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KODAI MATH. J.
36 (2013), 409–427

FUNCTIONAL CENTRAL LIMIT THEOREM FOR TAGGED PARTICLE DYNAMICS IN STOCHASTIC RANKING PROCESS

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Abstract

In this paper we consider "parabolically" scaled centered tagged particle dynamics for a stochastic ranking process (regarded as a particle system), which is driven according to an algorithm for self-organizing linear list of a finite number of items. We let the number of items to infinity and show that the scaled tagged particle weakly converges to a "diffusion" processes with occasional jumps, in which the particle jumps to 0 when its own Poisson clock rings and behaves as a "diffusion" process otherwise. The "diffusion" is decomposed into a sum of independent continuous Markov Gaussian processes with a random covariance. Intuitively, each component process is constructed by infinitely many particles having the same intensity behind the tagged particle. This random covariance depends only on its own last Poisson time. In multi-tagged particle system, "hyperbolically" scaled tagged particles decompose $[0, 1]$ interval into $L+1$ layers, where L is a number of tagged particles. Intuitively, infinitely many particles in each layer construct a "diffusion" processes, which is interpreted as a shrunk version of that in the single tagged particle case. Each "parabolically" scaled centered tagged particle holds in common these "diffusion" processes if the corresponding layer is behind the corresponding "hyperbolically" scaled tagged particle.

1. Introduction

We consider a stochastic ranking process (or Poisson embedding of the move-to-front rules), which is driven according to an algorithm for a self-organizing linear list of a finite number of items. The list is updated in the following way. Each item has an independent Poisson clock, whose rate depends on type of the item. If the Poisson clock of the i-th item rings, then it is transferred to the top of the list and each of the items located in front of the i-th item accordingly descend simultaneously by one rank; those behind do not move at all. In this paper, we treat this process as an "interacting particle system". We fasten a tag to a "particle" (or tags to "particles") and observe the

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2010 *Mathematics Subject Classification*. 60K35, 60F05.

Key words and phrases. Stochastic ranking process, tagged particle, functional central limit theorem.

Received December 25, 2012; revised March 5, 2013.

motion of the “tagged particle” (or “tagged particles”). In the previous paper [23], we obtained a scaling limit of tagged particle dynamics and multi-tagged particle dynamics as the number of the particles tends to infinity under a “hyperbolic” scaling. In this scaling limit the limit process of the scaled tagged particle jumps to the top of the list when its own Poisson clock rings and moves deterministically along a curve otherwise. The tagged particles in the multi-tagged particle system are independent. These results may be regarded as a law of large numbers.

In this paper we consider “parabolically” scaled centered tagged particle dynamics. In the limit under this scaling we obtain a sum of diffusion processes with occasional jump in which the particle jumps to 0 when its own Poisson clock rings and behaves as a sum of continuous Markov Gaussian processes with a random covariance otherwise. We can interpret that each of these Gaussian processes is constructed by infinitely many particles having the same intensity behind the tagged particle. This random covariance depends only on its own last Poisson time (Theorem 2.2). In multi-tagged particle system, “hyperbolically” scaled tagged particles decompose the unit interval $[0,1]$ into $L+1$ layers, where L is a number of tagged particles. Intuitively, infinitely many particles in each layer construct a sum of continuous Markov Gaussian processes, which is interpreted as a shrunk version of that in the single tagged particle case. Each “parabolically” scaled centered tagged particle holds in common these Gaussian processes if the corresponding layer is behind the corresponding “hyperbolically” scaled tagged particle (Theorem 2.5).

The move-to-front rule is introduced by Tsetlin [27] and studied in many papers [5, 16, 20, 21, 22]. It is also studied as least-recently-used cashing [1, 3, 4, 6, 7, 8, 9, 10, 17, 18, 25, 26]. Recently it is reintroduced and studied as a mathematical model of the ranking in the web page of online bookstores or in the posting web pages [11, 12, 13, 14, 15, 23].

This paper is organized as follows: In section 2, we define our model and state main results. In section 3, we prove Theorem 2.2. In section 4, we prove Lemma 2.4. In section 5, we prove Theorem 2.5.

