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A MYERS THEOREM VIA m-BAKRY-ÉMERY CURVATURE

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Abstract

In this paper, we prove that a complete manifold whose m-Bakry-Émery curvature satisfies

$$\mathrm{Ric}_{f,m}(x) \geq -(m-1)\frac{K_0}{(1+r(x))^2}$$

for some constant $K_0 < -\frac{1}{4}$ should be compact. We also get an upper bound estimate for the diameter.

1. Introduction

Let (M, g) be an n-dimensional complete manifold and f a smooth potential function on M. We use $\mathrm{d}\mu = e^{-f}\,\mathrm{d}V$ to denote the weighted measure, where $\mathrm{d}V$ is the classical Riemann-Lebesgue measure. Then the weighted Laplacian is defined by

$$\triangle_f = \triangle - \nabla f \cdot \nabla,$$

which is symmetric with respect to the weighted measure $\mathrm{d}\mu$. In fact, it is easy to verify that

$$\int_M \nabla u \cdot \nabla v \,\mathrm{d}\mu = -\int_M u \triangle_f v \,\mathrm{d}\mu$$

holds for any $u, v \in C_0^\infty(M)$.

When studying the weighted Laplacian, we always use the m-Bakry-Émery curvature

$$\mathrm{Ric}_{f,m} = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{\mathrm{d}f \otimes \mathrm{d}f}{m-n}$$

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to replace the Ricci curvature, where $m \ge n$ and $m = n$ if and only if f is constant. This curvature relates to the weighted Laplacian via the following weighted Bochner formula [3, 5, 9]

$$\frac{1}{2}\triangle_f |\nabla u|^2 = |\nabla^2 u|^2 + \nabla u \cdot \nabla \triangle_f u + \mathrm{Ric}_{f,m}(\nabla u, \nabla u) + \frac{1}{m-n}(\nabla f \cdot \nabla u)^2.$$

Myers' theorem is a beautiful result, which states that the manifold is compact if the Ricci curvature has a positive lower bound. This theorem has been generalized by many people in various situations. The authors of [2] prove a Myers theorem when the Ricci curvature is bounded from below by

$$\mathrm{Ric}(x) \ge -(n-1)\frac{K_0}{r^2(x)} \tag{1.1}$$

for $r(x) \ge d > 0$, where $K_0 < -\frac{1}{4}$ and $r(x)$ is the geodesic distance function associated with some fixed point. A counterexample given in [2] shows that Myers' theorem may not be true if $K_0 = -\frac{1}{4}$. Further refinements are given in [7], where the authors prove that Myers' theorem holds when the Ricci curvature is bounded from below by

$$\mathrm{Ric}(x) \ge (n-1)\left[\frac{1}{4r^2(x)} - \frac{K_0}{r^2(x)\ln r(x)}\right] \tag{1.2}$$

for $r(x) \ge d > 0$ and $K_0 < -\frac{1}{4}$. We should point out that all these discussions rely on the using of the classical index lemma [1]. In [9], Qian proves a Myers theorem when the m-Bakry-Émery curvature is bounded from below by a positive constant. Recently, the authors of [12] also get a Myers theorem when the ∞-Bakry-Émery curvature

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\, f$$

with a bounded potential function is bounded from below by a positive constant. The method they used is the excess function. For other Myers' theorems see [4, 6, 8, 13].

We call the m-Bakry-Émery curvature almost positive at infinity if for some $K_0 < -\frac{1}{4}$,

$$\mathrm{Ric}_{f,m}(x) \ge -(m-1)\frac{K_0}{(1+r(x))^2} \tag{1.3}$$

holds for any $x \in M$. Now we state the main result in this paper.

THEOREM 1.1. *We assume that the m-Bakry-Émery curvature satisfies* (1.3) *for some constant* $K_0 < -\frac{1}{4}$. *Then the manifold is compact and the diameter satisfies*

$$\mathrm{diam}_{\mathrm{M}} < 2(e^{2\pi/\bar{K}} - 1), \tag{1.4}$$

where

$$\bar{K} = \sqrt{-K_0 - \frac{1}{4}}. \tag{1.5}$$

Remark 1.2. Note that the assumption in Theorem 1.1 has a different form from (1.1). This difference is unessential if one is interested only in the compactness, but only bring different expressions for the diameter estimate.

