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ZEROS OF WITTEN ZETA FUNCTIONS AND ABSOLUTE LIMIT

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1. Introduction

The Witten zeta function

$$\zeta_G^W(s) = \sum_{\rho \in \hat{G}} \deg(\rho)^{-s}$$

was introduced by Witten [W] in 1991, where G is a compact topological group and $\hat{G}$ denotes the unitary dual, that is, the set of equivalence classes of irreducible unitary representations. The example

$$\zeta_{SU(2)}^W(s) = \sum_{m=0}^{\infty} \deg(\mathrm{Sym}^m)^{-s} = \sum_{n=1}^{\infty} n^{-s} = \zeta(s),$$

where $\zeta(s)$ denotes the Riemann zeta function, suggests fine properties for general case. In fact, Witten showed arithmetical interpretation for $\zeta_{SU(n)}^W(2m)$ $(m = 1, 2, 3, \ldots)$ containing Euler's result ([E1] 1735)

$$\zeta_{SU(2)}^W(2m) \in \pi^{2m}\mathbf{Q}.$$

In this paper we look at the opposite side: special values at negative integers such as

$$\zeta_{SU(2)}^W(-1) = \text{“}\sum_{n=1}^{\infty} n\text{”} = -\frac{1}{12}, \tag{1}$$

$$\zeta_{SU(2)}^W(-2) = \text{“}\sum_{n=1}^{\infty} n^2\text{”} = 0 \tag{2}$$

due to Euler [E2] (1749). We notice that the value

$$\text{“}\sum_{n=1}^{\infty} n\text{”} = -\frac{1}{12}$$

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appears as the one-dimensional Casimir energy: see Casimir [C] and Hawking [H]. The equality

$$\text{“}\sum_{n=1}^{\infty} n^2\text{”} = 0$$

means the vanishing of the two-dimensional Casimir energy.

We notice that

$$\zeta_G^W(-2) = |G|$$

when G is a finite group. We conjecture that

$$\zeta_G^W(-2) = 0 \tag{3}$$

for infinite groups G.

For deeper understanding of the situation, we introduce a new zeta function (Witten L-function)

$$\zeta_G^W(s,g) = \sum_{\rho\in\hat{G}} \frac{\mathrm{trace}(\rho(g))}{\deg(\rho)} \deg(\rho)^{-s} \tag{4}$$

where G is a compact topological group, g is an element of G, $\hat{G}$ is the set of equivalence classes of irreducible ($\mathbf{C}$-valued) representations of G, $\deg(\rho)$ is the degree (the dimension) of an irreducible representation $\rho \in \hat{G}$. Note that $\mathrm{trace}(\rho(g))$ is the character of the representation ρ. This Witten zeta function $\zeta_G^W(s,g)$ reduces to the (usual) Witten zeta function when we specialize g to the identity element $1 \in G$:

$$\zeta_G^W(s) = \zeta_G^W(s,1).$$

In the case of a finite group G we have

$$\zeta_G^W(-2,g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We conjecture that

$$\zeta_G^W(-2,g) = 0 \tag{5}$$

when G is an infinite group. The following result treats the case $G = SU(2)$.

Theorem 1. *Suppose $g \in SU(2)$ is conjugate to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ with $0 \le \theta \le \pi$.*
(1) *We have an expression*

$$\zeta_{SU(2)}^W(s,g) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n \sin\theta} n^{-s}$$

in $\mathrm{Re}(s) > 1$. The function $\zeta_{SU(2)}^W(s,g)$ in s has a meromorphic continuation to the whole complex plane.

