Euclidean space and its focal submanifold and obtained the following facts.

Fact 1 ([LT]). *Let M be a compact isoparametric submanifold in a Euclidean space. Then the following statements* (i) *and* (ii) *hold*:

(i) *The mean curvature flow M_t for M collapses to a focal submanifold F of M in finite time. If the natural fibration of M onto F is spherical, then the mean curvature flow M_t has type I singularity, that is,* $\lim_{t \to T-0} \max_{v \in S^\perp M_t} \|A_v^t\|_\infty^2 (T - t)$

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$< \infty$, *where* A_v^t *is the shape operator of* M_t *for* v, $\|A_v^t\|_\infty$ *is the sup norm of* A_v^t *and* $S^\perp M_t$ *is the unit normal bundle of* M_t.

(ii) *For any focal submanifold* F *of* M, *the set of all parallel submanifolds of* M *collapsing to* F *along the mean curvature flow is a one-parameter* C^∞*-family.*

FACT 2 ([LT]). *Let* M *be as in Fact* 1, C *the Weyl domain of* M *at* $x_0 (\in M)$ *and* σ *a simplex of dimension greater than zero of* ∂C. *Then the following statements* (i) *and* (ii) *hold*:

(i) *For any focal submanifold* F (*of* M) *through* σ, *the maen curvature flow* F_t *for* F *collapses to a focal submanifold* F' (*of* M) *through* $\partial\sigma$ *in finite time. If the natural fibration of* F *onto* F' *is spherical*, *then the mean curvature flow* F_t *has type I singularity.*

(ii) *For any focal submanifold* F (*of* M) *through* $\partial\sigma$, *the set of all focal submanifolds of* M *through* σ *collapsing to* F *along the mean curvature flow is a one-parameter* C^∞*-family.*

Since the focal submanifold of M through the only 0-dimensional simplex of ∂C is a one-point set, it follows from the statement (i) of Facts 1 and 2 that M collapses to a one-point set after finitely many times of collapsings along the mean curvature flows.

As a generalized notion of compact isoparametric hypersurfaces in a sphere and a hyperbolic space, and a compact isoparametric submanifolds in a Euclidean space, Terng-Thorbergsson [TT] introduced the notion of an equifocal submanifold in a symmetric space G/K. This notion is defined as a compact submanifold (which we denote by M) in G/K with flat section, trivial normal holonomy group and parallel focal structure. Here the parallel focal structure means that the tangential focal structures of M move to one another under the parallel translations with respect to the normal connection of M. For a compact submanifold M with flat section and trivial normal holonomy group, it is equifocal if and only if, for any parallel normal vector field $\tilde{v}$ of M, the set of all the focal radii of M along the normal geodesic $\gamma_{\tilde{v}_x}$ with $\gamma'_{\tilde{v}_x}(0) = \tilde{v}_x$ is independent of the choice of $x \in M$. On the other hand, Heintze-Liu-Olmos [HLO] introduced the notion of isoparametric submanifold with flat section in a (general) Riemannian manifold as a (properly embedded) submanifold with flat section and trivial normal holonomy group whose sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction. In the sequel, we assume that all isoparametric submanifolds with flat section are complete.

TERMINOLOGY. In this paper, we shall call an isoparametric submanifold with flat section an *isoparametric submanifold* simply.

For a compact submanifold in a symmetric space G/K of compact type, the isoparametricness is equivalent to the equifocality (see [HLO]). The author has recently investigated the mean curvature flows for an equifocal submanifold in a symmetric space of compact type and its focal submanifold, and obtained the following facts.

Fact 3 ([Koi10]). *Let M be an equifocal submanifold in a symmetric space G/K of compact type. Then the following statements* (i) *and* (ii) *hold*:

(i) *If M is not minimal, then the mean curvature flow M_t for M collapses to a focal submanifold F of M in finite time. Furthermore, if M is irreducible, if the codimension of M is geater than one and if the natural fibration of M onto F is spherical, then M_t has type I singularity.*

(ii) *For any focal submanifold F of M, the set of all parallel submanifolds of M collapsing to F along the mean curvature flow is a one-parameter C^∞-family.*

Fact 4 ([Koi10]). *Let M be as in Fact* 3, *C the image of the fundamental domain of the Coxeter group of M at $x_0(\in M)$ by the normal exponential map and σ a stratum of dimension greater than zero of ∂C (which is a stratified space). Then the following statements* (i) *and* (ii) *hold*:

(i) *For any non-minimal focal submanifold F of M through σ, the mean curvature flow F_t for F collapses to a focal submanifold F' of M through $\partial\sigma$ in finite time. If M is irreducible, if the codimension of M is greater than one and if the natural fibration of F onto F' is spherical, then the mean curvature flow F_t has type I singularity.*

(ii) *For any focal submanifold F of M through $\partial\sigma$, the set of all focal submanifolds of M through σ collapsing to F along the mean curvature flow is a one-parameter C^∞-family.*

Since focal submanifolds of M through the lowest dimensional stratum of ∂C are minimal, it follows from the statement (i) of Facts 3 and 4 that M collapses to a minimal focal submanifold of M after finitely many times of collapsings along the mean curvature flows.

Assumption. In the sequel, we assume that all submanifolds are real analytic.

We [Koi1,2] introduced the notion of a complex equifocal submanifold as a (properly embedded) complete submanifold with flat section, trivial normal holonomy group and parallel complex focal structure, where the parallel complex focal structure means that the tangential focal structures of the complexification $M^{\mathbb{C}}(\subset G^{\mathbb{C}}/K^{\mathbb{C}})$ of $M(\subset G/K)$ move to one another under the parallel translations with respect to the normal connection of $M^{\mathbb{C}}$. For a submanifold M with flat section and trivial normal holonomy group, it is complex equifocal if and only if, for any parallel normal vector field $\tilde{v}$ of M, the set of all the complex focal radii of M along the normal geodesic $\gamma_{\tilde{v}_x}$ with $\gamma'_{\tilde{v}_x}(0)=\tilde{v}_x$ is independent of the choice of $x\in M$.

We ([Koi3]) introduced the notion of a proper complex equifocal submanifold as a complex equifocal submanifold having a good complex focal structure, where "good complex focal structure" means that the focal structure of the complexification of the submanifold at any point x_0 consists of infinitely many complex hyperplanes in the normal space at x_0 and that the group generated by the complex reflections of order two with respect to the complex hyperplanes is discrete.

Next we recall the notion of a focal point of non-Euclidean type on the ideal boundary $N(\infty)$ of a submanifold M in a Hadamard manifold N which was introduced in [Koi6]. Let v be a unit normal vector of M and $\gamma_v : [0, \infty) \to N$ the normal geodesic of M of direction v. If there exists a M-Jacobi field Y along γ_v satisfying $\lim_{t\to\infty} \frac{\|Y(t)\|}{t} = 0$, then we call $\gamma_v(\infty)(\in N(\infty))$ a *focal point* of M *on the ideal boundary* $N(\infty)$ *along* γ_v, where $\gamma_v(\infty)$ is the asymptotic class of γ_v. Also, if there exists a M-Jacobi field Y along γ_v satisfying $\lim_{t\to\infty} \frac{\|Y(t)\|}{t} = 0$ and $\mathrm{Sec}(v, Y(0)) \neq 0$, then we call $\gamma_v(\infty)$ a *focal point of non-Euclidean type of* M *on* $N(\infty)$ *along* γ_v, where $\mathrm{Sec}(v, Y(0))$ is the sectional curvature for the 2-plane spanned by v and $Y(0)$. If, for any unit normal vector v of M, $\gamma_v(\infty)$ is not a focal point of non-Euclidean type of M on $N(\infty)$, then we say that *M has no focal point of non-Euclidean type on the ideal boundary* $N(\infty)$. It is known that principal orbits of Hermann actions on symmetric spaces of non-compact type are curvature-adapted isoparametric submanifolds and they have no focal point of non-Euclidean type on the ideal boundary (see Theorem B in [Koi3] and its proof and so on). According to Theorem 15 in [Koi2] and Theorem A in [Koi6], we have the following fact.

FACT 5. *For a curvature-adapted isoparametric submanifold M in a symmetric space N of non-compact type, it has no focal point of non-Euclidean type on the ideal boundary $N(\infty)$ if and only if it is proper complex equifocal.*

Let M be a curvature-adapted isoparametric submanifold in a symmetric space $N = G/K$ of non-compact type having no focal point of non-Euclidean type on the ideal boundary $N(\infty)$ of N. Assume that a focal submanifold of M exists. Note that a focal submanifold of M exists if G/K is other than a hyperbolic space. Let F_l be one of the lowest dimensional focal submanifolds of M. Without loss of generality, we may assume that $eK \in F_l$. Note that F_l passes through $\exp^{\perp}(\widetilde{\sigma})$ for one $\widetilde{\sigma}$ of the lowest dimensional simplex of the boundary $\partial\widetilde{C}$ of the fundamental domain $\widetilde{C}$ of the real Coxeter group associated with M. See the next paragraph about the definition of the real Coxeter group associated with M. Set $\mathfrak{p} := T_{eK}(G/K)$ and $\mathfrak{p}' := T^{\perp}_{eK}F_l$. Take a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{p}'$ and a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ containing $\mathfrak{b}$. Let $\triangle$ be the root system of G/K with respect to $\mathfrak{a}$ and $\triangle'$ be that of $F_l^{\perp}$ with respect to $\mathfrak{b}$. Also, let $\mathfrak{p}_\alpha$ be the root space for $\alpha \in \triangle$. Note that, if $\mathrm{rank}\, F_l^{\perp} = \mathrm{rank}(G/K)$, then we have $\mathfrak{a} = \mathfrak{b}$ and $\triangle' \subset \triangle$. Since M is curvature-adapted, so is also F_l. Hence we have $\mathfrak{p}' = \sum_{\alpha\in\triangle_+}(\mathfrak{p}_\alpha \cap \mathfrak{p}')$, where $\triangle_+$ is the positive root system of $\triangle$ with respect to a lexicographic ordering of $\mathfrak{b}^*$. In this paper, we prove the following fact for the mean curvature flows for a curvature-adapted isoparametric submanifold in a symmetric space $N = G/K$ of non-compact type having no focal point of non-Euclidean type on the ideal boundary $N(\infty)$ and its focal submanifold.

THEOREM A. *Let M be a curvature-adapted isoparametric submanifold in a symmetric space $N = G/K$ of non-compact type having no focal point of non-Euclidean type on the ideal boundary $N(\infty)$, M_t $(0 \le t < T)$ the mean curvature flow for M, $\triangle$, $\mathfrak{p}_\alpha$ and $\mathfrak{p}'$ be as above. Assume that* $\operatorname{codim} M = \operatorname{rank} N$ *and that* $\dim(\mathfrak{p}_\alpha \cap \mathfrak{p}') \ge \frac{1}{2} \dim \mathfrak{p}_\alpha$ $(\alpha \in \triangle)$. *Then the following statements* (i), (ii) *and* (iii) *hold.*

(i) *M is not minimal and M_t collapses to a focal submanifold of M in finite time.*

(ii) *If M_t collapses to a focal submanifold F of M in finite time and if the natural fibration of M onto F is spherical, then M_t has type I singularity.*

(iii) *For any focal submanifold F of M, the set of all parallel submanifolds of M collapsing to F along the mean curvature flow is a one-parameter C^∞-family.*

Remark 1.1. The principal orbits of the isotropy action (of a symmetric space of non-compact type) and Hermann actions in Table 1 (see Section 5) satisfy all the conditions in Theorem A.

The focal set of a curvature-adapted proper complex equifocal submanifold M at any point $x(\in M)$ consists of the images of finitely many (real) hyperplanes in the normal space $T_x^\perp M$ by the normal exponential map $\exp^\perp$ of M and the group generated by the reflections with respect to the hyperplanes is a (finite) Coxeter group. In [Koi6], we called this group the *real Coxeter group associated with M*.