We always use the index lemma [1] to prove Myers' theorem in the Ricci curvature case. However it seems that the index lemma is not easy to be generalized to the Bakry-Émery curvature case. So we shall prove Theorem 1.1 by using the weighted Laplacian comparison theorem and the excess function. In this paper, we always use $r_p(x) = \mathrm{dist}(p, x)$ to denote the geodesic distance function determined by $p \in M$. In particular, for a given fixed point $O \in M$, we let $r(x) = r_O(x)$.

2. Weighted Laplacian comparison theorem

The Laplacian comparison theorem is a fundamental tool in geometric analysis [10]. The weighted Laplacian comparison theorem can be found in [3, 5, 11]. Let $T_O(M)$ be the tangent space at O. For $r > 0$ and a unit tangent vector $v \in T_O(M)$, we use

$$\mathrm{d}V(\exp(rv)) = J(O, r, v)\, \mathrm{d}r\mathrm{d}v$$

to denote the volume form in geodesic coordinate centered at O. It is easy to see that if $x \in M$ is any point such that $x = \exp_O(rv)$ [10], then

$$H(r(x)) = \triangle r(x) = \frac{J'(O, r, v)}{J(O, r, v)}$$

and

$$H_f(r(x)) = \triangle_f r(x) = \frac{J_f'(O, r, v)}{J_f(O, r, v)},$$

where $J_f(O, r, v)\, \mathrm{d}r\mathrm{d}v$ is the weighted volume form in geodesic coordinate, $H(r(x))$ is the mean curvature of the geodesic sphere with inward pointing normal vector and $H_f(r(x))$ is the mean curvature associated with the weighted measure $\mathrm{d}\mu = e^{-f}\, \mathrm{d}V$.

We first introduce the following weighted Laplacian comparison theorem, which can be found in [3, 5, 11].

Lemma 2.1. *We assume that*

$$\mathrm{Ric}_{f,m}(x) \geq -(m-1)K(r(x)), \tag{2.1}$$

where $K(r(x))$ is a function of $r(x)$. If a_K solves the following Riccati equation

$$\begin{cases} \dfrac{\partial a_K}{\partial r} = (m-1)K(r) - \dfrac{a_K^2}{m-1}, \\ \lim\limits_{r \searrow 0} r a_K = m-1, \end{cases} \tag{2.2}$$

then

$$\triangle_f r(x) \le a_K(r(x)) \tag{2.3}$$

holds at $x \notin \mathrm{Cut}(O)$.

Remark 2.2. In order to solve (2.2), we first solve the following differential equation

$$\varphi''(r) = K(r)\varphi(r) \tag{2.4}$$

with the initial value

$$\varphi(0) = 0, \quad \varphi'(0) = 1.$$

It is easy to see that

$$a_K(r) = (m-1)\frac{\varphi'(r)}{\varphi}$$

solves (2.2) if $\varphi(r)$ solves (2.4).

The following two lemmas are useful for proving Theorem 1.1.

Lemma 2.3. *We assume that the m-Bakry-Émery curvature satisfies* (1.3) *for some constant* $K_0 < -\frac{1}{4}$. *Then*

$$\triangle_f r(x) \le \frac{m-1}{r(x)+1}\left(\frac{1}{2} + \bar{K}\cot(\bar{K}\ln(r(x)+1))\right) \tag{2.5}$$

holds for $x \in M$ *satisfying*

$$0 < r(x) < e^{\pi/\bar{K}} - 1,$$

where $\bar{K}$ *is defined in* (1.5).