(2) *For a positive even integer m, we have* $\zeta^W_{SU(2)}(-m,g)=0$ *for all* $g\in SU(2)$. *Moreover,* $s=-2$ *is a simple zero of* $\zeta^W_{SU(2)}(s,g)$, *and the first derivative at* $s=-2$ *is given as*

$$\frac{\partial\zeta^W_{SU(2)}}{\partial s}(-2,g)=\begin{cases}-\dfrac{\zeta(3)}{4\pi^2} & \text{if } \theta=0,\\[2ex] \dfrac{1}{4\pi\sin\theta}\left(\zeta\left(2,\dfrac{\theta}{2\pi}\right)-\dfrac{\pi^2}{2\sin^2\dfrac{\theta}{2}}\right)>0 & \text{if } 0<\theta<\pi,\\[2ex] \dfrac{7\zeta(3)}{4\pi^2} & \text{if } \theta=\pi.\end{cases}$$

Here $\zeta(s,x)$ *denotes the Hurwitz zeta function.*

(3) *The special value at* $s=-1$ *is given as*

$$\zeta^W_{SU(2)}(-1,g)=\begin{cases}-\dfrac{1}{12} & \text{if } \theta=0,\\[2ex] \dfrac{1}{4\sin^2\dfrac{\theta}{2}} & \text{if } 0<\theta<\pi,\\[2ex] \dfrac{1}{4} & \text{if } \theta=\pi.\end{cases}$$

We now introduce a 'multi'-version of Witten L-function. For $g_1,\dots,g_r\in G$, we define

$$\begin{aligned}\zeta^W_G(s;g_1,\dots,g_r)&:=\sum_{\rho\in\hat{G}}\frac{\operatorname{trace}(\rho(g_1))}{\deg(\rho)}\cdots\frac{\operatorname{trace}(\rho(g_r))}{\deg(\rho)}\times\deg(\rho)^{-s}\\&=\sum_{\rho\in\hat{G}}\frac{\operatorname{trace}(\rho(g_1))\cdots\operatorname{trace}(\rho(g_r))}{\deg(\rho)^{s+r}}.\end{aligned}$$

It is natural to ask whether the vanishing $\zeta^W_G(-2;g_1,\dots,g_r)\stackrel{?}{=}0$ of the special value at $s=-2$ for this generalization holds. We have a partial answer to this question.

Theorem 2. *We have* $\zeta^W_{SU(2)}(-m;g_1,g_2)=0$ *for* $g_1,g_2\in SU(2)$, *and a positive even integer m.*

We also give an example of the non-vanishing for the case $r=3$: for some $g\in SU(2)$, we prove that $\zeta^W_{SU(2)}(-2;g,g,g)\neq 0$. These results related with the Lie group $SU(2)$ are given in Section 2.

We report further examples of zeros of Witten zeta functions for infinite groups.

THEOREM 3. $\zeta^W_{SU(3)}(s) = 0$ *for* $s = -1, -2, \ldots.$

The proof of this theorem is given in Section 3.

The next example is not a Lie group, but a totally disconnected group. Let $\mathbf{Z}_p$ be the p-adic integer ring for a prime number p.

THEOREM 4. *Suppose* $p \neq 2$. *Then* $\zeta^W_{SL_2(\mathbf{Z}_p)}(s) = 0$ *for* $s = -1, -2.$

Now we consider the congruence subgroups. For a positive integer m, we define a subgroup of $SL_3(\mathbf{Z}_p)$ of finite index by

$$SL_3(\mathbf{Z}_p)[p^m] = \ker(SL_3(\mathbf{Z}_p) \to SL_3(\mathbf{Z}_p/(p^m))).$$

THEOREM 5. *Suppose* $p \neq 3$.

(1)

$$\zeta^W_{SL_3(\mathbf{Z}_p)[p^m]}(s) = p^{8m}\frac{(1-p^{-2-s})(1-p^{-1-s})}{(1-p^{1-2s})(1-p^{2-3s})} \times (1 + (p^{-1}+p^{-2})p^{-s} + (1+p^{-1})p^{-2s} + p^{-2-3s}).$$

(2) $\zeta^W_{SL_3(\mathbf{Z}_p)[p^m]}(s) = 0$ *for* $s = -1, -2.$

(3)

$$\zeta^W_{SL_3(\mathbf{Z}_1)[1^m]}(s) = \frac{(s+1)(s+2)}{\left(s-\frac{1}{2}\right)\left(s-\frac{2}{3}\right)}.$$

Here we interpret that if $\zeta^W_{SL_3(\mathbf{Z}_p)[p^m]}(s)$ has an expression as an analytic function on p, and there is a limit $p \to 1$, then its limit is denoted by

$$\zeta^W_{SL_3(\mathbf{Z}_1)[1^m]}(s) = \lim_{p\to 1} \zeta^W_{SL_3(\mathbf{Z}_p)[p^m]}(s).$$

These results on totally disconnected groups are given in Section 4.