THEOREM B. *Let M be a curvature-adapted isoparametric submanifold in a symmetric space $N = G/K$ of non-compact type having no focal point of non-Euclidean type on the ideal boundary $N(\infty)$ and M_t $(0 \le t < T)$ the mean curvature flow for M. Assume that* $\operatorname{codim} M = \operatorname{rank} N$ *and that the lowest dimensional focal submanifold of M is an one-point set. Let $\tilde{\sigma}$ be a stratum of dimension greater than zero of the fundamental domain $\tilde{C}$ (which is a stratified space) of the real Coxeter group of M. Then, the following statements* (i) *and* (ii) *hold.*

(i) *Any focal submanifold F of M through $\exp^\perp(\tilde{\sigma})$ is not minimal and the mean curvature flow F_t for F collapses to a focal submanifold F' of M through $\exp^\perp(\partial\tilde{\sigma})$ in finite time. If the natural fibration of F onto F' is spherical, then F_t has type I singularity.*

(ii) *For any focal submanifold F of M through $\exp^\perp(\partial\tilde{\sigma})$, the set of all focal submanifolds of M through $\exp^\perp(\tilde{\sigma})$ collapsing to F along the mean curvature flow is a one-parameter C^∞-family.*

According to the statement (i) of Theorems A and B, if M is a curvature-adapted isoparametric submanifold having no focal point of non-Euclidean type on the ideal boundary, if $\operatorname{codim} M = \operatorname{rank}(G/K)$ and if F_l is one-point set, then M collapses to one-point set after finitely many times of collapsings along the mean curvature flows.

$$\begin{array}{llll} M_t \xrightarrow[(t\to T_1)]{} F^1 & & & \\ & F^1_t \xrightarrow[(t\to T_2)]{} F^2 & & \\ & & \ddots & \\ & & & F^{k-1}_t \xrightarrow[(t\to T_k)]{} \text{one point set} \end{array}$$

$$\left(\begin{array}{l} F^1 : \text{a focal submanifold of } M \\ F^i : \text{a focal submanifold of } F^{i-1} \ (i = 2, \dots, k-1) \end{array}\right)$$

2. Basic notions and facts

In this section, we briefly recall the notions of a proper complex equifocal submanifold in a symmetric space G/K of non-compact type and a proper complex isoparametric submanifold in an (infinite dimensional) pseudo-Hilbert space. First we recall the notion of a complex equifocal submanifold in G/K. Let M be a submanifold with flat section in G/K, where "M has flat section" means that, for each $x = gK \in M$, $\exp^\perp(T^\perp_{gK}M)$ is a flat totally geodesic submanifold in G/K. Denote by A the shape tensor of M and R the curvature tensor of G/K. Let $v \in T^\perp_x M$ and $X \in T_x M$ $(x = gK)$. Set $R(v) := R(\cdot, v)v$. Denote by γ_v the geodesic in G/K with $\gamma_v'(0) = v$. The strongly M-Jacobi field Y along γ_v with $Y(0) = X$ (hence $Y'(0) = -A_v X$) is given by

$$Y(s) = \left(P_{\gamma_v|_{[0,s]}} \circ \left(\cos(s\sqrt{R(v)}) - \frac{\sin(s\sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v\right)\right)(X),$$

where $Y'(0) = \widetilde{\nabla}_v Y$, $P_{\gamma_v|_{[0,s]}}$ is the parallel translation along $\gamma_v|_{[0,s]}$, and $\cos(s\sqrt{R(v)})$ and $\dfrac{\sin(s\sqrt{R(v)})}{s\sqrt{R(v)}}$ are defined by

$$\cos(s\sqrt{R(v)}) := \sum_{k=0}^{\infty} \frac{(-1)^k s^{2k} R(v)^k}{(2k)!} \quad \text{and} \quad \frac{\sin(s\sqrt{R(v)})}{s\sqrt{R(v)}} := \sum_{k=0}^{\infty} \frac{(-1)^k s^{2k} R(v)^k}{(2k+1)!},$$

respectively. Since M has flat section, all focal radii of M along γ_v are given as zero points of strongly M-Jacobi fields along γ_v. Hence all focal radii of M along γ_v coincide with the zero points of the real-valued function F_v over $\mathbb{R}$ defined by

$$F_v(s) := \det\left(\cos(s\sqrt{R(v)}) - \frac{\sin(s\sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v\right).$$

So we defined the notion of a complex focal radius of M along γ_v as the zero points of the complex-valued function $F_v^{\mathbb{C}}$ over $\mathbb{C}$ defined by

$$F_v^{\mathbb{C}}(z) := \det\left(\cos(z\sqrt{R(v)^{\mathbb{C}}}) - \frac{\sin(z\sqrt{R(v)^{\mathbb{C}}})}{\sqrt{R(v)^{\mathbb{C}}}} \circ A_v^{\mathbb{C}}\right),$$

where $R(v)^{\mathbb{C}}$ (resp. $A_v^{\mathbb{C}}$) is the complexification of $R(v)$ (resp. A_v). Also, for a complex focal radius z of M along γ_v, we call

$$\dim_{\mathbb{C}} \operatorname{Ker}\left(\cos(z\sqrt{R(v)^{\mathbb{C}}}) - \frac{\sin(z\sqrt{R(v)^{\mathbb{C}}})}{\sqrt{R(v)^{\mathbb{C}}}} \circ A_v^{\mathbb{C}}\right)$$

the *multiplicity* of the complex focal radius z. Here we note that complex focal radii along γ_v indicate the positions of focal points of the extrinsic complexification $M^{\mathbb{C}}(\hookrightarrow G^{\mathbb{C}}/K^{\mathbb{C}})$ of M along the complexified geodesic $\gamma_{\iota_* v}^{\mathbb{C}}$, where $G^{\mathbb{C}}/K^{\mathbb{C}}$ is the anti-Kaehlerian symmetric space associated with G/K and ι is the natural embedding of G/K into $G^{\mathbb{C}}/K^{\mathbb{C}}$. See [Koi2] about the definitions of $G^{\mathbb{C}}/K^{\mathbb{C}}$, $M^{\mathbb{C}}(\hookrightarrow G^{\mathbb{C}}/K^{\mathbb{C}})$ and $\gamma_{\iota_* v}^{\mathbb{C}}$. Furthermore, assume that the normal holonomy group of M is trivial. Let $\tilde{v}$ be a parallel unit normal vector field of M. Assume that the number (which may be ∞) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$. Let $\{r_{i,x} \mid i = 1, 2, \dots\}$ be the set of all complex focal radii along $\gamma_{\tilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\operatorname{Re} r_{i,x} > \operatorname{Re} r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\operatorname{Re} r_{i,x} = \operatorname{Re} r_{i+1,x}$ & $\operatorname{Im} r_{i,x} = -\operatorname{Im} r_{i+1,x} < 0$". Let r_i $(i = 1, 2, \dots)$ be complex valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions r_i $(i = 1, 2, \dots)$ *complex focal radius functions for* $\tilde{v}$. If, for each parallel unit normal vector field $\tilde{v}$ of M, the set of all complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, if each complex focal radius function for $\tilde{v}$ is constant on M and if it has constant multiplicity, then we call M a *complex equifocal submanifold*.

Next we recall the notion of a proper complex isoparametric submanifold in an (infinite dimensional) pseudo-Hilbert space. Let M be a pseudo-Riemannian submanifold of finite codimension in a pseudo-Hilbert space $(V, \langle\,,\,\rangle)$. See [Koi1] about this definition. We call M a *Fredholm pseudo-Riemannian submanifold* (or simply *Fredholm submanifold*) if there exists an orthogonal time-space decomposition $V = V_- \oplus V_+$ such that $(V, \langle\,,\,\rangle_{V_\pm})$ is a Hilbert space and that, for each $v \in T^\perp M$, A_v is a compact operator with respect to $f^*\langle\,,\,\rangle_{V_\pm}$, where an orthogonal time-space decomposition $V = V_- \oplus V_+$ means that $\langle\,,\,\rangle|_{V_- \times V_-}$ is negative definite, $\langle\,,\,\rangle|_{V_+ \times V_+}$ is positive definite and that $\langle\,,\,\rangle|_{V_- \times V_+} = 0$, and $\langle\,,\,\rangle_{V_\pm} := -\pi_{V_-}^*\langle\,,\,\rangle + \pi_{V_+}^*\langle\,,\,\rangle$ (π_{V_-} (resp. π_{V_+}): the orthogonal projection of V onto V_- (resp. V_+)). Since A_v is a compact operator with respect to $f^*\langle\,,\,\rangle_{V_\pm}$, for each $v \in T^\perp M$, the operator $\mathrm{id} - A_v$ is a Fredholm operator with respect to $f^*\langle\,,\,\rangle_{V_\pm}$ and hence the normal exponential map $\exp^\perp : T^\perp M \to V$ of M is a Fredholm map with respect to the metric of $T^\perp M$ naturally defined from $f^*\langle\,,\,\rangle_{V_\pm}$ and $\langle\,,\,\rangle_{V_\pm}$, where id is the identity transformation of TM. In the sequel, set $\langle\,,\rangle := f^*\langle\,,\rangle$. The set of all eigenvalues of the complexification $A_v^{\mathbb{C}}$ of A_v is described as $\{0\}\cup\{\mu_i \,|\, i=1,2,\dots\}$, where "$|\mu_i| > |\mu_{i+1}|$" or "$|\mu_i| = |\mu_{i+1}|$ & $\mathrm{Re}\,\mu_i > \mathrm{Re}\,\mu_{i+1}$" or "$|\mu_i| = |\mu_{i+1}|$ & $\mathrm{Re}\,\mu_i = \mathrm{Re}\,\mu_{i+1}$ & $\mathrm{Im}\,\mu_i = -\mathrm{Im}\,\mu_{i+1} > 0$". We call μ_i the i-th complex principal curvature for v. Assume that the normal holonomy group M is trivial. Fix a parallel normal vector field $\tilde{v}$ on M. Assume that the number (which may be ∞) of distinct complex principal curvatures of $\tilde{v}_x$ is independent of $x \in M$. Then we define functions $\tilde{\mu}_i$ $(i=1,2,\dots)$ on M by assigning the i-th complex principal curvature for $\tilde{v}_x$ to each $x \in M$. We call this function $\tilde{\mu}_i$ the i-th *complex principal curvature function for* $\tilde{v}$. A Fredholm submanifold M is called a *complex isoparametric submanifold* if the normal holonomy group of M is trivial and if, for each parallel normal vector field $\tilde{v}$, the number of distinct complex principal curvatures of direction $\tilde{v}_x$ is independent of the choice of $x \in M$ and if each complex principal curvature function of direction $\tilde{v}$ is constant on M. Assume that M is a complex isoparametric submanifold. If, for each $v \in T^\perp M$, the complexified shape operator $A_v^{\mathbb{C}}$ is diagonalizable with respect to a pseudo-orthonormal base of $(T_xM)^{\mathbb{C}}$ (x : the base point of v), that is, there exists a pseudo-orthonormal base of $(T_xM)^{\mathbb{C}}$ consisting of the eigenvectors of $A_v^{\mathbb{C}}$, then we call M a *proper complex isoparametric submanifold*, where a pseudo-orthonormal base means a linearly independent system $\{e_i\}_{i=1}^{\infty}$ of a pseudo-Hilbert space $(T_xM, \langle\,,\rangle)$ such that, for each $i \in \mathbb{N}$, there exists $\hat{i} \in \mathbb{N}$ satisfying $|\langle v_i, v_j\rangle| = \delta_{\hat{i}j}$ $(j \in \mathbb{N})$ ($\delta_{\cdot\cdot}$: the Kronecker's symbol) and that $\overline{\bigoplus_{i=1}^{\infty}\mathrm{Span}\{v_i\}} = T_xM$ ($\bar{\cdot}$: the closure of $\cdot$ with respect to the original topology of T_xM). Then, for each $x \in M$, $A_v^{\mathbb{C}}$'s $(v \in T_x^\perp M)$ are simultaneously diagonalizable with respect to a pseudo-orthonormal base of $(T_xM)^{\mathbb{C}}$ because $A_v^{\mathbb{C}}$'s commute. There exists a family $\{E_i \,|\, i \in I\}$ $(I \subset \mathbb{N})$ of parallel subbundles of $(TM)^{\mathbb{C}}$ such that, for each $x \in M$, $(T_xM)^{\mathbb{C}} = \overline{\bigoplus_{i\in I}(E_i)_x}$ holds and that this decomposition is a common-eigenspace decomposition of $A_v^{\mathbb{C}}$'s $(v \in T_x^\perp M)$. Also, there exist smooth sections λ_i $(i \in I)$ of $((T^\perp M)^{\mathbb{C}})^*$ such that $A_v^{\mathbb{C}} = \lambda_i(v)\,\mathrm{id}$ on $(E_i)_x$ for each $v \in (T^\perp M)^{\mathbb{C}}$, where x is the base point of v. The subbundles E_i $(i \in I)$ are called *complex curvature distributions of* M and λ_i $(i \in I)$ are called *complex principal curvatures of* M. Define a complex normal vector field $\mathbf{n}_i$ $(i \in I)$ by $\lambda_i(\cdot) = \langle \mathbf{n}_i, \cdot\rangle^{\mathbb{C}}$, where $\langle\,,\rangle^{\mathbb{C}}$ is the complexification of $\langle\,,\rangle$. Note that each $\mathbf{n}_i$ is parallel with respect to the complexification $\nabla^{\perp\mathbb{C}}$ of $\nabla^\perp$. The normal vector fields $\mathbf{n}_i$ $(i \in I)$ are called *complex curvature normals of* M.