2. Model and results

Let $\{\nu_i; i \in \mathbf{N}\}$ be independent Poisson random measures on $[0,\infty)$ with intensity $w_i(s)\,ds$. We suppose that the set of intensities is finite, i.e., there exists K such that $\{w_i; i \in \mathbf{N}\} = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_K\}$. Let $(x_1^N, x_2^N, \ldots, x_N^N)$ be a permutation of $1, 2, \ldots, N$. We define a stochastic ranking process $X^N = (X_1^N, X_2^N, \ldots, X_N^N)$ as a unique solution of the system of stochastic integral equation

$$(1)\qquad X_i^N(t) = x_i^N + \sum_{j=1}^{N} \int_0^t \mathbf{1}(X_j(s-) > X_i(s-))\nu_j(ds) + \int_0^t (1 - X_i(s-))\nu_i(ds)$$

where $\mathbf{1}(A)$ is the indicator function of A. We regard $X_i^N(t)$ and x_i^N as positions of the i-th particle at time t and at time 0 respectively. The i-th particle has the Poisson clock v_i with intensity w_i and at each time when the clock v_i rings, then i-th particle jumps to the top. If a Poisson clock of a particle located behind the i-th particle rings, then the i-th particle jumps backward by one step.

We define the normalized position of X^N by

$$Y_i^N(t) = \frac{1}{N}(X_i^N(t) - 1).$$

We set $y_i^N := (x_i^N - 1)/N$ for $1 \le i \le N$. Let us consider the Cauchy problem for a system of quasi linear PDE

$$\begin{aligned}
&\frac{\partial}{\partial t}u_l(x,t) = -u_l(x,s)\tilde{w}_l(s) - \sum_{m=1}^{K} u_m(x,s)\tilde{w}_m(s)\frac{\partial}{\partial x}u_l(x,s),\\
&u_l(0,t) = f_l(0),\\
&u_l(1,t) = 0,\\
&u_l(x,0) = f_l(x),
\end{aligned} \tag{2}$$

for $l = 1, 2, \ldots, K$ where the initial functions f_l, $1 \le l \le K$ are smooth and decreasing, and satisfy that $f_l \ge 0$ and $\sum_{l=1}^K f_l(0) = 1$. In [13] it is proved that this system of PDE has a unique global classical solution. From now on, we denote by $u(x,t) = (u_1(x,t), \ldots, u_K(x,t))$ the unique global solution of (2).

We refer following results [23].

Proposition 2.1 ([23, Theorem 1.2]). *Assume that*

$$\begin{aligned}
&y_1^N \to y_1, \quad (N \to \infty), \text{ almost surely},\\
&\frac{1}{N}\sum_{j=1}^{N} \mathbf{1}(w_j = \tilde{w}_l)\mathbf{1}(y_j^N \ge x) \to f_l(x),\\
&\qquad (N \to \infty), \text{ uniformly in } x \in [0,1] \text{ almost surely}.
\end{aligned}$$

Then the scaled tagged particle motion $Y_1^N(t)$ converges to $Y_1(t)$ uniformly in $t \in [0,T]$ almost surely for all $T \ge 0$, where Y_1 is the solution of

$$Y_1(t) = y_1 + \sum_{l=1}^{K}\int_0^t u_l(Y_1(s-), s)\tilde{w}_l(s)\,ds - \int_0^t Y_1(s-)v_1(ds).$$

Furthermore, assume that for some L,

$$(y_1^N, y_2^N, \dots, y_L^N) \to (y_1, y_2, \dots, y_L), \quad (N \to \infty), \textit{ almost surely},$$

$$\frac{1}{N}\sum_{j=1}^{N} \mathbf{1}(w_j = \tilde{w}_l)\mathbf{1}(y_j^N \geq x) \to f_l(x),$$

$$(N \to \infty), \textit{ uniformly in } x \in [0,1] \textit{ almost surely}.$$

Then the scaled tagged particle system $(Y_1^N(t), Y_2^N(t), \dots, Y_L^N(t))$ *converges to* $(Y_1(t), Y_2(t), \dots Y_L(t))$ *uniformly in* $t \in [0, T]$ *almost surely for all* $T \geq 0$, *where* Y_i, $i = 1, 2, \dots, L$ *are the solutions of*

$$Y_i(t) = y_i + \sum_{l=1}^{K} \int_0^t u_l(Y_i(s-), s)\tilde{w}_l(s)\, ds - \int_0^t Y_i(s-)v_i(ds).$$