2. $SU(2)$

2.1. Parametrization of irreducible representations of $SU(2)$

The set of equivalence classes, $\hat{G}$, of irreducible unitary representations of $G = SU(2)$ is parametrized by the set of natural numbers. For a natural number n, we denote by $\rho = \rho_n \in \hat{G}$, the corresponding irreducible representation of G. For a $g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in G$, we have the character formula

$$\text{trace}(\rho(g)) = e^{i(n-1)\theta} + e^{i(n-3)\theta} + \cdots + e^{i(3-n)\theta} + e^{i(1-n)\theta} \tag{6}$$

and the degree

$$\deg(\rho) = \text{trace}(\rho(I_2)) = n, \tag{7}$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SU(2)$ is the identity matrix. We also see that $\mathrm{trace}(\rho(-I_2)) = (-1)^{n-1} n$.

We start from $g = \pm I_2 \in SU(2)$. In these cases, $\zeta^W_{SU(2)}(s, g)$ is written in terms of the Riemann zeta function. We see that $\zeta^W_{SU(2)}(s, I_2) = \zeta(s)$, and

$$\zeta^W_{SU(2)}(s, -I_2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s). \tag{8}$$

2.2. Poly-logarithm function

We recall the poly-logarithm

$$Z(s, x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s},$$

which is written also as $\mathrm{Li}_s(x)$ in literature. This series converges if $|x| < 1$ and $s \in \mathbf{C}$, or $|x| = 1$ and $\mathrm{Re}(s) > 1$. In the following, we restrict to the case $|x| = 1$.

THEOREM 6. *Suppose $|x| = 1$ and $x \neq 1$. Then $Z(s, x)$ is analytically continued to a holomorphic function on $s \in \mathbf{C}$. Moreover, for every non-negative integer m, the function $Z(-m, x)$ can be expressed by a rational function in x. The first several examples are*

$$Z(0, x) = \frac{x}{1-x}, \quad Z(-1, x) = \frac{x}{(1-x)^2}, \quad Z(-2, x) = \frac{x(1+x)}{(1-x)^3}, \ldots.$$

Proof. For $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned}
Z(s, x) &= x + \frac{x^2}{2^s} + \sum_{n=3}^{\infty} \frac{x^n}{n^s} \\
&= x + \frac{x^2}{2^s} + \sum_{n=2}^{\infty} \frac{x^{n+1}}{(n+1)^s} \\
&= x + \frac{x^2}{2^s} + \sum_{n=2}^{\infty} x^{n+1} n^{-s} (1 + n^{-1})^{-s} \\
&= x + \frac{x^2}{2^s} + \sum_{n=2}^{\infty} x^{n+1} n^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} n^{-k} \\
&= x + \frac{x^2}{2^s} + x \sum_{k=0}^{\infty} \binom{-s}{k} (Z(s+k, x) - x) \\
&= x + \frac{x^2}{2^s} + x(Z(s, x) - x) + x \sum_{k=1}^{\infty} \binom{-s}{k} (Z(s+k, x) - x).
\end{aligned}$$

This shows

$$(1-x)Z(s,x) = x + x^2(2^{-s}-1) + x\sum_{k=1}^{\infty}\binom{-s}{k}(Z(s+k,x)-x). \tag{9}$$

By the estimates of binomial coefficients, the right-hand side converges absolutely on the right-half plane $\mathrm{Re}(s) > 0$. This shows the analytic continuation of $Z(s,x)$ to $\mathrm{Re}(s) > 0$. Repeating this argument, we obtain the analytic continuation to whole $s \in \mathbf{C}$. To substitute $s = -m$ with $m = 0, 1, \ldots,$ we have the recursion equation

$$(1-x)Z(-m,x) = x + x^2(2^m-1) + x\sum_{k=1}^{m}\binom{m}{k}(Z(-(m-k),x)-x). \tag{10}$$ □