Let G/K be a symmetric space of non-compact type and $\pi : G \to G/K$ be the natural projection. The parallel transport map ϕ for the semi-simple Lie group G is defined by $\phi(u) := g_u(1)$ $(u \in H^0([0,1],\mathfrak{g}))$, where g_u is the element of $H^1([0,1],G)$ with $g_u(0) = e$ (e, : the identity element of G) and $g_{u*}^{-1}g_u' = u$. Here we note that $H^0([0,1],\mathfrak{g})$ is a pseudo-Hilbert space. See [Koi1] the detail of the definition of the pseudo-Hilbert space $H^0([0,1],\mathfrak{g})$ and ϕ. Let M be a complex equifocal submanifold in G/K. Since M is complex equifocal, $\tilde{M} := (\pi\circ\phi)^{-1}(M)$ is complex isoparametric. In particular, if $\tilde{M}$ is proper complex isoparametric, then M is called a *proper complex equifocal submanifold*. Let M

be a proper complex equifocal submanifold in a symmetric space G/K of non-compact type. Denote by A (resp. $\widetilde{A}$) the shape tensor of M (resp. $\widetilde{M}$). Since $\widetilde{M}$ is proper complex isoparametric, the complexified shape operators of $\widetilde{M}$ is simultaneously diagonalizeble with respect to a pseudo-orthonormal base. Hence the complex focal set of $\widetilde{M}$ at any point $u(\in \widetilde{M})$ consists of infinitely many complex hyperplanes in the complexified normal space $(T_u^\perp \widetilde{M})^{\mathbf{C}}$ and the group generated by the complex reflections of order two with respect to the complex hyperplanes is discrete. From this fact, it follows that, for the complex focal set of the proper complex equifocal submanifold M, the following fact holds:

($*$) *The complex focal set of M at any point $x(\in M)$ consists of infinitely many complex hyperplanes in the complexified normal space $(T_x^\perp M)^{\mathbf{C}}$ and the group generated by the complex reflections of order two with respect to the complex hyperplanes is discrete.*

Let H be a symmetric subgroup of G (i.e., there exists an involution of G with $(\mathrm{Fix}\,\tau)_0 \subset H \subset \mathrm{Fix}\,\tau$), where $\mathrm{Fix}\,\tau$ is the fixed point group of τ and $(\mathrm{Fix}\,\tau)_0$ is the identity component of $\mathrm{Fix}\,\tau$. The natural action H on G/K is called a *Hermann type action*. It is shown that a principal orbit of a Hermann type ation is a proper complex equifocal and curvature-adapted ([Koi3]), where the curvature-adaptedness means that, for each normal vector v of M, $R(\cdot, v)v$ preserves T_xM (x : the base point of v) invariantly and that $[R(\cdot, v)v, A_v] = 0$ (R : the curvature tensor of G/K). Let $P(G, H \times K) := \{g \in H^1([0,1], G) \mid (g(0), g(1)) \in H \times K\}$, where $H^1([0,1], G)$ is a pseudo-Hilbert Lie group of all H^1-paths in G having $[0,1]$ as the domain. See [Koi1] about the detail of the definition of $H^1([0,1], G)$. This group $P(G, H \times K)$ acts on $H^0([0,1], \mathfrak{g})$ as gauge action. It is shown that orbits of the $P(G, H \times K)$-action are the inverse images of the H-orbits by $\pi \circ \phi$ (see [Koi2]).

3. The regularized mean curvature vector of Fredholm submanifold with proper shape operators

In this section, we shall define the regularized mean curvature vector of a certain kind of Fredholm submanifold in a pseudo-Hlbert space. Let M be a Fredholm submanifold in a pseudo-Hilbert space $(V, \langle\ ,\ \rangle)$. Denote by A the shape tensor of M. Fix $v \in T^\perp M$. If the complexified shape operator $A_v^{\mathbf{C}}$ is diagoalizable with respect to a pseudo-orthonormal base, then $A_v^{\mathbf{C}}$ is said to be *proper*. If $A_v^{\mathbf{C}}$ is proper for any $v \in T^\perp M$, then we say that M *has proper shape operators*. Assume that M has proper shape operators. Fix $v \in T_x^\perp M$. Let $\{\mu_i \mid i = 1, 2, \dots\}$ ("$|\mu_i| > |\mu_{i+1}|$" or "$|\mu_i| = |\mu_{i+1}|$ & $\mathrm{Re}\,\mu_i > \mathrm{Re}\,\mu_{i+1}$" or "$|\mu_i| = |\mu_{i+1}|$ & $\mathrm{Re}\,\mu_i = \mathrm{Re}\,\mu_{i+1}$ & $\mathrm{Im}\,\mu_i = -\mathrm{Im}\,\mu_{i+1} > 0$") be the set of all eigenvalues of $A_v^{\mathbf{C}}$ other than zero and m_i the multiplicity of μ_i. Then we define the regularized trace $\mathrm{Tr}_r\, A_v^{\mathbf{C}}$ of $A_v^{\mathbf{C}}$ by $\mathrm{Tr}_r\, A_v^{\mathbf{C}} := \sum_i m_i \mu_i$. Also, we define the trace $\mathrm{Tr}_{\mathrm{abs}}(A_v^{\mathbf{C}})^2$ by $\mathrm{Tr}_{\mathrm{abs}}(A_v^{\mathbf{C}})^2 := \sum_i m_i |\mu_i|^2$. If there exist $\mathrm{Tr}_r\, A_v^{\mathbf{C}}$ and $\mathrm{Tr}_{\mathrm{abs}}(A_v^{\mathbf{C}})^2$ for each $v \in T^\perp M$, then we say that M is *regularizable*. It is shown that, if μ is an eigenvalue of $A_v^{\mathbf{C}}$ with multiplicity m, then so is also the conjugate $\bar{\mu}$ of

μ. Hence we have $\mathrm{Tr}_r\, A_v^{\mathbb{C}} \in \mathbb{R}$. Define $H_x \in T_x^{\perp} M$ by $\langle H_x, v\rangle = \mathrm{Tr}_r\, A_v^{\mathbb{C}}$ ($\forall v \in T_x^{\perp} M$). We call the normal vector field $H(: x \mapsto H_x)$ of M the *regularized mean curvature vector* of M. Let $f_t : M \hookrightarrow V$ $(0 \le t < T)$ be a C^∞-family of regularizable Fredholm submanifolds with proper shape operators and H_t be the regularized mean curvature vector of f_t. Define by $\widetilde{f} : M \times [0, T) \to V$ by $\widetilde{f}(x, t) := f_t(x)$ $((x, t) \in M \times [0, T))$. If $\widetilde{f}_*\left(\frac{\partial}{\partial t}\right) = H_t$, then we call f_t $(0 \le t < T)$ the *regularized mean curvature flow*. In the sequel, we call this flow the mean curvature flow for simplicity.

Let G/K be a symmetric space of non-compact type, $\pi : G \to G/K$ the natural projection and $\phi : H^0([0,1], \mathfrak{g}) \to G$ the parallel tansport map for G. Let M be a curvature-adapted isoparametric submanifold in G/K having no focal point of non-Euclidean type on the ideal boundary of G/K and set $\widetilde{M} := (\pi \circ \phi)^{-1}(M)$, which is proper complex isoparametric (hence has proper shape operators). Denote by H the mean curvature vector of M. Then we have the following fact.

Lemma 3.1. *The submanifold $\widetilde{M}$ is regularizable and the regularized mean curvature vector $\widetilde{H}$ is equal to the horizontal lift H^L of H.*

Proof. Without loss of generality, we may assume that $eK \in M$. For simplicity, set $\mathfrak{m} := T_{eK}M$ and $\mathfrak{b} := T_{eK}^{\perp} M$. Since M is flat section (hence $\mathfrak{b}$ is abelian), the normal connection of M is flat and since M is curvature-adapted, the operators $R(\cdot, v)v$'s $(v \in \mathfrak{b})$ and A_v's $(v \in \mathfrak{b})$ commute to one another. Also they are diagonalizable with respect to an orthonormal base, respectively. Therefore they are simultaneously diagonalizable with respect to an orthonormal base. Let $\mathfrak{m} = \mathfrak{m}_0^R + \sum_{i \in I^R} \mathfrak{m}_i^R$ be the common eigenspace decomposition of $R(\cdot, v)v$'s $(v \in \mathfrak{b})$ and $\mathfrak{m} = \mathfrak{m}_0^A + \sum_{i \in I^A} \mathfrak{m}_i^A$ be the common eigenspace decomposition of A_v's $(v \in \mathfrak{b})$, where $\mathfrak{m}_0^R := \bigcap_{v \in \mathfrak{b}} \mathrm{Ker}\, R(\cdot, v)v$ and $\mathfrak{m}_0^A := \bigcap_{v \in \mathfrak{b}} \mathrm{Ker}\, A_v$. Set $m_i^R := \dim \mathfrak{m}_i^R$ and $m_i^A := \dim \mathfrak{m}_i^A$. Also, set $I_i^A := \{j \in I^A \cup \{0\} \mid \mathfrak{m}_j^A \cap \mathfrak{m}_i^R \neq \{0\}\}$ $(i \in I^R \cup \{0\})$. Since $R(\cdot, v)v$ $(v \in \mathfrak{b})$ and A_v's $(v \in \mathfrak{b})$ are simultaneously diagonalizable, we have $\mathfrak{m} = \sum_{i \in I^R \cup \{0\}} \sum_{j \in I_i^A} (\mathfrak{m}_j^A \cap \mathfrak{m}_i^R)$. Let $\beta_i (\ge 0)$ $(i \in I^R)$ and λ_i $(i \in I^A)$ be linear functions over $\mathfrak{b}$ defined by $R(\cdot, v)v|_{\mathfrak{m}_i^R} = -\beta_i(v)^2\, \mathrm{id}$ $(v \in \mathfrak{b})$ and $A_v|_{\mathfrak{m}_i^A} = \lambda_i(v)\, \mathrm{id}$ $(v \in \mathfrak{b})$. Denote by $\mathfrak{b}_r$ the set of all $v \in \mathfrak{b}$ such that $\beta_i(v) \neq 0$, $\lambda_i(v) \neq 0$, $\beta_i(v)$'s $(i \in I^R)$ are mutually distinct and that so are also $\lambda_i(v)$'s $(i \in I^A)$. Note that $\mathfrak{b}_r$ is open and dense in $\mathfrak{b}$. Fix $v \in \mathfrak{b}_r$. Denote by $\widetilde{A}$ the shape tensor of $\widetilde{M}$ and $\mathrm{Spec}\, \widetilde{A}_{v^L}^{\mathbb{C}}$ the spectrum of $\widetilde{A}_{v^L}^{\mathbb{C}}$, where v^L is the horizontal lift of v to the constant path $\hat{0}$ at the zero vector 0 of $\mathfrak{g}$. Set $I_{i,v,+}^A := \{j \in I_i^A \mid |\lambda_j(v)| > |\beta_i(v)|\}$, $I_{i,v,-}^A := \{j \in I_i^A \mid |\lambda_j(v)| < |\beta_i(v)|\}$ and $I_{i,v,0}^A := \{j \in I_i^A \mid |\lambda_j(v)| = |\beta_i(v)|\}$. Since M is a curvature-adapted isoparametric submanifold having no focal point of non-Euclidean type on the ideal boundary of G/K, we can show that $I_{i,v,0}^A = \emptyset$ (see Theorem A of [Koi1]) and that $I_{i,v,+}^A$ and $I_{i,v,-}^A$ are at most one point sets, respectively (see the proof of Theorems B and C of [Koi6]). When $I_{i,v,+}^A \neq \emptyset$ (resp. $I_{i,v,-}^A \neq \emptyset$), denote by $j_{i,v}^+$ (resp. $j_{i,v}^-$) the only element. Set $I_{v,+}^R := \{i \in I^R \mid I_{i,v,+}^A \neq \emptyset\}$ and $I_{v,-}^R := \{i \in I^R \mid I_{i,v,-}^A \neq \emptyset\}$. For simplicity, set $\mathfrak{m}_{i,v}^{\pm} := \mathfrak{m}_{j_{i,v}^{\pm}}^A \cap \mathfrak{m}_i^R$ and $m_{i,v}^{\pm} := \dim \mathfrak{m}_{i,v}^{\pm}$. According to Theorem 5.9

in [Koi1], Spec $\widetilde{A}^{\mathbf{C}}_{v^L}\setminus\{0\}$ is given by

$$\text{(3.1)}\quad \operatorname{Spec} \widetilde{A}^{\mathbf{C}}_{v^L}\setminus\{0\} = \left\{ \frac{\beta_i(v)}{\operatorname{arctanh}(\beta_i(v)/\lambda_{j^+_{i,v}}(v)) + k\pi\sqrt{-1}} \,\middle|\, i \in I^R_{v,+},\ k \in \mathbf{Z} \right\}$$
$$\cup \left\{ \frac{\beta_i(v)}{\operatorname{arctanh}(\lambda_{j^-_{i,v}}(v)/\beta_i(v)) + \left(k+\frac{1}{2}\right)\pi\sqrt{-1}} \,\middle|\, i \in I^R_{v,-},\ k \in \mathbf{Z} \right\}$$

Hence we have

$$\begin{aligned} \operatorname{Tr}_r \widetilde{A}^{\mathbf{C}}_{v^L} &= \sum_{i\in I^R_{v,+}} \sum_{k\in\mathbf{Z}} \frac{\beta_i(v)}{\operatorname{arctanh}(\beta_i(v)/\lambda_{j^+_{i,v}}(v)) + k\pi\sqrt{-1}} \times m^+_{i,v} \\ &\quad + \sum_{i\in I^R_{v,-}} \sum_{k\in\mathbf{Z}} \frac{\beta_i(v)}{\operatorname{arctanh}(\lambda_{j^-_{i,v}}(v)/\beta_i(v)) + \left(k+\frac{1}{2}\right)\pi\sqrt{-1}} \times m^-_{i,v} \\ &= \sum_{i\in I^R_{v,+}} m^+_{i,v}\lambda_{j^+_{i,v}}(v) + \sum_{i\in I^R_{v,-}} m^-_{i,v}\lambda_{j^-_{i,v}}(v) \\ &= \sum_{j\in I^A} m_j\lambda_j(v) = \operatorname{Tr} A_v \ \ (\in \mathbf{R}) \end{aligned}$$

in terms of $\coth\theta = \sum_{j\in\mathbf{Z}} \dfrac{1}{\theta + j\pi\sqrt{-1}}$ and $\coth\left(\theta + \dfrac{\pi\sqrt{-1}}{2}\right) = \tanh\theta$. Also, we have

$$\operatorname{Tr}_{\mathrm{abs}}(\widetilde{A}^{\mathbf{C}}_{v^L})^2 \le \sum_{i\in I^R_{v,+}} \sum_{k\in\mathbf{Z}} \frac{|\beta_i(v)|^2}{k^2} + \sum_{i\in I^R_{v,-}} \sum_{k\in\mathbf{Z}} \frac{|\beta_i(v)|^2}{k^2} < \infty.$$

Hence $\widetilde{M}$ is regularizable and $\operatorname{Tr}_r \widetilde{A}^{\mathbf{C}}_{v^L} = \operatorname{Tr} A_v$ holds. This implies $\langle \widetilde{H}_{\hat{0}}, v^L\rangle = \langle H_{eK}, v\rangle (= \langle (H^L)_{\hat{0}}, v^L\rangle)$. Since this relation holds for any $v \in \mathfrak{b}_r$ and $\mathfrak{b}_r$ is dense in $\mathfrak{b}$, we obtain $\widetilde{H}_{\hat{0}} = (H^L)_{\hat{0}}$. Similarly we can show $\widetilde{H}_u = (H^L)_u$ for any $u \in \widetilde{M}$. Thus we obtain $\widetilde{H} = H^L$. q.e.d.