Proposition 2.1 expresses a scaled particle moves deterministically obeying the same ODE as for the corresponding characteristic curve of the system of PDE (2) except for its successive Poisson epochs at each of which it jumps to the top independently of the motion of the other tagged particles. We recall that Proposition 2.1 can be regarded as a law of large numbers.

The process Y^N may be considered as a "hyperbolic" scaling of X^N, and we are going to consider a "parabolic" scaling. Here is given a reason why we use the words "hyperbolic" and "parabolic" in spite of no scale change of time in both cases. First we consider

$$W^N(t) := \sum_{j=1}^{N} \int_0^t v_j(ds).$$

It is obvious that this is a (time inhomogeneous) continuous time random walk. To simplify our notation, we assume that $K = 1$ and $\tilde{w}_1 = 1$, so that W^N is a time homogeneous random walk with mean waiting time $1/N$. It is standard to see that $\frac{1}{N}W^N(t)$ converges to t and $\frac{1}{\sqrt{N}}(W^N(t) - Nt)$ converges to a standard Brownian motion as a law of large numbers and an invariance principle respectively. In this case, we have scaled the process in space, not in time. Nevertheless we have a law of large numbers and an invariance principle, since adding independent Poisson measures plays a speed up (time scaling) role. The situation for Y^N and Z^N (defined below) is the same, so the expressions "hyperbolic" and "parabolic" scaling according as the space scaling factors $1/N$ and $1/\sqrt{N}$ respectively. (In the "parabolic" scaling case, we may also need some centering.)

Let $\{\tau_n^1\}_{n\geq 1}$ be successive Poisson epochs of v_1 with $\tau_n^1 < \tau_{n+1}^1$ for all $n \geq 1$. To simplify our notation, we set $\tau_0^1 = 0$. We define $\mathscr{F}^1$ by σ-algebra generated by Poisson epochs $\{\tau_n^1; n \geq 0\}$. By the direct computation, we have the following formula for $Y_1(t)$;

$$(3)\quad Y_1(t) = \begin{cases} y_1 + \sum_{l=1}^{K} f_l(y_1)\left\{1 - \exp\left(-\int_0^t \tilde{w}_l(s)\, ds\right)\right\} & \text{if } 0 \le t < \tau_1^1, \\ \sum_{l=1}^{K} f_l(0)\left\{1 - \exp\left(-\int_{\tau_n^1}^t \tilde{w}_l(s)\, ds\right)\right\} & \text{if } \tau_n^1 \le t < \tau_{n+1}^1,\ n \ge 1. \end{cases}$$

We define "parabolically" scaled centered tagged particle dynamics Z_1^N by

$$Z_1^N(t) = \sqrt{N}(Y_1^N(t) - Y_1(t)).$$

Set p_l, q_l for $1 \le l \le K$ by

$$p_l(s,t) = \exp\left(-\int_s^t \tilde{w}_l(u)\, du\right), \quad q_l(s,t) = 1 - p_l(s,t)$$

for $s \le t$. We also set $\tau^1(t) := \max\{\tau_n^1 \le t; n \ge 0\}$. Furthermore, we set $a_1(t) = (a_{1,1}(t), a_{1,2}(t), \dots, a_{1,K}(t))$, $\rho_1(t) = (\rho_{1,1}(t), \rho_{1,2}(t), \dots, \rho_{1,K}(t))$ by

$$a_{1,k}(t) := \begin{cases} \sqrt{f_k(y_1)}\, p_k(0,t) & \text{if } \tau^1(t) = 0, \\ \sqrt{f_k(0)}\, p_k(\tau^1(t), t) & \text{if } \tau^1(t) > 0, \end{cases}$$