First several examples show

$$Z(-3,x) = \frac{x(1+4x+x^2)}{(1-x)^4}, \quad Z(-4,x) = \frac{x(1+x)(1+10x+x^2)}{(1-x)^5},$$

$$Z(-5,x) = \frac{x(1+26x+66x^2+26x^3+x^4)}{(1-x)^6}.$$

These examples seem to show

LEMMA 7. *Suppose* $|x| = 1$ *with* $x \neq 1$. *Then*

$$Z(0,x) + Z(0,x^{-1}) = -1, \tag{11}$$

and for every positive integer m,

$$Z(-m,x) + (-1)^m Z(-m,x^{-1}) = 0. \tag{12}$$

Proof. We start from [Jonquière 1880]

$$e^{-\pi is/2}Z(s,e^{i\theta}) + e^{\pi is/2}Z(s,e^{-i\theta}) = \frac{(2\pi)^s}{\Gamma(s)}\zeta\left(1-s,\frac{\theta}{2\pi}\right) \tag{13}$$

in Milnor [M]. Putting $s = -m$ with $m = 1, 2, \ldots,$ we have

$$e^{\pi im/2}Z(-m,e^{i\theta}) + e^{-\pi im/2}Z(-m,e^{-i\theta}) = 0.$$ □

We remark that $Z(0,1) = \zeta(0) = -1/2$. In this sense, the formula (11) is valid also for $x = 1$.

2.3. An example

$$Z(-1,e^{i\theta}) = \frac{1}{(e^{-i\theta/2}-e^{i\theta/2})^2} = -\frac{1}{4\sin^2(\theta/2)}. \tag{14}$$

and this shows

$$\mathrm{Li}_{-1}(e^{-i\theta}) = \mathrm{Li}_{-1}(e^{i\theta}), \tag{15}$$

an even function in θ.

2.4. Proof of Theorem 1(1) and analytic continuation

Now we consider regular elements in $SU(2)$. Suppose $0 < \theta < \pi$. Then we have, for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\zeta_{SU(2)}^{W}\left(s, \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) &= \sum_{n=1}^{\infty} \frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}} \frac{1}{n} n^{-s} \\
&= \frac{1}{e^{i\theta} - e^{-i\theta}} \sum_{n=1}^{\infty} \left(\frac{e^{in\theta}}{n^{s+1}} - \frac{e^{-in\theta}}{n^{s+1}} \right) \\
&= \frac{1}{e^{i\theta} - e^{-i\theta}} \{ Z(s+1, e^{i\theta}) - Z(s+1, e^{-i\theta}) \} \\
&= \frac{1}{2i \sin \theta} \{ Z(s+1, e^{i\theta}) - Z(s+1, e^{-i\theta}) \},
\end{aligned}$$

and the right-hand side has meromorphic continuation to whole $s \in \mathbf{C}$.

Note that we interpret

$$\frac{\sin(n\theta)}{n \sin \theta} = \begin{cases} 1 & \text{if } \theta = 0, \\ (-1)^{n-1} & \text{if } \theta = \pi. \end{cases} \tag{16}$$

2.5. Proof of Theorem 1(2); vanishing

For $g = \pm I_2$ and for positive even integer m, we obtain $\zeta_{SU(2)}^{W}(-m, \pm I_2) = 0$ from $\zeta(-m) = 0$.

For $g \neq \pm I_2$, suppose $0 < \theta < \pi$. Then for a positive integer m, we have

$$\zeta_{SU(2)}^{W}(-m, g) = \frac{1}{2i \sin \theta} (Z(1-m, e^{i\theta}) - Z(1-m, e^{-i\theta})). \tag{17}$$

This is zero for even m by the formula (12).