By using Lemma 3.1 and imitating the proof of Lemma 3.1 of [Koi10], we can show the following fact.

Lemma 3.2. *The mean curvature flow $\widetilde{M}_t$ (resp. M_t) for $\widetilde{M}$ (resp. M) exists in short time and $\widetilde{M}_t = (\pi\circ\phi)^{-1}(M_t)$ holds.*

4. Proofs of Theorems A and B

Let M be a curvature-adapted isoparametric submanifold in a symmetric space G/K of non-compact type as in Theorem A. Without loss of generality, we may assume that $eK \in M$. We use the notations in the previous section.

Denote by $\mathfrak{F}$ the focal set of M at eK. Since $\pi \circ \phi$ is a pseudo-Riemannian submersion, the focal set $\widetilde{\mathfrak{F}}$ of $\widetilde{M}$ at $\hat{0}$ is equal to $\{v^L_{\hat{0}} \mid \exp^\perp(v) \in \mathfrak{F}\}$, where $\exp^\perp$ is the normal exponential map of M and $v^L_{\hat{0}}$ is the horizontal llft of v to $\hat{0}$. Here we regard the normal space $T^\perp_{\hat{0}}\widetilde{M}$ of $\widetilde{M}$ at $\hat{0}$ as a subspace of $H^0([0,1],\mathfrak{g})$. In the sequel, we identify $v^L_{\hat{0}}$ with v through $(\pi\circ\phi)_{*\hat{0}}$. The focal set $\widetilde{\mathfrak{F}}$ is equal to $\{v \mid \mathrm{Ker}(\widetilde{A}_v - \mathrm{id}) \neq \{0\}\}(\subset \mathfrak{b})$. The complex focal structure $\widetilde{\mathfrak{F}}^{\mathbf{C}}$ of $\widetilde{M}$ at $\hat{0}$ is defined by $\widetilde{\mathfrak{F}}^{\mathbf{C}} := \{v \mid \mathrm{Ker}(\widetilde{A}^{\mathbf{C}}_v - \mathrm{id}) \neq \{0\}\}(\subset \mathfrak{b}^{\mathbf{C}})$. According to the proof of Theorems B and C of [Koi6], by using (3.1) and discussing delicately, we can show that $\beta_i(v)/\lambda_{j^+_{i,v}}(v)$ and $\lambda_{j^-_{i,v}}(v)/\beta_i(v)$ are independent of the choice of v (in the sequel, we denote these constants by c_i^+ and c_i^-, respectively), $I^R_{v,+}$ and $I^R_{v,-}$ are independent of the choice of v (in the sequel, we denote these sets by I^R_+ and I^R_-) and that $\widetilde{\mathfrak{F}}^{\mathbf{C}}$ is described as follows:

$$\widetilde{\mathfrak{F}}^{\mathbf{C}} = \left(\bigcup_{i\in I^R_+}\bigcup_{j\in\mathbf{Z}} \beta_i^{\mathbf{C}-1}(\operatorname{arctanh} c_i^+ + j\pi\sqrt{-1})\right) \cup \left(\bigcup_{i\in I^R_-}\bigcup_{j\in\mathbf{Z}} \beta_i^{\mathbf{C}-1}\left(\operatorname{arctanh} c_i^- + \left(j+\frac{1}{2}\right)\pi\sqrt{-1}\right)\right). \tag{4.1}$$

Since $\lambda_{j^+_{i,v}} = \dfrac{1}{c_i^+}\beta_i$ and $\lambda_{j^-_{i,v}} = c_i^-\beta_i$, they are independent of the choice of $v \in \mathfrak{b}$. Hence we denote $\lambda_{j^\pm_{i,v}}$ by $\lambda_i^\pm$. Therefore, $\widetilde{\mathfrak{F}}$ is given by

$$\widetilde{\mathfrak{F}} = \bigcup_{i\in I^R_+} \beta_i^{-1}(\operatorname{arctanh} c_i^+). \tag{4.2}$$

Denote by Λ the set of all complex principal curvatures of $\widetilde{M}$. According to (3.1), Λ is given by

$$\Lambda = \left\{\frac{\widetilde{\beta_i^{\mathbf{C}}}}{\operatorname{arctanh} c_i^+ + j\pi\sqrt{-1}} \,\middle|\, i \in I^R_+,\ j\in\mathbf{Z}\right\} \cup \left\{\frac{\widetilde{\beta_i^{\mathbf{C}}}}{\operatorname{arctanh} c_i^- + \left(j+\frac{1}{2}\right)\pi\sqrt{-1}} \,\middle|\, i \in I^R_-,\ j\in\mathbf{Z}\right\},$$

where $\widetilde{\beta_i^{\mathbf{C}}}$ is the parallel section of $((T^\perp\widetilde{M})^{\mathbf{C}})^*$ with $(\widetilde{\beta_i^{\mathbf{C}}})_{\hat{0}} = \beta_i^{\mathbf{C}}$. From the assumption, M admits a focal submanifold. Hence we have $I^R_+ \neq \emptyset$ and $\bigcap_{i\in I^R_+}\beta_i^{-1}(\operatorname{arctanh} c_i^+) \neq \emptyset$ (see the proof of Theorems B and C in [Koi6]). Fix $O \in \bigcap_{i\in I^R_+}\beta_i^{-1}(\operatorname{arctanh} c_i^+)$. Set

$$\triangle' := \{\beta_i \mid i\in I^R\} \cup \{-\beta_i \mid i \in I^R\},$$
$$\triangle'^V := \{\beta_i \mid i\in I^R_+\} \cup \{-\beta_i \mid i \in I^R_+\},$$
$$\triangle'^H := \{\beta_i \mid i\in I^R_-\} \cup \{-\beta_i \mid i \in I^R_-\}.$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{b}(=T^{\perp}_{eK}M)$ and $\triangle$ be the root system of G/K with respect to $\mathfrak{a}$. Then we have $\triangle' = \{\alpha|_{\mathfrak{b}} \mid \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}} \neq 0\}$. Let F_l be the focal submanifold of M through $x_0 := \exp^{\perp}(O)$, which is one of the lowest dimensional focal submanifolds of M. For simplicity, we set

$$\widetilde{\lambda}_i^+ := \frac{\widetilde{\beta_i^{\mathbb{C}}}}{\operatorname{arctanh} c_i^+} \qquad (i \in I_+^R)$$

$$\widetilde{\lambda}_i^- := \frac{\widetilde{\beta_i^{\mathbb{C}}}}{\operatorname{arctanh} c_i^- + \frac{1}{2}\pi\sqrt{-1}} \qquad (i \in I_-^R)$$

and

$$b_i^+ := \frac{\pi}{\operatorname{arctanh} c_i^+} \qquad (i \in I_+^R)$$

$$b_i^- := \frac{\pi}{\operatorname{arctanh} c_i^- + \frac{1}{2}\pi\sqrt{-1}} \qquad (i \in I_-^R).$$

Then we have

$$\Lambda = \left\{ \frac{\widetilde{\lambda}_i^+}{1 + b_i^+ j\sqrt{-1}} \,\middle|\, i \in I_+^R,\ j \in \mathbb{Z} \right\} \cup \left\{ \frac{\widetilde{\lambda}_i^-}{1 + b_i^- j\sqrt{-1}} \,\middle|\, i \in I_-^R,\ j \in \mathbb{Z} \right\}.$$

For simplicity, we set $\widetilde{\lambda}_{ij}^{\pm} := \dfrac{\widetilde{\lambda}_i^{\pm}}{1 + b_i^{\pm} j\sqrt{-1}}$ $(i \in I_{\pm}^R,\ j \in \mathbb{Z})$. Take $v \in \mathfrak{b}_r \setminus \widetilde{\mathfrak{F}}$, where we note that $\widetilde{\mathfrak{F}} = \bigcup_{i \in I_+^R} (\widetilde{\lambda}_i^+)_{\hat{0}}^{-1}(1)$. We have $\dim \operatorname{Ker}(\widetilde{A}_v^{\mathbb{C}} - (\widetilde{\lambda}_{ij}^{\pm})_{\hat{0}}(v)\ \mathrm{id}) = m_i^{\pm}$ $(i \in I_{\pm}^R,\ j \in \mathbb{Z})$. Set $E_{ij}^{\pm} := \operatorname{Ker}(\widetilde{A}_v^{\mathbb{C}} - (\widetilde{\lambda}_{ij}^{\pm})_{\hat{0}}(v)\ \mathrm{id})$ $(i \in I_{\pm}^R,\ j \in \mathbb{Z})$, which are independent of the choice of $v \in \mathfrak{b}_r \setminus \widetilde{\mathfrak{F}}$. Take another $w \in \mathfrak{b}_r \setminus \widetilde{\mathfrak{F}}$. Let $\widetilde{w}$ be the parallel normal vector field of $\widetilde{M}$ with $\widetilde{w}_{\hat{0}} = w$. Denote by $\eta_{\widetilde{w}}$ the end-point map for $\widetilde{w}$ and $\widetilde{M}_w := \eta_{\widetilde{w}}(\widetilde{M})$, which is a parallel submanifold of $\widetilde{M}$. We have

$$\begin{aligned} T_w \widetilde{M}_w &= \eta_{\widetilde{w}*}(T_{\hat{0}}\widetilde{M}) \\ &= \left(\bigoplus_{i \in I_+^R} \bigoplus_{j \in \mathbb{Z}} \eta_{\widetilde{w}*}(E_{ij}^+) \right) \oplus \left(\bigoplus_{i \in I_-^R} \bigoplus_{j \in \mathbb{Z}} \eta_{\widetilde{w}*}(E_{ij}^-) \right). \end{aligned}$$

Denote by $\widetilde{A}^w$ the shape tensor of $\widetilde{M}_w$. We have

$$(\widetilde{A}^w)_v^{\mathbb{C}}|_{\eta_{\widetilde{w}*}(E_{ij}^{\pm})} = \frac{(\widetilde{\lambda}_{ij}^{\pm})_{\hat{0}}(v)}{1 - (\widetilde{\lambda}_{ij}^{\pm})_{\hat{0}}(w)}\ \mathrm{id} = \frac{(\widetilde{\lambda}_i^{\pm})_{\hat{0}}(v)}{(1 + b_i^{\pm} j\sqrt{-1}) - (\widetilde{\lambda}_i^{\pm})_{\hat{0}}(w)}\ \mathrm{id}.$$