Proof. When $K_0 < -\frac{1}{4}$, we can verify easily that

$$\varphi(r) = \sqrt{r+1}\,\sin(\bar{K}\ln(r+1))$$

and

$$\varphi(r) = \sqrt{r+1}\,\cos(\bar{K}\ln(r+1))$$

are two solutions of

$$\varphi''(r) - \frac{K_0}{(1+r)^2}\varphi(r) = 0. \tag{2.6}$$

Hence

$$\varphi(r) = \frac{\sqrt{r+1}}{\bar{K}} \sin(\bar{K} \ln(r+1))$$

is a solution of (2.6) with initial value $\varphi(0) = 0$, $\varphi'(0) = 1$. Lemma 2.3 follows from Lemma 2.1 and Remark 2.2. □

LEMMA 2.4. *We assume that the m-Bakry-Émery curvature satisfies* (1.3) *for some constant* $K_0 < -\frac{1}{4}$. *Let* $r_0 = r(p)$ *for some fixed* $p \neq O$. *We consider a minimizing geodesic* γ *jointing* O *and* p. *Then*

$$\triangle_f(r_p)(x) \le \frac{m-1}{1+r_0-r_p(x)}\left[-\frac{1}{2} + \bar{K}\cot\left(\bar{K}\ln\frac{1+r_0}{1+r_0-r_p(x)}\right)\right] \tag{2.7}$$

holds for $x \in \gamma$ *satisfying*

$$0 < r_p(x) < (1+r_0)(1-e^{-\pi/\bar{K}}).$$

Proof. Note that $r(x) = r_0 - r_p(x)$ for $x \in \gamma$. As before, we solve

$$\varphi'' - \frac{K_0}{(1+r_0-r)^2}\varphi = 0$$

with initial value $\varphi(0) = 0$, $\varphi'(0) = 1$, where

$$0 \le r < (1+r_0)(1-e^{-\pi/\bar{K}}).$$

If we let

$$\varphi(r) = \sqrt{1+r_0-r}[C_1 \sin(\bar{K}\ln(1+r_0-r)) + C_2\cos(\bar{K}\ln(1+r_0-r))].$$

Then the initial value condition tells us that

$$C_1 = -\frac{1}{\bar{K}}\sqrt{1+r_0}\cos(\bar{K}\ln(1+r_0))$$

and

$$C_2 = \frac{1}{\bar{K}}\sqrt{1+r_0}\sin(\bar{K}\ln(1+r_0)).$$

Then

$$\varphi(r) = \frac{\sqrt{(1+r_0)(1+r_0-r)}}{\bar{K}}\sin\left(\bar{K}\ln\frac{1+r_0}{1+r_0-r}\right).$$

Lemma 2.4 follows from Lemma 2.1 and Remark 2.2. □

We also need the following result.

Lemma 2.5. *We assume that the m-Bakry-Émery curvature satisfies* (1.3) *for some constant* $K_0 < -\frac{1}{4}$. *Let* $\gamma(t)$ *be a minimizing geodesic ray starting from* O *and* $y_1, y_2 \in \gamma$ *satisfying* $B = r(y_2) - r(y_1) > 0$. *Then*

$$\triangle_f(r)(y_2) \le \triangle_f(r)(y_1) + \frac{(m-1)K_0 B}{(r(y_1)+1)(r(y_2)+1)}. \tag{2.8}$$

Proof. By the Riccati inequality

$$(\triangle r)' \le -\frac{(\triangle r)^2}{n-1} - \mathrm{Ric}(\partial r, \partial r),$$

we get that [3]

$$(\triangle_f r)' \le -\frac{(\triangle_f r)^2}{m-1} - \mathrm{Ric}_{f,m}(\partial r, \partial r).$$

Hence,

$$(\triangle_f(r))' \le \frac{(m-1)K_0}{(1+r)^2}.$$

Integrating this inequality from $r(y_1)$ to $r(y_2)$ leads to (2.8). □

3. Proof of Theorem 1.1

We introduce the definition of excess function.