2.6. Proof of Theorem 1(2), first derivative

We see that

$$\frac{1}{\Gamma(s)} = \frac{s(s+1)}{\Gamma(s+2)} \tag{18}$$

shows that

$$\frac{1}{\Gamma(s)} = -(s+1) + O((s+1)^2), \quad (s \to -1). \tag{19}$$

We again start from the formula (13)

$$e^{-\pi i s/2} Z(s,x) + e^{\pi i s/2} Z(s,x^{-1}) = \frac{(2\pi)^s}{\Gamma(s)} \zeta\left(1-s, \frac{\theta}{2\pi}\right)$$

with $x = e^{i\theta}$. Taking $\left.\frac{\partial}{\partial s}\right|_{s=-1}$ in this formula, we have

$$i\frac{\partial Z}{\partial s}(-1,x) + (-i)\frac{\partial Z}{\partial s}(-1,x^{-1}) + (-\pi i/2)(i)Z(-1,x) + (\pi i/2)(-i)Z(-1,x^{-1})$$
$$= (2\pi)^{-1}(-1)\zeta\left(2, \frac{\theta}{2\pi}\right).$$

Then

$$i \times 2i \sin\theta \times \frac{\partial \zeta^W_{SU(2)}}{\partial s}(-2,g) = -\pi Z(-1, e^{i\theta}) - \frac{1}{2\pi}\zeta\left(2, \frac{\theta}{2\pi}\right), \tag{20}$$

and

$$4\pi \sin\theta \times \frac{\partial \zeta^W_{SU(2)}}{\partial s}(-2,g) = 2\pi^2 Z(-1, e^{i\theta}) + \zeta\left(2, \frac{\theta}{2\pi}\right). \tag{21}$$

We have

$$\zeta(2,t) + \zeta(2,1-t) = \frac{\pi^2}{\sin^2(\pi t)} \tag{22}$$

since the left-hand side is equal to

$$\sum_{n=0}^{\infty} \frac{1}{(n+t)^2} + \sum_{n=0}^{\infty} \frac{1}{(n+1-t)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(n+t)^2} \tag{23}$$

which is equal to the right-hand side. This shows

$$8\pi \sin\theta \times \frac{\partial \zeta^W_{SU(2)}}{\partial s}(-2,g) = \zeta\left(2, \frac{\theta}{2\pi}\right) - \zeta\left(2, 1-\frac{\theta}{2\pi}\right) > 0 \tag{24}$$

since $\frac{\theta}{2\pi} < 1 - \frac{\theta}{2\pi}$.

2.7. Proof of Theorem 1(3)

$$\zeta^W_{SU(2)}(-1, I_2) = \zeta(-1) = -\frac{1}{12} \tag{25}$$

and

$$\zeta^W_{SU(2)}\left(-1, \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) = \frac{Z(0,x) - Z(0,x^{-1})}{x - x^{-1}} = \frac{-x}{(1-x)^2} = \frac{1}{4\sin^2(\theta/2)}, \tag{26}$$

where $x = e^{i\theta}$ for all $0 < \theta \le \pi$.

2.8. An average over the group

Let G be a finite group. The normalized Haar measure dg on G is, by definition,

$$\int_G f(g)\, dg = \frac{1}{|G|}\sum_{g \in G} f(g). \tag{27}$$

Then we see that, for all $s \in \mathbf{C}$,

$$\int_G \zeta^W_G(s,g)\, dg = 1, \tag{28}$$

since the left-hand side is equal to

$$\sum_{\rho \in \hat{G}} \left(\int_G \mathrm{trace}(\rho(g))\, dg\right) \deg(\rho)^{-s-1}, \tag{29}$$

where the average is non-zero only for the trivial representation ρ.

Now we consider the case where G is a compact group which is not necessarily a finite group. Again let dg be the normalized Haar measure of G so that $\int_G dg = 1$. We ask the value

$$\int_G \zeta^W_G(s,g)\, dg. \tag{30}$$

We can give some example;

$$\int_{SU(2)} \zeta^W_{SU(2)}(-2,g)\, dg = 0, \tag{31}$$

$$\int_{SU(2)} \zeta^W_{SU(2)}(-1,g)\, dg = 1. \tag{32}$$

The latter formula is proved by the Weyl integral formula;

$$\int_{SU(2)} \zeta^W_{SU(2)}(-1,g)\, dg = \int_0^\pi \zeta^W_{SU(2)}\left(-1, \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) \frac{2}{\pi} \sin^2\theta\, d\theta = 1. \tag{33}$$

2.9. $r = 2$

We now discuss the properties of a generalization of Witten zeta functions with several characters. We give a proof of Theorem 2.