Hence the set Λ^w of all complex principal curvatures of $\widetilde{M}_w$ is given by

$$\begin{aligned}
\Lambda^w &= \left\{ \frac{\widetilde{\lambda}_{ij}^+}{1-(\widetilde{\lambda}_{ij}^+)_{\hat{0}}(w)} \;\middle|\; i \in I_+^R,\ j \in \mathbb{Z} \right\} \\
&\quad \cup \left\{ \frac{\widetilde{\lambda}_{ij}^-}{1-(\widetilde{\lambda}_{ij}^-)_{\hat{0}}(w)} \;\middle|\; i \in I_-^R,\ j \in \mathbb{Z} \right\} \\
&= \left\{ \frac{\widetilde{\lambda}_{i}^+}{1+b_i^+ j\sqrt{-1}-(\widetilde{\lambda}_{i}^+)_{\hat{0}}(w)} \;\middle|\; i \in I_+^R,\ j \in \mathbb{Z} \right\} \\
&\quad \cup \left\{ \frac{\widetilde{\lambda}_{i}^-}{1+b_i^- j\sqrt{-1}-(\widetilde{\lambda}_{i}^-)_{\hat{0}}(w)} \;\middle|\; i \in I_-^R,\ j \in \mathbb{Z} \right\} \\
&= \left\{ \frac{\widetilde{\beta_i^{\mathbb{C}}}}{\operatorname{arctanh} c_i^+ + j\pi\sqrt{-1}-\beta_i(w)} \;\middle|\; i \in I_+^R,\ j \in \mathbb{Z} \right\} \\
&\quad \cup \left\{ \frac{\widetilde{\beta_i^{\mathbb{C}}}}{\operatorname{arctanh} c_i^- + \left(j+\frac{1}{2}\right)\pi\sqrt{-1}-\beta_i(w)} \;\middle|\; i \in I_-^R,\ j \in \mathbb{Z} \right\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
(4.3)\qquad \operatorname{Tr}(\widetilde{A}_v^w)^{\mathbb{C}} &= \sum_{i \in I_+^R} \sum_{j \in \mathbb{Z}} \frac{\beta_i(v)}{\operatorname{arctanh} c_i^+ + j\pi\sqrt{-1}-\beta_i(w)} \times m_i^+ \\
&\quad + \sum_{i \in I_-^R} \sum_{j \in \mathbb{Z}} \frac{\beta_i(v)}{\operatorname{arctanh} c_i^- + \left(j+\frac{1}{2}\right)\pi\sqrt{-1}-\beta_i(w)} \times m_i^- \\
&= \sum_{i \in I_+^R} m_i^+ \coth(\operatorname{arctanh} c_i^+ - \beta_i(w))\beta_i(v) \\
&\quad + \sum_{i \in I_-^R} m_i^- \tanh(\operatorname{arctanh} c_i^+ - \beta_i(w))\beta_i(v) \quad (v \in \mathfrak{b}),
\end{aligned}$$

where we use $\sum_{j \in \mathbb{Z}} \dfrac{1}{\theta + j\pi\sqrt{-1}} = \coth\theta$ and $\coth\left(\theta + \dfrac{\pi\sqrt{-1}}{2}\right) = \tanh\theta$. Hence we have

$$\begin{aligned}
\langle (\widetilde{H}^w)_w, v \rangle &= \left\langle \sum_{i \in I_+^R} m_i^+ \coth(\operatorname{arctanh} c_i^+ - \beta_i(w))\beta_i^\#, v \right\rangle \\
&\quad + \left\langle \sum_{i \in I_-^R} m_i^- \tanh(\operatorname{arctanh} c_i^- - \beta_i(w))\beta_i^\#, v \right\rangle,
\end{aligned}$$

where $\beta_i^\#$ is defined by $\beta_i(\cdot) = \langle \beta_i^\#, \cdot \rangle$ $(i \in I^R)$. Since this relation holds for any $v \in \mathfrak{b}_r \setminus \widetilde{\mathfrak{F}}$, we have

$$(4.4) \qquad (\widetilde{H}^w)_w = \sum_{i\in I_+^R} m_i^+ \coth(\operatorname{arctanh} c_i^+ - \beta_i(w))\beta_i^{\#} + \sum_{i\in I_-^R} m_i^- \tanh(\operatorname{arctanh} c_i^- - \beta_i(w))\beta_i^{\#}.$$

Set

$$\widetilde{C} := \{w\in\mathfrak{b} \mid (\widetilde{\lambda}_i^+)_{\hat{0}}(w) < 1 \ \ (i\in I_+^R)\} = \{w\in\mathfrak{b} \mid \beta_i(w) < \operatorname{arctanh} c_i^+ \ \ (i\in I_+^R)\},$$

which is a fundamental domain of the real Coxeter group associated with $\widetilde{M}$. Each parallel submanifold of M passes through the only one point of $\exp^\perp(\widetilde{C})$ and each focal submanifold of M passes through the only one point of $\exp^\perp(\partial\widetilde{C})$. Define a vector field X on $\widetilde{C}$ by $X_w := (\widetilde{H}^w)_w$ $(w\in\widetilde{C})$. Let $\{\psi_t\}$ be the local one-parameter transformation group of X. Now we shall prove the statements (i) and (iii) of Theorem A.

Proof of (i) *and* (iii) *of Theorem* A. First we shall show the statement (i). Denote by $\widetilde{\sigma}_i$ $(i\in I_+^R)$ the maximal dimensional simplex of $\partial\widetilde{C}$ contained in $\beta_i^{-1}(\operatorname{arctanh} c_i^+)$. Fix $i_0\in I_+^R$. Take $w_0\in\widetilde{\sigma}_{i_0}$ and $w_0'\in\widetilde{C}$ near w_0 such that $w_0 - w_0'$ is normal to $\widetilde{\sigma}_{i_0}$. Set $w_0^\varepsilon := \varepsilon w_0' + (1-\varepsilon)w_0$ for $\varepsilon\in(0,1)$. Then we have $\lim_{\varepsilon\to+0}\beta_{i_0}(w_0^\varepsilon) = \operatorname{arctanh} c_{i_0}^+$ and $\sup_{0<\varepsilon<1}\beta_{i_0}(w_0^\varepsilon) < \operatorname{arctanh} c_i^+$ for each $i\in I_+^R\backslash\{i_0\}$. Hence we have

$$\lim_{\varepsilon\to+0}\coth(\operatorname{arctanh} c_{i_0}^+ - \beta_{i_0}(w_0^\varepsilon)) = \infty$$

and

$$\sup_{0<\varepsilon<1}\coth(\operatorname{arctanh} c_i^+ - \beta_i(w_0^\varepsilon)) < \infty \quad (i\in I_+^R\backslash\{i_0\}).$$

Therefore, we have $\lim_{\varepsilon\to+0}\dfrac{X_{w_0^\varepsilon}}{\|X_{w_0^\varepsilon}\|}$ is the outward unit normal vector of $\widetilde{\sigma}_{i_0}$. Also we have $\lim_{\varepsilon\to+0}\|X_{w_0^\varepsilon}\| = \infty$. From these facts, X is as in the first figure of Figure 1 on a sufficiently small collar neighborhood of $\widetilde{\sigma}_{i_0}$. Define a function ρ over $\widetilde{C}$ by

$$\rho(w) := -\sum_{i\in I_+^R} m_i^+ \log\sinh(\operatorname{arctanh} c_i^+ - \beta_i(w)) - \sum_{i\in I_-^R} m_i^- \log\cosh(\operatorname{arctanh} c_i^- - \beta_i(w)) \quad (w\in\widetilde{C}).$$

From the definition of X and (4.4), we have $\operatorname{grad}\rho = X$. For simplicity, set $\partial_i := \frac{\partial}{\partial x_i}$ $(i = 1,\dots,r)$. Then we have

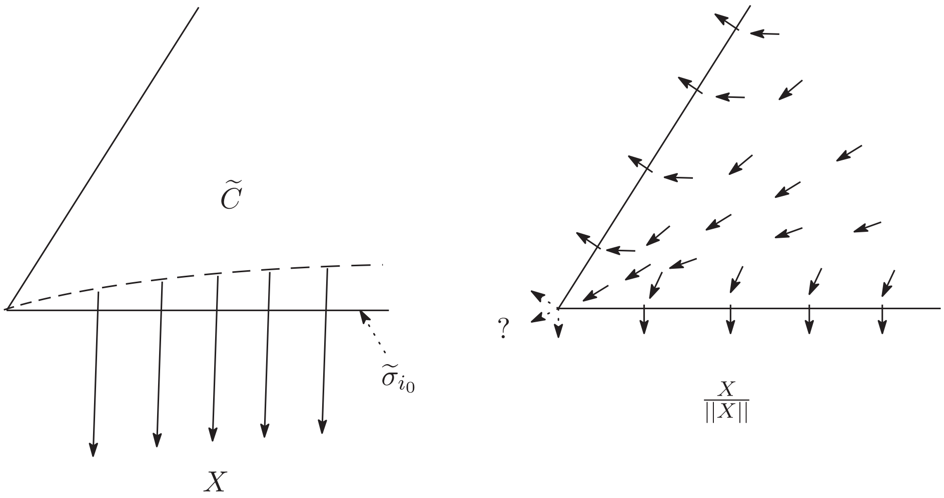

FIGURE 1

$$(\partial_j\partial_k\rho)(w) = \sum_{i\in I_+^R}\frac{m_i^+}{\sinh^2(\operatorname{arctanh} c_i^+ - \beta_i(w))}\beta_i(\partial_j)\beta_i(\partial_k) - \sum_{i\in I_-^R}\frac{m_i^-}{\cosh^2(\operatorname{arctanh} c_i^- - \beta_i(w))}\beta_i(\partial_j)\beta_i(\partial_k). \tag{4.5}$$

As $w \to \partial\widetilde{C}$, $\frac{1}{\sinh^2(\operatorname{arctanh} c_i^+ - \beta_i(w))} \to \infty$ for at least one $i \in I_+^R$ and $\frac{1}{\cosh^2(\operatorname{arctanh} c_i^- - \beta_i(w))} \le 1$ for all $i \in I_-^R$. Hence we see that ρ is downward convex on a sufficiently small collar neighborhood of $\partial\widetilde{C}$. Furthermore, since $\operatorname{codim} M = \operatorname{rank}(G/K)$ and $\dim(\mathfrak{p}_\alpha \cap \mathfrak{p}') \ge \frac{1}{2}\dim \mathfrak{p}_\alpha$ $(\alpha \in \triangle)$ by the assumption, we have $I_+^R = I^R$, $m_i^+ \ge m_i^-$ and $c_i^+ = c_i^-$ $(i \in I_-^R)$. From the relation (4.5), we have

$$(\partial_j\partial_k\rho)(w) \ge \sum_{i\in I^R\setminus I_-^R}\frac{m_i^+}{\sinh^2(\operatorname{arctanh} c_i^+ - \beta_i(w))}\beta_i(\partial_j)\beta_i(\partial_k) + \sum_{i\in I_-^R}\frac{4m_i^+}{\sinh^2 2(\operatorname{arctanh} c_i^+ - \beta_i(w))}\beta_i(\partial_j)\beta_i(\partial_k).$$

Hence we see that ρ is downward convex on the whole of $\widetilde{C}$. Also, it is clear that $\rho(w) \to \infty$ as $w \to \partial\widetilde{C}$ and that $\rho(tw) \to -\infty$ as $t \to \infty$ for each $w \in \widetilde{C}$. From these facts, ρ and X are as in Figure 2. Hence $t \mapsto \psi_t(\hat{0})$ converges to a point w_2 of $\partial\widetilde{C}$ in a finite time T. Therefore M is not minimal and the mean curvature flow M_t collapses to the focal submanifold of M through $\exp^\perp(w_2)$ in finite time. Thus the statement (i) is shown.

Next we shall show the statement (iii) of Theorem A. Since X is as in the second figure of Figure 2, we obtain the following fact:

($*_1$) For each $w \in \partial \widetilde{C}$, there exists $w' \in \widetilde{C}$ such that the flow $\psi_t(w')$ converges to w.

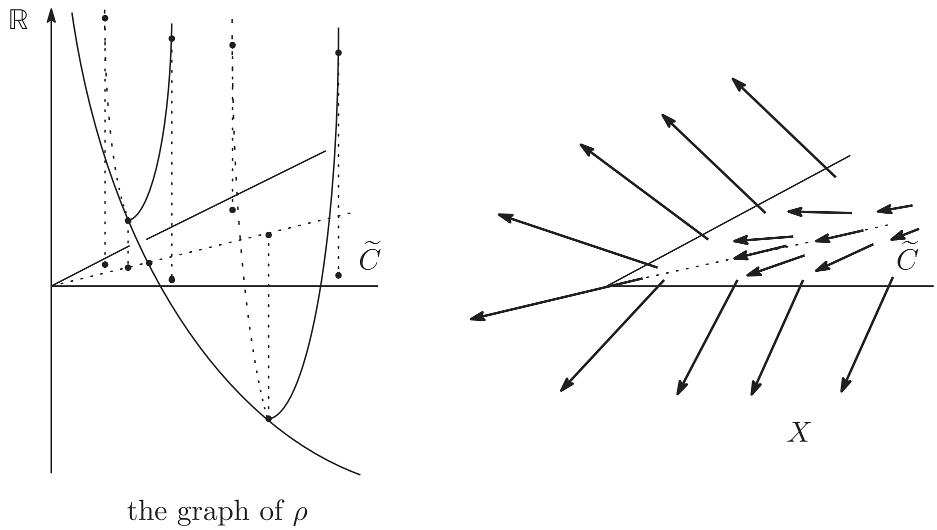

FIGURE 2

Now we shall show the following statement:

($*_2$) For any $w \in \partial \widetilde{C}$, the set $\{w' \in \widetilde{C} \mid \text{the flow } \psi_t(w') \text{ converges to } w\}$ is equal to the image of a flow of X.