$$\rho_{1,k}(t) := \frac{q_k(\tau^1(t), t)}{p_k(\tau^1(t), t)},$$

for $1 \le k \le K$. Note that a_1, ρ_1 are $\mathscr{F}^1$ measurable random functions. Furthermore both of them depend only on the last Poisson time, i.e., if $\tau_n^1 \le t < \tau_{n+1}^1$, then they depend only on τ_n^1. We also note that $\rho_1(\tau_n^1) = 0$ for all $n \ge 0$. Furthermore, each component of a_1, ρ_1 is a smooth function except $\{\tau_n^1; n \ge 0\}$ and each component of ρ_1 is a strictly increasing function for each interval $[\tau_n^1, \tau_{n+1}^1)$ for $n \ge 0$.

Let $\{B_j; 1 \le j \le K\}$ be independent Brownian motions, which are independent of the Poisson random measures. We denote by $\rho'_{1,k}$ the derivative of the absolutely continuous part of $\rho_{1,k}$. We set a jump diffusion process $\Theta_1(t) = (\Theta_{1,1}(t), \Theta_{1,2}(t), \dots, \Theta_{1,K}(t))$ by

$$\Theta_{1,k}(t) := \int_0^t \sqrt{\rho'_{1,k}(s)}\, dB_k(s) - \int_0^t \Theta_{1,k}(s)\nu_1(ds),$$

for $1 \le k \le K$. Note that $\Theta_{1,k}(t)$ has the expression

$$\Theta_{1,k}(t) = \int_{\tau^1(t)}^t \sqrt{\rho'_{1,k}(s)}\, dB_k(s).$$

We define a jump diffusion process $\Psi_1(t) = (\Psi_{1,1}(t), \Psi_{1,2}(t), \dots, \Psi_{1,K}(t))$ by

$$(4)\qquad \Psi_{1,k}(t) := a_{1,k}(t)\Theta_{1,k}(t).$$

By applying Ito formula to $f(a_{1,k}(t), \Theta_{1,k}(t))$ with $f(x,y) = xy$, we have

$$\begin{aligned}\Psi_{1,k}(t) &= \int_0^t a'_{1,k}(s)\Theta_{1,k}(s)\,ds + \int_0^t a_{1,k}(s)\sqrt{\rho'_{1,k}(s)}\,dB_k(s) - \int_0^t \Psi_{1,k}(s)\nu_1(ds)\\ &= \int_0^t \frac{a'_{1,k}(s)}{a_{1,k}(s)}\Psi_{1,k}(s)\,ds + \int_0^t a_{1,k}(s)\sqrt{\rho'_{1,k}(s)}\,dB_k(s) - \int_0^t \Psi_{1,k}(s)\nu_1(ds),\end{aligned}$$

for $1 \le k \le K$. Then we define Z_1 by

$$Z_1(t) = \sum_{k=1}^{K} \Psi_{1,k}(t). \tag{5}$$

These processes jump to 0 at each Poisson time τ_n^1 for $n \ge 1$. By the definition of Θ, the law of $\Psi_{1,k}$ in the time interval $[\tau_n^1, \tau_{n+1}^1)$ coincides with that of $a_{1,k}(t)\tilde{B}_{k,n}(\rho_{1,k}(t))$, where $\tilde{B}_{k,n}$ is another standard Brownian motion. This is a typical continuous Markov Gaussian process (see Propositions 3.3, 3.4 below). By the definition, the components of Ψ_1 are independent. Therefore $Z_1(t)$ in the time interval $[\tau_n^1, \tau_{n+1}^1)$ is a Gaussian process with covariance $\Gamma(s,t) = \sum_{k=1}^K a_{1,k}(s)a_{1,k}(t)\rho_{1,k}(s)$ for $\tau_n^1, s \le t < \tau_{n+1}^1$. If $K \ge 2$, then $Z_1(t)$ is not a Markov process in general.