Proof.

$$\mathrm{trace}(\rho(g_1)) = \frac{x^n - x^{-n}}{x - x^{-1}}, \quad \mathrm{trace}(\rho(g_2)) = \frac{y^n - y^{-n}}{y - y^{-1}}$$

with $x = e^{i\theta_1}$, $y = e^{i\theta_2}$. In the cases $g_2 = \pm I_2$, we have

$$\zeta^W_{SU(2)}(s, g_1, I_2) = \zeta^W_{SU(2)}(s, g_1), \tag{34}$$

$$\zeta^W_{SU(2)}(s, g_1, -I_2) = \zeta^W_{SU(2)}(s, -g_1). \tag{35}$$

Then the problem on the special values is reduced to the case treated in Theorem 1(2).

Now we may suppose $x, y \neq \pm 1$. Then

$$\begin{aligned} &\zeta^W_{SU(2)}(s, g_1, g_2) \\ &= \frac{1}{(x - x^{-1})(y - y^{-1})} \sum_{n=1}^{\infty} \frac{(xy)^n + (x^{-1}y^{-1})^n - (xy^{-1})^n - (x^{-1}y)^n}{n^{s+2}} \\ &= \frac{Z(s+2, xy) + Z(s+2, x^{-1}y^{-1}) - Z(s+2, xy^{-1}) - Z(s+2, x^{-1}y)}{(x - x^{-1})(y - y^{-1})}. \end{aligned} \tag{36}$$

This shows

$$\begin{aligned} \zeta^W_{SU(2)}(-2, g_1, g_2) &= \frac{(Z(0, xy) + Z(0, x^{-1}y^{-1})) - (Z(0, xy^{-1}) + Z(0, x^{-1}y))}{(x - x^{-1})(y - y^{-1})} \\ &= 0, \end{aligned} \tag{37}$$

where we have used the formula (11). □

2.10. $r = 3$

By the similar computation, we obtain

$$\begin{aligned} &\zeta^W_{SU(2)}(s; g, g, g) \\ &= \frac{Z(s+3, x^3) - 3Z(s+3, x) + 3Z(s+3, x^{-1}) - Z(s+3, x^{-3})}{(x - x^{-1})^3}. \end{aligned} \tag{38}$$

If $x = i$, then

$$\zeta^W_{SU(2)}(-2; g, g, g) = \frac{4Z(1, -i) - 4Z(1, i)}{(2i)^3} = \frac{-2\pi i}{-8i} = \frac{\pi}{4} \neq 0.$$

3. $SU(3)$

3.1. On analytic continuation

Let G be a compact semisimple Lie group. Then the Witten zeta $\zeta_G^W(s)$ has a meromorphic continuation to $\mathbf{C}$. This is a special case of

$$\sum_{m_1,\dots,m_r\geq 1} Q(m_1,\dots,m_r)P(m_1,\dots,m_r)^{-s}. \tag{39}$$

Analytic continuation of these zeta functions is discussed in [Mellin 1900], [Mahler 1928].

3.2. A special value at a negative integer

Let n be a positive integer. Let $M = 2n+2$, and suppose $\mathrm{Re}(s) > -n - \frac{1}{2} + \frac{\varepsilon}{2}$, with $\varepsilon > 0$. By [Ma], we have

$$\begin{aligned}\zeta_{SU(3)}^W(s) &= 2^s \sum_{m,n\geq 1} \frac{1}{m^s n^s (m+n)^s} \\ &= 2^s \frac{\Gamma(2s-1)\Gamma(1-s)}{\Gamma(s)}\zeta(3s-1) \\ &\quad + 2^s \sum_{k=0}^{M-1} (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}\zeta(2s+k)\zeta(s-k) \\ &\quad + 2^s \frac{1}{2\pi\sqrt{-1}} \int_{\mathrm{Re}(z)=2n+2-\varepsilon} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\zeta(2s+z)\zeta(s-z)\,dz.\end{aligned} \tag{40}$$