That is, we shall show that the situation as in Figure 3 cannot happen. Let W be the real Coxeter group of $\widetilde{M}$ at $\hat{0}$, that is, the group generated by the reflections with respect to the (real) hyperplanes l_i's $(i \in I_+^R)$ in $\mathfrak{b}$ containing $\widetilde{\sigma}_i$. This group W is a finite Coxeter group. Set $V := \mathrm{Span}\{\beta_i^{\#} \mid i \in I_+^R\}$ and $\widetilde{C}_V := \widetilde{C} \cap V$ (see Figure 4). This space V is W-invariant and W acts trivially on the orthogonal complement $V^{\perp}$ of V. Let $\{\phi_1, \dots, \phi_{r'}\}$ be a base of the space of all W-invariant polynomial functions over V, where we note that $r' = \dim V$. Set $\Phi := (\phi_1, \dots, \phi_{r'})$, which is a polynomial map from V to $\mathbb{R}^{r'}$. It is shown that Φ is a homeomorphism of the closure $\overline{\widetilde{C}_V}$ of $\widetilde{C}_V$ onto $\Phi(\overline{\widetilde{C}_V})$. Set $\xi_w(t) := \psi_t(w)$ and $\bar{\xi}_w(t) := \Phi(\psi_t(w))$, where $w \in \widetilde{C}_V$. Let $(x_1, \dots, x_{r'})$ be a Euclidean coordinate of V and $(y_1, \dots, y_{r'})$ the natural coordinate of $\mathbb{R}^{r'}$. Set $\xi_w^i(t) := x_i(\xi_w(t))$ and $\bar{\xi}_w^i(t) := y_i(\bar{\xi}_w(t))$ $(i = 1, \dots, r')$. Then we have

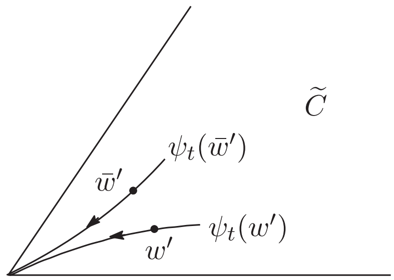

FIGURE 3

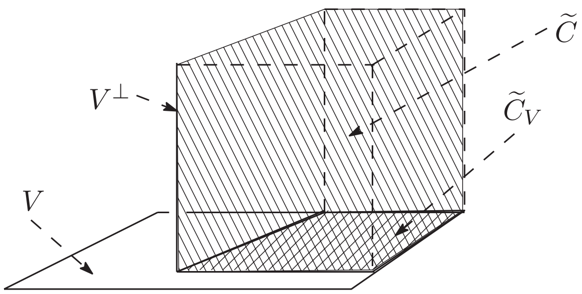

FIGURE 4

$$\begin{aligned}(\bar{\xi}^i_w)'(t) &= \langle \mathrm{grad}(y_i \circ \Phi)_{\xi_w(t)}, X_{\xi_w(t)} \rangle \\ &= \sum_{j \in I^R_+} m^+_j \coth(\mathrm{arctanh}\, c^+_j - \beta_j(\xi_w(t)))\beta_j(\mathrm{grad}(y_i \circ \Phi)_{\xi_w(t)}) \\ &\quad + \sum_{j \in I^R_-} m^-_j \tanh(\mathrm{arctanh}\, c^-_j - \beta_j(\xi_w(t)))\beta_j(\mathrm{grad}(y_i \circ \Phi)_{\xi_w(t)}).\end{aligned}$$

Let f_i be the W-invariant C^∞-function over V such that

$$\begin{aligned}f_i(v) := &\sum_{j \in I^R_+} m^+_j \coth(\mathrm{arctanh}\, c^+_j - \beta_j(v))\beta_j(\mathrm{grad}(y_i \circ \Phi)_v) \\ &+ \sum_{j \in I^R_-} m^-_j \tanh(\mathrm{arctanh}\, c^-_j - \beta_j(v))\beta_j(\mathrm{grad}(y_i \circ \Phi)_v)\end{aligned}$$

for all $v \in \widetilde{C}_V$. It is easy to show that such a W-invariant C^∞-function exists uniquely. According to the Schwarz's theorem in [S], we can describe f_i as

$f_i = Y_i \circ \Phi$ in terms of some C^∞-function Y_i over $\mathbb{R}^{r'}$. Set $Y := (Y_1, \dots, Y_{r'})$, which is regarded as a C^∞-vector field on $\mathbb{R}^{r'}$. Then we have $Y_{\Phi(w)} = \Phi_*(X_w)$ $(w \in \widetilde{C}_V)$, that is, $Y|_{\Phi(\widetilde{C}_V)} = \Phi_*(X)$. Also we can show that $Y|_{\partial\Phi(\widetilde{C}_V)}$ has no zero point. From these facts, we see that, for any $w \in \partial\widetilde{C}_V$, the set $\{w' \in \widetilde{C}_V \mid$ the flow $\psi_t(w')$ converges to $w\}$ is equal to the image of a flow of X (see Figure 5). In more general, we obtain the statement $(*_2)$ from this fact.

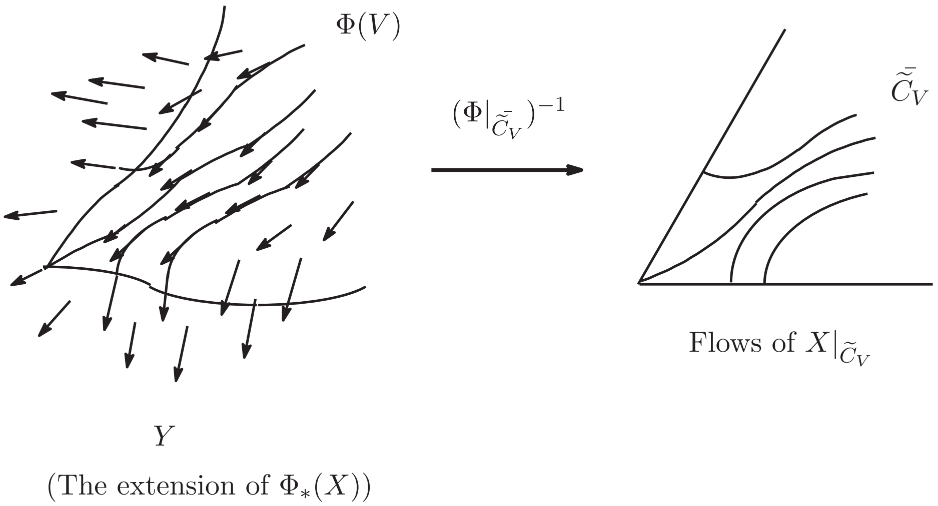

Figure 5

Take an arbitrary focal submanifold F of M. Let $\exp^\perp(w_1)$ be the only intersection point of F and $\exp^\perp(\partial\widetilde{C})$. According to the above fact $(*_2)$, the set of all parallel submanifolds of M collapsing to F along the mean curvature flow is a one-parameter C^∞-family. Thus the statement (iii) of Theorem A is shown. q.e.d.

Next we prove the statement (ii) of Theorem A.

Proof of (ii) *of Theorem* A. Let M and F be as in (ii) of Theorem A. Since the natural fibration of M onto F is spherical, so is also the natural fibration of $\widetilde{M}$ onto $\widetilde{F}$. Hence $\widetilde{F}$ meets one of $(\partial\widetilde{C} \cap \beta_i^{-1}(\operatorname{arctanh} c_i^+))^\circ$'s $(i \in I_+^R)$ (at one point). Assume that $\widetilde{F}$ meets $(\partial\widetilde{C} \cap \beta_{i_0}^{-1}(\operatorname{arctanh} c_{i_0}^+))^\circ$. Let u_0 be the intersection point. Let T be the explosion time of the flow M_t. Denote by A^t (rep. $\widetilde{A}^t$) the shape tensor of M_t (resp. $\widetilde{M}_t$), H^t the mean curvature vector of M_t and $\widetilde{H}^t$ the regularized mean curvature vector of $\widetilde{M}_t$. We have

(4.6) $$\mathrm{Spec}(\widetilde{A}^t_v)^{\mathbb{C}}\setminus\{0\} = \left\{ \frac{\beta_i(v)}{\mathrm{arctanh}\ c_i^+ + j\pi\sqrt{-1} - \beta_i(\psi_t(\hat{0}))} \,\middle|\, i \in I_+^R,\ j \in \mathbb{Z} \right\} \cup \left\{ \frac{\beta_i(v)}{\mathrm{arctanh}\ c_i^- + \left(j+\frac{1}{2}\right)\pi\sqrt{-1} - \beta_i(\psi_t(\hat{0}))} \,\middle|\, i \in I_-^R,\ j \in \mathbb{Z} \right\}$$

for each $v \in T^\perp_{\psi_t(\hat{0})}\widetilde{M}_t (= T_{\hat{0}}\widetilde{M})$. Since

$$\lim_{t\to T-0} \psi_t(\hat{0}) = u_0 \in (\partial\widetilde{C} \cap \beta_{i_0}^{-1}(\mathrm{arctanh}\ c_{i_0}^+))^\circ,$$

we have $\lim_{t\to T-0} \beta_{i_0}(\psi_t(\hat{0})) = \mathrm{arctanh}\ c_{i_0}^+$ and $\lim_{t\to T-0} \beta_i(\psi_t(\hat{0})) < \mathrm{arctanh}\ c_i^+$ $(i \in I_+^R\setminus\{i_0\})$. Hence we have

(4.7) $$\begin{aligned} &\lim_{t\to T-0} \|(\widetilde{A}^t_v)^{\mathbb{C}}\|_\infty^2 (T-t) \\ &= \lim_{t\to T-0} \frac{\beta_{i_0}(v)^2}{(\mathrm{arctanh}\ c_{i_0}^+ - \beta_{i_0}(\psi_t(\hat{0})))^2}(T-t) \\ &= \frac{1}{2}\beta_{i_0}(v)^2 \lim_{t\to T-0} \frac{1}{(\mathrm{arctanh}\ c_{i_0}^+ - \beta_{i_0}(\psi_t(\hat{0})))\beta_{i_0}\left(\frac{d}{dt}\psi_t(\hat{0})\right)}. \end{aligned}$$

Since $\frac{d}{dt}\psi_t(\hat{0}) = (\widetilde{H}^t)_{\psi_t(\hat{0})}$, it follows from (4.4) that

$$\begin{aligned} &\lim_{t\to T-0} (\mathrm{arctanh}\ c_{i_0}^+ - \beta_{i_0}(\psi_t(\hat{0}))))\beta_{i_0}\left(\frac{d}{dt}\psi_t(\hat{0})\right) \\ &= \lim_{t\to T-0} \left(\sum_{i\in I_+^R} m_i^+ \coth(\mathrm{arctanh}\ c_i^+ - \beta_i(\psi_t(\hat{0})))\langle\beta_i^\#, \beta_{i_0}^\#\rangle(\mathrm{arctanh}\ c_{i_0}^+ - \beta_{i_0}(\psi_t(\hat{0}))) \right. \\ &\qquad \left. + \sum_{i\in I_-^R} m_i^- \tanh(\mathrm{arctanh}\ c_i^- - \beta_i(\psi_t(\hat{0})))\langle\beta_i^\#, \beta_{i_0}^\#\rangle(\mathrm{arctanh}\ c_{i_0}^+ - \beta_{i_0}(\psi_t(\hat{0}))) \right) \\ &= m_{i_0}^+\langle\beta_{i_0}^\#, \beta_{i_0}^\#\rangle \lim_{t\to T-0} \coth(\mathrm{arctanh}\ c_{i_0}^+ - \beta_{i_0}(\psi_t(\hat{0})))(\mathrm{arctanh}\ c_{i_0}^+ - \beta_{i_0}(\psi_t(\hat{0}))) \\ &= m_{i_0}^+\langle\beta_{i_0}^\#, \beta_{i_0}^\#\rangle \lim_{t\to T-0} \cosh^2(\mathrm{arctanh}\ c_{i_0}^+ - \beta_{i_0}(\psi_t(\hat{0}))) \\ &= m_{i_0}^+\langle\beta_{i_0}^\#, \beta_{i_0}^\#\rangle, \end{aligned}$$

which together with (4.7) deduces

$$\lim_{t\to T-0} \|(\widetilde{A}^t_v)^{\mathbb{C}}\|_\infty^2 (T-t) = \frac{\beta_{i_0}(v)^2}{2m_{i_0}^+\langle\beta_{i_0}^\#, \beta_{i_0}^\#\rangle}$$

and hence

$$\lim_{t\to T-0}\max_{v\in S^{\perp}_{\psi_t(\hat{0})}\widetilde{M}_t}\|(\widetilde{A}^t_v)^{\mathbf{C}}\|^2_{\infty}(T-t)=\frac{1}{2m^+_{i_0}}. \tag{4.8}$$