Theorem 2.2. *Assume that the following relations hold true with probability one*:

$$y_1^N - y_1 = o\left(\frac{1}{\sqrt{N}}\right), \quad (N \to \infty),$$

$$\frac{1}{N}\sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_k)\mathbf{1}(y_j^N \ge y_1^N) - f_k(y_1) = o\left(\frac{1}{\sqrt{N}}\right), \quad (N \to \infty), \text{ for } 1 \le k \le K,$$

$$\frac{1}{N}\sum_{j=1}^N \mathbf{1}(w_j = \tilde{w}_k) - f_k(0) = o\left(\frac{1}{\sqrt{N}}\right), \quad (N \to \infty), \text{ for } 1 \le k \le K.$$

Then Z_1^N conditioned on $\mathscr{F}^1$ converges weakly in the Skorohod topology to Z_1 conditioned on $\mathscr{F}^1$.

We interpret Z_1, Ψ_1 as follows: For each Poisson time $\{\tau_n^1 : n \ge 1\}$, the processes are reset to 0. In each time interval $[\tau_n^1, \tau_{n+1}^1)$, $n = 0, 1, 2, \dots$, the process Z_1 is a centered continuous Gaussian process with covariance $\Gamma(s,t) = \sum_{k=1}^K a_{1,k}(s)a_{1,k}(t)\rho_{1,k}(s)$ for $s \le t$, which is decomposed into a sum of independent centered continuous Markov Gaussian processes. We note that $a_{1,k}(t)$ corresponds to the expected number of particles behind the tagged particle with the same intensity $\tilde{w}_k$. Hence each centered continuous Markov Gaussian process is constructed by particles behind the tagged particle with the same intensity.

Remark 2.3. We suppose that the set of intensities is infinite. If we add some extra assumptions on intensities and initial configurations, then our proof below is applicable and Z^N converges to some centered Gaussian process, which is continuous except for the Poisson time $\{\tau_n^1; n \geq 1\}$.

We also consider a "parabolically" scaled centered multi-tagged particle system. We define $Z^N = Z^{N,L} = (Z_1^N, Z_2^N, \dots, Z_L^N)$ by

$$Z_l^N(t) = \sqrt{N}(Y_l^N(t) - Y_l(t)),$$

for $1 \leq l \leq L$. Let $\{\tau_n^l\}_{n \geq 1}$ be successive Poisson epochs with respect to v_l with $\tau_n^l < \tau_{n+1}^l$ for all $n \geq 1$ for $1 \leq l \leq L$. For $1 \leq l \leq L$ set $\tau_0^l = 0$,

$$\tau^l(t) = \max\{\tau_n^l \leq t; n \geq 0\}$$

and

$$\tau^{\mathbf{L}}(t) = \max\{\tau^l(t); 1 \leq l \leq L\}.$$

We also set

$$\rho_k(s,t) = \frac{q_k(s,t)}{p_k(s,t)}, \quad \rho_k'(s,t) = \frac{\partial}{\partial t}\rho_k(s,t).$$

Let $\{B_{i,j}; 1 \leq i \leq L, 1 \leq j \leq K\}$ be independent Brownian motions, which are independent of the Poisson random measures. Then we define a jump diffusion process $\Xi(t) = (\Xi_{i,j}(t))_{1 \leq i \leq L, 1 \leq j \leq K}$ by

$$\Xi_{i,j}(t) := \int_{\tau^{\mathbf{L}}(t)}^{t} \sqrt{\rho_j'(\tau^{\mathbf{L}}(t), s)}\, dB_{i,j}(s).$$

We set $R(t) := (R_1(t), R_2(t), \dots, R_L(t))$ a permutation of $(1, 2, \dots, L)$ such that

$$Y_{R_1(t)}(t) \geq Y_{R_2(t)}(t) \geq \cdots \geq Y_{R_L(t)}(t),$$

where if $Y_{R_i(t)}(t) = Y_{R_{i+1}(t)}(t)$, then we set $R_i(t) < R_{i+1}(t)$. Namely, by means of $R(t)$, we arrange $Y(t) = (Y_1(t), Y_2(t), \dots, Y_L(t))$ in descending order. We set $S(t) = (S_1(t), S_2(t), \dots, S_L(t))$ a permutation of $(1, 2, \dots, L)$ such that $R_{S_l(t)} = l$ for $1 \leq l \leq L$.