Reminding

$$\left.\frac{\Gamma(2s-1)}{\Gamma(s)}\right|_{s=-n} = (-1)^{n-1}\frac{n!}{2(2n+1)!}, \tag{41}$$

we can put $s = -n$ in this identity and obtain

$$\begin{aligned}\zeta_{SU(3)}^W(-n) &= 2^{-n}(-1)^{n-1}\frac{n!n!}{2(2n+1)!}\zeta(-3n-1) \\ &\quad + 2^{-n}\sum_{k=0}^{2n}(-1)^k \frac{(-n)(1-n)\cdots(k-1-n)}{k!}\zeta(-2n+k)\zeta(-n-k) \\ &\quad + 2^{-n}(-1)\frac{(-n)(1-n)\cdots(-1)\cdot 1\cdots n}{(2n+1)!}\frac{1}{2}\zeta(-3n-1).\end{aligned} \tag{42}$$

This shows $\zeta_{SU(3)}^W(-n) = 0$ for a positive odd integer n, since $\zeta(-3n-1) = 0$ and $\zeta(-2n+k)\zeta(-n-k) = 0$ for $k = 0, 1, \dots, n$. On the other hand, for a positive even integer n, we have

$$\zeta^W_{SU(3)}(-n) = -2^{-n}\frac{(n!)^2}{(2n+1)!}\zeta(-3n-1) + 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\zeta(-2n+k)\zeta(-n-k) = 0, \tag{43}$$

where the last equality follows from the following lemma:

LEMMA 8. *For a positive even integer n, we have*

$$\sum_{k+l=n,k,l\geq 0}\frac{1}{k!l!}\zeta(-n-k)\zeta(-n-l) = \frac{n!}{(2n+1)!}\zeta(-3n-1). \tag{44}$$

Equivalently,

$$\sum_{k+l=n,k,l\geq 0}\frac{1}{k!l!}\frac{B_{n+1+k}}{n+1+k}\frac{B_{n+1+l}}{n+1+l} = -\frac{n!}{(2n+1)!}\frac{B_{3n+2}}{3n+2}. \tag{45}$$

This follows from [CW, Theorem 2] when we substitute $\alpha = \gamma = n-1$ and $\delta = \varepsilon = 1$. □

This concludes the proof of Theorem 3.

4. The groups over $\mathbf{Z}_p$

4.1. SL_2

Let p be an odd prime. We denote by $\mathbf{Z}_p$ the ring of integers in the non-archimedean local field $\mathbf{Q}_p$. Jaikin-Zapirain [J] obtains the following explicit formula:

$$\zeta^W_{SL_2(\mathbf{Z}_p)}(s) = Z_0(s) + Z_\infty(s), \tag{46}$$

with

$$\begin{aligned} Z_0(s) &= \zeta^W_{SL_2(\mathbf{F}_p)}(s) \\ &= 1 + 2\left(\frac{p-1}{2}\right)^{-s} + 2\left(\frac{p+1}{2}\right)^{-s} + \frac{p-1}{2}(p-1)^{-s} \\ &\quad + p^{-s} + \frac{p-3}{2}(p+1)^{-s}, \end{aligned} \tag{47}$$

$$Z_\infty(s) = \frac{1}{1-p^{-s+1}}\left(4p\left(\frac{p^2-1}{2}\right)^{-s} + \frac{p^2-1}{2}(p^2-p)^{-s} + \frac{(p-1)^2}{2}(p^2+p)^{-s}\right). \tag{48}$$

This deduces

$$Z_0(-2) = p(p^2 - 1) = |SL_2(\mathbf{F}_p)| = p(p+1)(p-1), \tag{49}$$

$$Z_\infty(-2) = -p(p^2 - 1), \tag{50}$$

$$Z_0(-1) = p(p+1), \tag{51}$$

$$Z_\infty(-1) = -p(p+1), \tag{52}$$

$$Z_0(0) = p + 4, \tag{53}$$

$$Z_\infty(0) = -\frac{4}{p-1} - p - 4. \tag{54}$$

This shows

$$\zeta^W_{SL_2(\mathbf{Z}_p)}(-2) = 0, \tag{55}$$

$$\zeta^W_{SL_2(\mathbf{Z}_p)}(-1) = 0, \tag{56}$$

$$\zeta^W_{SL_2(\mathbf{Z}_p)}(0) = -\frac{4}{p-1}, \tag{57}$$

which concludes the proof of Theorem 4.