Thus $\widetilde{M}_t$ has type I singularity. Denote by $\exp_G$ the exponential map of G and Exp the exponential map of G/K at eK. Also, denote by $S(1)$ the unit hypersphere in $\mathfrak{b}$ centered at 0. Set $g_t:=\exp_G(\psi_t(\hat{0}))$ and $\bar{v}_t:=g_{t*}(v)$ for each $v\in S(1)$. The relation $\bar{v}_t=(\pi\circ\phi)_{*\psi_t(\hat{0})}(v)$ holds. Since M is proper complex equifocal and curvature-adapted by the assumption and since M_t is a parallel submanifold of M, M_t is also proper complex equifocal and curvature-adapted (see Lemma 3.4 of [Koi9]). It is easy to show that $T_{\mathrm{Exp}(\psi_t(\hat{0}))}M_t=g_{t*}(\mathfrak{m})$ and that $T_{\mathrm{Exp}(\psi_t(\hat{0}))}M_t=g_{t*}(\mathfrak{m}^R_0)+\sum_{i\in I^R}g_{t*}(\mathfrak{m}^R_i)$ is the common-eigenspace decomposition of $R(\cdot,\bar{v}_t)\bar{v}_t$'s $(v\in\mathfrak{b})$. In similar to β_i $(i\in I^R)$, λ^+_i $(i\in I^R_+)$ and λ^-_i $(i\in I^R_-)$, we define linear functions β^t_i $(i\in I^R)$, $(\lambda^t_i)^+$ $(i\in I^R_+)$ and $(\lambda^t_i)^-$ $(i\in I^R_-)$ on $T^{\perp}_{\psi_t(\hat{0})}M_t=g_{t*}\mathfrak{b}$ by

$$\begin{aligned}
&R(\cdot,\bar{v}_t)\bar{v}_t|_{g_{t*}(\mathfrak{m}^R_i)}=\beta^t_i(\bar{v}_t)^2\,\mathrm{id}\quad(v\in\mathfrak{b}),\\
&\{\lambda\in\mathrm{Spec}(A^t_{\bar{v}_t}|_{g_{t*}(\mathfrak{m}^R_i)})\,|\,|\lambda|>|\beta^t_i(\bar{v}_t)|\}=\{(\lambda^t_i)^+(\bar{v}_t)\}\quad(v\in\mathfrak{b})\\
&\{\lambda\in\mathrm{Spec}(A^t_{\bar{v}_t}|_{g_{t*}(\mathfrak{m}^R_i)})\,|\,|\lambda|<|\beta^t_i(\bar{v}_t)|\}=\{(\lambda^t_i)^-(\bar{v}_t)\}\quad(v\in\mathfrak{b}).
\end{aligned}$$

It is clear that $\beta^t_i=\beta_i\circ g^{-1}_{t*}$ $(i\in I^R)$. The values $\beta^t_i(\bar{v}_t)/(\lambda^t_i)^+(\bar{v}_t)$ $(i\in I^R_+)$ and $(\lambda^t_i)^-(\bar{v}_t)/\beta^t_i(\bar{v}_t)$ $(i\in I^R_-)$ are independent of the choice of $v\in\mathfrak{b}$. Denote by $(c^t_i)^+$ and $(c^t_i)^-$ these constants, respectively. If $i\in I^R_+\cap I^R_-$, then we have $(c^t_i)^+=(c^t_i)^-$. Hence we shall denote $(c^t_i)^+$ $(i\in I^R_+)$ and $(c^t_i)^-$ $(i\in I^R_-)$ by c^t_i for simplicity. In the sequel, we use this notation. The spectrum of $(\widetilde{A}^t_v)^{\mathbf{C}}$ other than zero is given by

$$\begin{aligned}
\mathrm{Spec}(\widetilde{A}^t_v)^{\mathbf{C}}\setminus\{0\}=&\left\{\frac{\beta_i(v)}{\operatorname{arctanh}c^t_i+j\pi\sqrt{-1}}\,\middle|\,i\in I^R_+,\ j\in\mathbf{Z}\right\}\\
&\cup\left\{\frac{\beta_i(v)}{\operatorname{arctanh}c^t_i+\left(j+\frac{1}{2}\right)\pi\sqrt{-1}}\,\middle|\,i\in I^R_-,\ j\in\mathbf{Z}\right\}.
\end{aligned}$$

On the other hand, we have $\lim_{t\to T-0}\max_{v\in S(1)}|(\lambda^t_{i_0})^+(\bar{v}_t)|=\infty$ and hence $\lim_{t\to T-0}c^t_{i_0}=0$. Also we have $\lim_{t\to T-0}\max_{v\in S(1)}|(\lambda^t_i)^+(\bar{v}_t)|<\infty$ and hence $\lim_{t\to T-0}|c^t_i|>0$ $(i\in I^R_+\setminus\{i_0\})$. Therefore we obtain

$$\begin{aligned}
&\lim_{t\to T-0}(T-t)\max_{v\in S(1)}\|(\widetilde{A}^t_v)^{\mathbf{C}}\|^2_{\infty} \qquad (4.9)\\
&\quad=\lim_{t\to T-0}(T-t)\max_{v\in S(1)}\left(\frac{\beta_{i_0}(v)}{\operatorname{arctanh}c^t_{i_0}}\right)^2
\end{aligned}$$

$$= \max_{v\in S(1)} \beta_{i_0}(v)^2 \lim_{t\to T-0} \frac{T-t}{\operatorname{arctanh}^2 c_{i_0}^t}$$

$$= \max_{v\in S(1)} \lim_{t\to T-0} \left(\frac{T-t}{\operatorname{arctanh}^2(\beta_{i_0}(v)/(\lambda_{i_0}^t)^+(\bar v_t))} \left(\frac{\beta_{i_0}(v)}{(\lambda_{i_0}^t)^+(\bar v_t)} \right)^2 (\lambda_{i_0}^t)^+(\bar v_t)^2 \right)$$

$$= \lim_{t\to T-0} (T-t) \max_{v\in S(1)} (\lambda_{i_0}^t)^+(\bar v_t)^2$$

$$= \lim_{t\to T-0} (T-t) \max_{v\in S(1)} \|A_{\bar v_t}^t\|_\infty^2 = \lim_{t\to T-0} (T-t) \max_{v\in S(1)} \|A_{\bar v_t}^t\|_\infty^2.$$

From this relation and (4.8), we obtain

$$\lim_{t\to T-0} (T-t) \max_{v\in S(1)} \|A_{\bar v_t}^t\|_\infty^2 = \frac{1}{2m_{i_0}^+} < \infty.$$

Thus the mean curvature flow M_t $(0 \le t < T)$ has type I singularity. q.e.d.

For each $S \subset I_+^R$, we set

$$\begin{aligned} \widetilde{\sigma}_S &:= \{ w \in \partial\widetilde{C} \mid (\widetilde{\lambda}_i^+)_{\hat 0}(w) < 1 \ \ (i \in I_+^R \setminus S) \ \& \ (\widetilde{\lambda}_i^+)_{\hat 0}(w) = 1 \ \ (i \in S) \} \\ &= \{ w \in \widetilde{C} \mid \beta_i(w) < \operatorname{arctanh} c_i^+ \ \ (i \in I_+^R \setminus S) \ \& \ \beta_i(w) = \operatorname{arctanh} c_i^+ \ \ (i \in S) \}, \end{aligned}$$

which is a simplex of $\widetilde{C}$. Take $w \in \widetilde{\sigma}_S$. Let $\widetilde{w}$ be the parallel normal vector field of $\widetilde{M}$ with $\widetilde{w}_{\hat 0} = w$. Denote by $\eta_{\widetilde{w}}$ the end-point map for $\widetilde{w}$ and $\widetilde{F}_w := \eta_{\widetilde{w}}(\widetilde{M})$, which is a focal submanifold of $\widetilde{M}$. We have

$$T_w\widetilde{F}_w = \left(\bigoplus_{i\in I_+^R\setminus S} \bigoplus_{j\in\mathbb{Z}} \eta_{\widetilde{w}*}(E_{ij}^+) \right) \oplus \left(\bigoplus_{i\in I_-^R} \bigoplus_{j\in\mathbb{Z}} \eta_{\widetilde{w}*}(E_{ij}^-) \right).$$

Denote by $\widetilde{A}^w$ the shape tensor of $\widetilde{F}_w$. In similar to (4.3), we have

$$\begin{aligned} \operatorname{Tr}(\widetilde{A}_v^w)^{\mathbb{C}} = &\sum_{i\in I^R\setminus S} m_i^+ \coth(\operatorname{arctanh} c_i^+ - \beta_i(w))\beta_i(v) \\ &+ \sum_{i\in I_-^R} m_i^- \tanh(\operatorname{arctanh} c_i^+ - \beta_i(w))\beta_i(v) \ \ (\in \mathbb{R}) \end{aligned} \tag{4.10}$$

for any $v \in \mathfrak{b}$, where $\mathfrak{b}$ is regarded as a subspace of $T_w^\perp\widetilde{F}_w$. Set $L := \widetilde{M} \cap T_w^\perp\widetilde{F}_w$, which is a focal leaf of $\widetilde{M}$. For any $u \in L$, let $\mathfrak{b}_u$ be the section of $\widetilde{M}$ through u. We can show $(\widetilde{H}^w)_w \in \bigcap_{u\in L} \mathfrak{b}_u$. Hence, from (4.10), the regularized mean curvature vector $\widetilde{H}^w$ of $\widetilde{F}_w$ exists and $(\widetilde{H}^w)_w$ is given by

$$\begin{aligned} (\widetilde{H}^w)_w = &\sum_{i\in I^R\setminus S} m_i^+ \coth(\operatorname{arctanh} c_i^+ - \beta_i(w))\beta_i^\# \\ &+ \sum_{i\in I_-^R} m_i^- \tanh(\operatorname{arctanh} c_i^+ - \beta_i(w))\beta_i^\#. \end{aligned} \tag{4.11}$$

Define a vector field $X^{\widetilde{\sigma}_S}$ on $\widetilde{\sigma}_S$ by $X^{\widetilde{\sigma}_S}_w := (\widetilde{H}^w)_w$ $(w \in \widetilde{\sigma}_S)$. This vector field $X^{\widetilde{\sigma}_S}$ is tangent to $\widetilde{\sigma}_S$. Let $\{\psi^{\widetilde{\sigma}_S}_t\}$ be the local one-parameter transformation group of $X^{\widetilde{\sigma}_S}$.

Proof of Theorem B. First we shall show the statement (i) of Theorem B. Let F be as in the satement (i) of Theorem B. Set $\widetilde{F} := (\pi \circ \phi)^{-1}(F)$. Since the lowest dimensional focal submanifold F_l of M is a one-point set by the assumption, we have $I^R_- = \emptyset$. Let w_0 be the intersection point of $\widetilde{F}$ and $\widetilde{\sigma}$. Set $S_0 := \{i \in I^R_+(= I^R) \,|\, \beta_i(w_0) = \operatorname{arctanh} c_i^+\}$. Since $\dim \widetilde{\sigma} \geq 1$, we have $I^R \setminus S_0 \neq \emptyset$. According to (4.11), we have

$$(4.12)\quad (X^{\widetilde{\sigma}})_w = (\widetilde{H}^w)_w = \sum_{i \in I^R \setminus S_0} m_i^+ \coth(\operatorname{arctanh} c_i^+ - \beta_i(w))\beta_i^{\#} \quad (w \in \widetilde{\sigma}).$$

We can show that $X^{\widetilde{\sigma}}$ is as in Figure 6 on a sufficiently small collar neighborhood of each maximal dimensional stratum of $\partial\widetilde{\sigma}$. Define a function $\rho_{\widetilde{\sigma}}$ over $\widetilde{\sigma}$ by

$$\rho_{\widetilde{\sigma}}(w) := -\sum_{i \in I^R \setminus S_0} m_i^+ \log \sinh(\operatorname{arctanh} c_i^+ - \beta_i(w)) \quad (w \in \widetilde{\sigma}).$$

Easily we can show $\operatorname{grad} \rho_{\widetilde{\sigma}} = X^{\widetilde{\sigma}}$. Let $(x_1, \dots, x_{r''})$ be the Euclidean coordinate of $\bigcap_{i \in S_0} \beta_i^{-1}(\operatorname{arctanh} c_i^+)$. For simplicity, set $\partial_i := \frac{\partial}{\partial x_i}$ $(i = 1, \dots, r'')$. Then we have

$$(\partial_j \partial_k \rho_{\widetilde{\sigma}})(w) = \sum_{i \in I^R \setminus S_0} \frac{m_i^+}{\sinh^2(\operatorname{arctanh} c_i^+ - \beta_i(w))} \beta_i(\partial_j)\beta_i(\partial_k).$$

Hence we see that $\rho_{\widetilde{\sigma}}$ is downward convex on $\widetilde{\sigma}$. Also, it is clear that $\rho_{\widetilde{\sigma}}(w) \to \infty$ as $w \to \partial\widetilde{\sigma}$ and that $\rho_{\widetilde{\sigma}}(tw) \to -\infty$ as $t \to \infty$ for each $w \in \widetilde{\sigma}$. From these facts, it follows that $\psi_t^{\widetilde{\sigma}}(w_0)$ converges to a point w_1 of $\partial\widetilde{\sigma}$ in a finite time. The mean curvature flow F_t collapses to the focal submanifold of M through $\exp^\perp(w_1)(\in \exp^\perp(\partial\widetilde{\sigma}))$. This completes the proof of the first-half part of the statement (i). The second-half part of the statement (i) is proved by imitating the proof of the statement (ii) of Theorem A.