We set

$$a^{l,0}(\tau^{\mathbf{L}}(t)) = 0,$$

$$a^{l,k}(\tau^{\mathbf{L}}(t)) = \begin{cases} f_k(y_l)p_k(0, \tau^{\mathbf{L}}(t)) & \text{if } \tau^{R_l(t)}(t) = 0, \text{ and } 1 \leq l \leq L, \\ f_k(0)p_k(\tau^{R_l(t)}(t), \tau^{\mathbf{L}}(t)) & \text{if } \tau^{R_l(t)}(t) \neq 0, \text{ and } 1 \leq l \leq L, \end{cases}$$

for $1 \leq k \leq K$. We define $\Psi(t) = (\Psi_{l,k}(t))_{1 \leq l \leq L, 1 \leq k \leq K}$ by

$$
(6)\quad
\begin{aligned}
\Psi_{l,k}(t) = \Psi_{l,k}(\tau^{\mathbf{L}}(t))p_k(\tau^{\mathbf{L}}(t),t) + \sum_{j=1}^{S_l(t)} \sqrt{a^{j,k}(\tau^{\mathbf{L}}(t)) - a^{j-1,k}(\tau^{\mathbf{L}}(t))} \\
\times \left\{ -\int_{\tau^{\mathbf{L}}(t)}^{t} \tilde{w}_k(s) p_k(\tau^{\mathbf{L}}(t),s)\Xi_{j,k}(s)\,ds \right. \\
\left. + \int_{\tau^{\mathbf{L}}(t)}^{t} p_k(\tau^{\mathbf{L}}(t),s)\,d\Xi_{j,k}(s) \right\},
\end{aligned}
$$

$$
\Psi_{l,k}(\tau^{\mathbf{L}}(t)) = \begin{cases} \Psi_{l,k}(\tau^{\mathbf{L}}(t)-) & \text{if } \tau^{\mathbf{L}}(t) \neq \tau^l(t), \\ 0 & \text{otherwise.} \end{cases}
$$

Then we define $Z(t) = (Z_1(t), Z_2(t), \ldots, Z_L(t))$ by

$$
(7)\qquad Z_l(t) = \sum_{k=1}^{K} \Psi_{l,k}(t).
$$

LEMMA 2.4. *The law of* $\Psi_{1,k}(t)$ *defined by* (4) *coincides with that by* (6) *for* $1 \le k \le K$. *Therefore the law of* $Z_1(t)$ *defined by* (5) *coincides with that by* (7).

We set $\mathscr{F}^{\mathbf{L}} := \sigma(\{\tau_n^1; n \ge 0\} \cup \{\tau_n^2; n \ge 0\} \cup \cdots \cup \{\tau_n^L; n \ge 0\})$.

THEOREM 2.5. *Assume that*

$$
y_l^N - y_l = o\left(\frac{1}{\sqrt{N}}\right), \quad (N \to \infty), \text{ almost surely for } 1 \le l \le L,
$$

$$
\frac{1}{N}\sum_{j=1}^{N} \mathbf{1}(w_j = \tilde{w}_k)\mathbf{1}(y_j^N \ge y_l^N) - f_k(y_l) = o\left(\frac{1}{\sqrt{N}}\right),
$$

$$
(N \to \infty) \text{ almost surely } 1 \le l \le L,\ 1 \le k \le K,
$$

$$
\frac{1}{N}\sum_{j=1}^{N} \mathbf{1}(w_j = \tilde{w}_k) - f_k(0) = o\left(\frac{1}{\sqrt{N}}\right), \quad (N \to \infty) \text{ almost surely for } 1 \le k \le K.
$$

Then $Z^N = (Z_1^N, Z_2^N, \ldots, Z_L^N)$ *conditioned on* $\mathscr{F}^{\mathbf{L}}$ *converges weakly in the Skorohod topology to* $Z = (Z_1, Z_2, \ldots, Z_L)$ *conditioned on* $\mathscr{F}^{\mathbf{L}}$.