Definition 3.1. For $p, q \in M$, the excess function associated with p, q is defined by

$$E_{p,q}(x) = r_p(x) + r_q(x) - r_p(q). \tag{3.1}$$

Lemma 3.2. *We assume that the m-Bakry-Émery curvature satisfies* (1.3) *for some constant* $K_0 < -\frac{1}{4}$. *Then for all* $p \in M$, *the geodesic distance from* O *to* p *is bounded by*

$$r(p) < e^{2\pi/\bar{K}} - 1. \tag{3.2}$$

Proof. If Lemma 3.2 is not correct, then for all

$$\theta \in \left(0, \frac{\pi}{\bar{K}}\right),$$

there exists some point $p \in M$ so that

$$r(p) = r_0 > L^2 + 2L = e^{2\theta} - 1, \tag{3.3}$$

where

$$L = L(\theta) = e^{\theta} - 1.$$

Let $\gamma(t)$ be a minimizing geodesic jointing O and p so that $\gamma(0) = O$ and $\gamma(r_0) = p$. Since r_0 satisfies (3.3), we can choose $q_1, q_2 \in \gamma$, so that

$$q_1 = \gamma(L), \quad q_2 = \gamma\left(\frac{r_0 - L}{L+1}\right).$$

Then

$$r(q_2) - r(q_1) = \frac{r_0 - (L^2 + 2L)}{L+1} > 0.$$

Lemma 2.5 tells us that

$$\triangle_f(r)(q_2) \le \triangle_f(r)(q_1) + \frac{(m-1)K_0(r_0 - (L^2 + 2L))}{(L+1)(r_0+1)}. \tag{3.4}$$

Recall that for O and p, the excess function is

$$E(x) = r(x) + r_p(x) - r_0.$$

By the triangle inequality we know that $E(x) \ge 0$. It is also easy to see that $E(x) = 0$ for $x \in \gamma$. Hence $E(x)$ achieves its minimal value on γ, which implies that $\triangle_f(E)(\gamma(t)) \ge 0$, in particular,

$$0 \le \triangle_f(r)(q_2) + \triangle_f(r_p)(q_2). \tag{3.5}$$

Since

$$r(q_1) = L < e^{\pi/\bar{K}} - 1.$$

Lemma 2.3 tells us that

$$\begin{aligned} \triangle_f(r)(q_1) &\le \frac{m-1}{2(L+1)}[1 + 2\bar{K}\cot(\bar{K}\ln(L+1))] \\ &= \frac{(m-1)e^{-\theta}}{2}(1 + 2\bar{K}\cot(\bar{K}\theta)). \end{aligned} \tag{3.6}$$

Note that

$$r_p(q_2) = r_0 - r(q_2) = \frac{L(r_0+1)}{L+1} < (r_0+1)(1 - e^{-\pi/\bar{K}})$$

and

$$1 + r_0 - r_p(q_2) = \frac{r_0+1}{L+1}.$$

Hence Lemma 2.4 tells us that

$$\begin{aligned} \triangle_f(r_p)(q_2) &\le \frac{(m-1)(L+1)}{2(r_0+1)}[-1 + 2\bar{K}\cot(\bar{K}\ln(L+1))] \\ &= \frac{(m-1)e^{\theta}}{2(r_0+1)}(-1 + 2\bar{K}\cot(\bar{K}\theta)). \end{aligned} \tag{3.7}$$

Plugging (3.4), (3.6) and (3.7) into (3.5) leads to

$$0 \le \frac{K_0(r_0 - (L^2 + 2L))}{(L+1)(r_0+1)} + \frac{e^{-\theta}}{2}(1 + 2\bar{K}\cot(\bar{K}\theta)) + \frac{e^{\theta}}{2(r_0+1)}(-1 + 2\bar{K}\cot(\bar{K}\theta)). \tag{3.8}$$

It is easy to see that

$$\lim_{\theta \nearrow \pi/\bar{K}} \cot \bar{K}\theta = -\infty.$$

Hence the right side of (3.8) tends to $-\infty$ as $\theta \nearrow \dfrac{\pi}{\bar{K}}$, which shows that

$$r(p) = r_0 \ge e^{2\pi/\bar{K}} - 1$$

can not happen and then Lemma 3.2 holds. □

Now we give the proof of Theorem 1.1.

Proof. By the triangle inequality, we conclude that for all $p, q \in M$, the geodesic distance between p and q satisfies

$$r(p,q) \le r(p) + r(q) \le 2(e^{2\pi/\bar{K}} - 1).$$

This finishes the proof of Theorem 1.1. □

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