Next we shall show the statement (ii) of Theorem B. Set $V := \operatorname{Span}\{\beta_i^{\#} \,|\, i \in I^R_+\}$ and $\widetilde{\sigma}_V := \widetilde{\sigma} \cap V$. Denote by $V_{\widetilde{\sigma}}$ be the affine subspace of V containing $\widetilde{\sigma}_V$ as an open subset. Let $W_{\widetilde{\sigma}}$ be a finite Coxeter group generated by the reflections with respect to the (real) hyperplanes $l_i^{\widetilde{\sigma}}$'s $(i \in I^R \setminus S_0)$ in $V_{\widetilde{\sigma}}$ containing $\widetilde{\sigma}_i \cap V_{\widetilde{\sigma}}$. Let $\{\phi_1^{\widetilde{\sigma}}, \dots, \phi_{r'}^{\widetilde{\sigma}}\}$ be a base of the space of all $W_{\widetilde{\sigma}}$-invariant polynomial functions over $V_{\widetilde{\sigma}}$, where we note that $r' = \dim V_{\widetilde{\sigma}}$. Set $\Phi_{\widetilde{\sigma}} := (\phi_1^{\widetilde{\sigma}}, \dots, \phi_{r'}^{\widetilde{\sigma}})$, which is a polynomial map from $V_{\widetilde{\sigma}}$ to $\mathbb{R}^{r'}$. It is shown that $\Phi_{\widetilde{\sigma}}$ is a homeomorphism of the closure $\overline{\widetilde{\sigma}_V}$ of $\widetilde{\sigma}_V$ onto $\Phi_{\widetilde{\sigma}}(\widetilde{\sigma}_V)$. Set $\xi_w(t) := \psi_t(w)$ and $\bar{\xi}_w(t) := \Phi_{\widetilde{\sigma}}(\psi_t^{\widetilde{\sigma}}(w))$, where $w \in \widetilde{\sigma}_V$. Let $(x_1, \dots, x_{r'})$ be a Euclidean coordinate of $V_{\tilde{\sigma}}$ and $(y_1, \ldots, y_{r'})$ the natural coordinate of $\mathbb{R}^{r'}$. Set $\xi_w^i(t) := x_i(\xi_w(t))$ and $\bar{\xi}_w^i(t) := y_i(\bar{\xi}_w(t))$ $(i = 1, \ldots, r')$. Then we have

$$\begin{aligned}(\bar{\xi}_w^i)'(t) &= \langle \mathrm{grad}(y_i \circ \Phi_{\tilde{\sigma}})_{\xi_w(t)}, X^{\tilde{\sigma}}_{\xi_w(t)} \rangle \\ &= \sum_{j \in I^R \setminus S_0} m_j^+ \coth(\operatorname{arctanh} c_j^+ - \beta_j(\xi_w(t)))\beta_j(\mathrm{grad}(y_i \circ \Phi_{\tilde{\sigma}})_{\xi_w(t)}).\end{aligned}$$

Let $f_i^{\tilde{\sigma}}$ be the $W_{\tilde{\sigma}}$-invariant C^∞-function over $V_{\tilde{\sigma}}$ such that

$$f_i^{\tilde{\sigma}}(v) := \sum_{j \in I^R \setminus S_0} m_j^+ \coth(\operatorname{arctanh} c_j^+ - \beta_j(v))\beta_j(\mathrm{grad}(y_i \circ \Phi_{\tilde{\sigma}})_v)$$

for all $v \in \tilde{\sigma}_V$. It is easy to show that such a $W_{\tilde{\sigma}}$-invariant C^∞-function exists uniquely. According to the Schwarz's theorem in [S], we can describe $f_i^{\tilde{\sigma}}$ as $f_i^{\tilde{\sigma}} = Y_i^{\tilde{\sigma}} \circ \Phi_{\tilde{\sigma}}$ in terms of some C^∞-function $Y_i^{\tilde{\sigma}}$ over $\mathbb{R}^{r'}$. Set $Y^{\tilde{\sigma}} := (Y_1^{\tilde{\sigma}}, \ldots, Y_r^{\tilde{\sigma}})$, which is regarded as a C^∞-vector field on $\mathbb{R}^{r'}$. Then we have $Y^{\tilde{\sigma}}_{\Phi_{\tilde{\sigma}}(w)} = (\Phi_{\tilde{\sigma}})_*(X_w^{\tilde{\sigma}})$ $(w \in \tilde{\sigma}_V)$, that is, $Y^{\tilde{\sigma}}|_{\Phi_{\tilde{\sigma}}(\tilde{\sigma}_V)} = (\Phi_{\tilde{\sigma}})_*(X^{\tilde{\sigma}})$. Also we can show that $Y^{\tilde{\sigma}}|_{\partial\Phi_{\tilde{\sigma}}(\tilde{\sigma}_V)}$ has no zero point. From these facts and the fact that $X^{\tilde{\sigma}}$ is as in Figure 6 on a sufficiently small collar neighborhood of each maximal dimensional stratum of $\partial\tilde{\sigma}_V$, we see that, for any $w \in \partial\tilde{\sigma}_V$, the set $\{w' \in \tilde{\sigma}_V \,|\, \text{the flow } \psi_t^{\tilde{\sigma}}(w') \text{ converges to } w\}$ is equal to the image of a flow of $X^{\tilde{\sigma}}$. In more general, the same fact holds for any $w \in \partial\tilde{\sigma}$. From this fact, the statement (ii) of Theorem B follows. q.e.d.

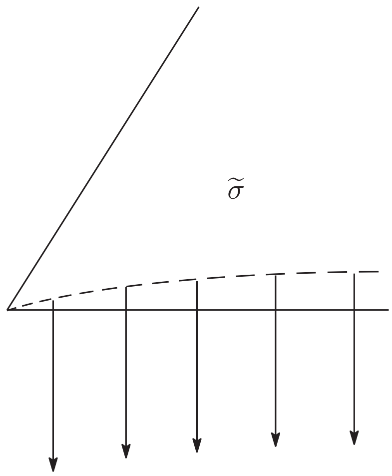

FIGURE 6

We shall explain that, in the statement of Theorem B, we cannot weaken the condition that F_l is a one-point set to the condition ($\triangle' = \triangle$ and $\dim(\mathfrak{p}_\alpha \cap \mathfrak{p}') \geq$

$\frac{1}{2}\dim \mathfrak{p}_\alpha$ $(\alpha \in \triangle))$ in the statement of Theorem A. Assume that M satisfies the condition in the statement of Theorem A. Let S_0 be as above and $\widetilde{\sigma} := \widetilde{\sigma}_{S_0}$. Define a function $\rho_{\widetilde{\sigma}}$ over $\widetilde{\sigma}$ by

$$\begin{aligned}\rho_{\widetilde{\sigma}}(w) := &- \sum_{i\in I^R\setminus S_0} m_i^+ \log\sinh(\operatorname{arctanh} c_i^+ - \beta_i(w)) \\ &- \sum_{i\in I_-^R} m_i^- \log\cosh(\operatorname{arctanh} c_i^+ - \beta_i(w)) \quad (w \in \widetilde{\sigma}).\end{aligned}$$

We have $\operatorname{grad}\rho_{\widetilde{\sigma}} = X^{\widetilde{\sigma}}$. Also, it follows from $m_i^+ \geq m_i^-$ and $c_i^+ = c_i^-$ $(i \in I_-^R)$ that

$$\begin{aligned}(\partial_j\partial_k\rho_{\widetilde{\sigma}})(w) \geq &\sum_{i\in I^R\setminus(S_0\cup I_-^R)} \frac{m_i^+}{\sinh^2(\operatorname{arctanh} c_i^+ - \beta_i(w))}\beta_i(\partial_j)\beta_i(\partial_k) \\ &+ \sum_{i\in(I^R\setminus S_0)\cap I_-^R} \frac{4m_i^+}{\sinh^2 2(\operatorname{arctanh} c_i^+ - \beta_i(w))}\beta_i(\partial_j)\beta_i(\partial_k) \\ &- \sum_{i\in S_0\cap I_-^R} \frac{m_i^+}{\cosh^2(\operatorname{arctanh} c_i^+ - \beta_i(w))}\beta_i(\partial_j)\beta_i(\partial_k).\end{aligned}$$

Thus we cannot conclude whether ρ is downward convex or not bacause of the existence of the third term in the right-hand side of this relation. Hence, in Theorem B, we cannot weaken the condition that F_l is a one-point set to the condition in the statement of Theorem A.

5. Examples

Principal orbits of Hermann actions on a symmetric space G/K of non-compact type are curvature-adapted isoparametric submanifolds and they have no focal point of non-Euclidean type on the ideal boundary of G/K. In particular, principal orbits of the isotropy action $K \curvearrowright G/K$ and those of Hermann actions $H \curvearrowright G/K$ as in Table 1 satisfy the additional conditions "$\operatorname{codim} M = \operatorname{rank} G/K$ and $\dim(\mathfrak{p}_\alpha \cap \mathfrak{p}') \geq \frac{1}{2}\dim \mathfrak{p}_\alpha$ $(\alpha \in \triangle)$" in the statement of Theorem A. In Table 1, L is the fixed point group of $\theta\circ\tau$, where θ is a Cartan involution of G with $(\operatorname{Fix}\theta)_0 \subset K \subset \operatorname{Fix}\theta$ and τ is an involution of G with $(\operatorname{Fix}\tau)_0 \subset H \subset \operatorname{Fix}\tau$. Then, for a Hermann action $H \curvearrowright G/K$, $F_l := H(eK)$ is one of the lowest dimensional focal submanifolds of principal orbits of $H \curvearrowright G/K$. The submanifolds F_l and $F_l^\perp := \exp^\perp(T_{eK}^\perp H(eK))$ are reflective and hence they are symmetric spaces. Explicitly they are described as $F_l = H/H\cap K$ and $F_l^\perp = L/H\cap K$, respectively (see Figure 7). In particular, in case of the isotropy action $K \curvearrowright G/K$, F_l is a one-point set. Hence the principal orbits of the isotropy action satisfy the conditions in the statement of Theorem B.

H	G/K	$F_l = H/H \cap K$	$F_l^{\perp} = L/H \cap K$
$SO^*(2n)$	$SU^*(2n)/Sp(n)$	$SO^*(2n)/U(n)$	$SL(n,\mathbf{C})/SU(n)$
$SO^*(2p)$	$SU(p,p)/S(U(p)\times U(p))$	$SO^*(2p)/U(p)$	$Sp(p,\mathbf{R})/U(p)$
$SO(n,\mathbf{C})$	$SL(n,\mathbf{C})/SU(n)$	$SO(n,\mathbf{C})/SO(n)$	$SL(n,\mathbf{R})/SO(n)$
$SU^*(2p)\cdot U(1)$	$Sp(p,p)/Sp(p)\times Sp(p)$	$SU^*(2p)/Sp(p)$	$Sp(p,\mathbf{C})/Sp(p)$
$SL(n,\mathbf{C})\cdot SO(2,\mathbf{C})$	$Sp(n,\mathbf{C})/Sp(n)$	$SL(n,\mathbf{C})/SU(n)$ $\times SO(2,\mathbf{C})/SO(2)$	$Sp(n,\mathbf{R})/U(n)$
$Sp(1,3)$	$E_6^2/SU(6)\cdot SU(2)$	$Sp(1,3)/Sp(1)\times Sp(3)$	$F_4^4/Sp(3)\cdot Sp(1)$
$SU(1,5)\cdot SL(2,\mathbf{R})$	$E_6^{-14}/Spin(10)\cdot U(1)$	$SU(1,5)/S(U(1)\times U(5))$ $\times SL(2,\mathbf{R})/SO(2)$	$SO^*(10)/U(5)$
$Sp(4,\mathbf{C})$	$E_6^{\mathbf{C}}/E_6$	$Sp(4,\mathbf{C})/Sp(4)$	$E_6^6/Sp(4)$
$SU(2,6)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	$SU(2,6)/S(U(2)\times U(6))$	$E_6^2/SU(6)\cdot SU(2)$
$SL(8,\mathbf{C})$	$E_7^{\mathbf{C}}/E_7$	$SL(8,\mathbf{C})/SU(8)$	$E_7^7/SU(8)$
$SO(16,\mathbf{C})$	$E_8^{\mathbf{C}}/E_8$	$SO(16,\mathbf{C})/SO(16)$	$E_8^8/SO(16)$
$Sp(3,\mathbf{C})\cdot SL(2,\mathbf{C})$	$F_4^{\mathbf{C}}/F_4$	$Sp(3,\mathbf{C})/Sp(3)$ $\times SL(2,\mathbf{C})/SU(2)$	$F_4^4/Sp(3)\cdot Sp(1)$
$SL(2,\mathbf{C})\times SL(2,\mathbf{C})$	$G_2^{\mathbf{C}}/G_2$	$SL(2,\mathbf{C})/SU(2)$ $\times SL(2,\mathbf{C})/SU(2)$	$G_2^2/SO(4)$

Table 1

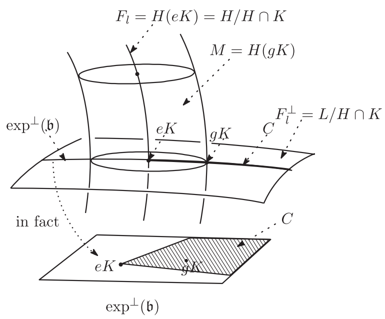

Figure 7

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