4.2. Congruence subgroups of SL_2

In this subsection, we assume that p is an odd prime. By [AKOV], we obtain

$$\zeta^W_{SL_2(\mathbf{Z}_p)[p^m]}(s) = p^{3m+2}\frac{1 - p^{-2-s}}{1 - p^{1-s}}. \tag{58}$$

This shows

$$\zeta^W_{SL_2(\mathbf{Z}_p)[p^m]}(-2) = 0, \tag{59}$$

$$\zeta^W_{SL_2(\mathbf{Z}_p)[p^m]}(-1) = -p^{3m+1}/(p+1). \tag{60}$$

By taking an "absolute limit" $p \to 1$, we obtain

$$\zeta^W_{SL_2(\mathbf{Z}_1)[1^m]}(s) = \frac{s+2}{s-1}. \tag{61}$$

4.3. Congruence subgroups of SL_3 and SU_3

In this subsection, we assume that p is a prime with $p \neq 3$. By [AKOV], we have

$$\zeta^W_{SL_3(\mathbf{Z}_p)[p^m]}(s) = p^{8m}\frac{1 + u(p)p^{-3-2s} + u(p^{-1})p^{-2-3s} + p^{-5-5s}}{(1 - p^{1-2s})(1 - p^{2-3s})}, \tag{62}$$

where $u(X) = X^3 + X^2 - X - 1 - X^{-1}$. We notice that it can be factorized as

$$\zeta^W_{SL_3(\mathbf{Z}_p)[p^m]}(s) = p^{8m}\frac{(1-p^{-2-s})(1-p^{-1-s})}{(1-p^{1-2s})(1-p^{2-3s})} \times (1+(p^{-1}+p^{-2})p^{-s}+(1+p^{-1})p^{-2s}+p^{-2-3s}). \tag{63}$$

We see that

$$\zeta^W_{SL_3(\mathbf{Z}_p)[p^m]}(-2) = \zeta^W_{SL_3(\mathbf{Z}_p)[p^m]}(-1) = 0.$$

The formula (64) shows

$$\lim_{p\to 1} \zeta^W_{SL_3(\mathbf{Z}_p)[p^m]}(s) = \frac{(s+1)(s+2)}{\left(s-\frac{1}{2}\right)\left(s-\frac{2}{3}\right)}, \tag{64}$$

which is considered to be "an absolute Witten zeta function $\zeta^W_{SL_3(\mathbf{Z}_1)[1^m]}(s)$".

Also by [AKOV],

$$\zeta^W_{SU_3(\mathbf{Z}_p)[p^m]}(s) = p^{8m}\frac{1+u(p)p^{-3-2s}+u(p^{-1})p^{-2-3s}+p^{-5-5s}}{(1-p^{1-2s})(1-p^{2-3s})} \tag{65}$$

$$= p^{8m}\frac{(1-p^{-2-s})(1-p^{-s})(1+p^{-1-s})}{(1-p^{1-2s})(1-p^{2-3s})} \times (1+(1-p^{-1}+p^{-2})p^{-s}+p^{-2-2s}), \tag{66}$$

where $u(X) = -X^3+X^2-X+1-X^{-1}$. This shows

$$\zeta^W_{SU_3(\mathbf{Z}_p)[p^m]}(-2) = \zeta^W_{SU_3(\mathbf{Z}_p)[p^m]}(0) = 0,$$

while

$$\zeta^W_{SU_3(\mathbf{Z}_p)[p^m]}(-1) = 2p^{8m-2}\frac{p-1}{p^5-1} = 2p^{8m-2}\frac{1}{[5]_p} \tag{67}$$

is non-zero where $[n]_p = \dfrac{p^n-1}{p-1}$ is a p-analogue of an integer n. This shows

$$\lim_{p\to 1} \zeta^W_{SU_3(\mathbf{Z}_p)[p^m]}(-1) = \frac{2}{5}. \tag{68}$$

By the formula (67), we have

$$\lim_{p\to 1} \zeta^W_{SU_3(\mathbf{Z}_p)[p^m]}(s) = \frac{s(s+2)}{\left(s-\frac{1}{2}\right)\left(s-\frac{2}{3}\right)}.$$

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