We interpret this process as follows: The jump times of each component coincide with its Poisson times. Each component of the process is given by a sum of independent centered Markov Gaussian processes. The "hyperbolically" scaled tagged particles Y divide $[0,1)$ interval into $L+1$ layers. For each Poisson time $\{\tau_n^{\mathbf{L}}; n \ge 1\}$, the layers are reset. The height of the layer is interpreted as expected normalized number of particles in the layer. In each layer, the height is divided into expected normalized number of particles with the same intensity in the layer. We interpret that each centered Markov Gaussian process is constructed by particles with the same intensity in corresponding layer. We also interpret these centered Markov Gaussian processes as shrunk version of that in the single tagged particle case. Each centered Markov Gaussian process is held in common by Z_l if the corresponding layer is behind $Y_l(\tau^{\mathbf{L}}(t))$.

3. Proof of Theorem 2.2

To prove the main theorem, we have only to show the tightness and the convergence of finite dimensional distribution. Namely, if we prove following two lemmas, then we conclude the proof of Theorem 2.2:

LEMMA 3.1. *The sequence* $\{Z_1^N; N \geq 1\}$ *conditioned on* $\mathscr{F}^1$ *is tight.*

LEMMA 3.2. *The finite dimensional characteristic function of* Z_1^N *conditioned on* $\mathscr{F}^1$ *converges to that of* Z_1 *conditioned on* $\mathscr{F}^1$. □

First we give fundamental results on Gaussian process.

PROPOSITION 3.3 [19, Proposition 13.7], [24, Chap. III Sec. 1, Exercise 1.13]. *Let* X *be a Gaussian process with covariance* Γ. *Then* X *is a Markov process if and only if* $\Gamma(s,u)\Gamma(t,t) = \Gamma(s,t)\Gamma(t,u)$ *for* $s < t < u$.

PROPOSITION 3.4 [24, Chap. III Sec. 1, Exercise 1.13]. *Suppose that* a *is continuous and does not vanish and* ρ *is a continuous strictly positive and non decreasing function. If we set* $Y(t) = a(t)B(\rho(t))$, *where* B *is a Brownian motion, then* Y *is a centered Gaussian process with covariance* $\Gamma(s,t) = a(s)a(t)\rho(s)$ *for* $s < t$.

Second we give the following formula for $Y_1^N(t)$ (cf [11, $Y_C^{(N)}$]), which plays an important role in the proof, and compute several quantities.

In the time interval $[\tau_n^1, \tau_{n+1}^1)$, for each $j \neq 1$, the order of the values Y_1^N and Y_j^N changes at most once at τ, the first Poisson time with respect to v_j in the time interval $[\tau_n^1, \tau_{n+1}^1)$. By the definition of Y_1^N, we have

$$(8) \quad Y_1^N(t) = \begin{cases} y_1^N + \dfrac{1}{N}\displaystyle\sum_{j=1}^N \mathbf{1}(Y_j^N(0) > Y_1^N(0))\mathbf{1}(v_j((0,t]) \geq 1) & \text{if } \tau^1(t) = 0, \\ \dfrac{1}{N}\displaystyle\sum_{j=1}^N \mathbf{1}(v_j((\tau^1(t), t]) \geq 1) & \text{if } \tau^1(t) > 0. \end{cases}$$

By simple computation, we have

$$E[Y_1^N(t) \mid \mathscr{F}^1]$$
$$= \begin{cases} y_1^N + \dfrac{1}{N}\displaystyle\sum_{j=2}^{N} \mathbf{1}(Y_j^N(0) > Y_1^N(0))\left\{1 - \exp\left(-\int_0^t w_j(s)\,ds\right)\right\} & \text{if } \tau^1(t) = 0, \\ \dfrac{1}{N}\displaystyle\sum_{j=2}^{N}\left\{1 - \exp\left(-\int_{\tau^1(t)}^t w_j(s)\,ds\right)\right\} & \text{if } \tau^1(t) > 0. \end{cases}$$

By the assumption of the Theorem 2.2, we have

$$\text{(9)} \qquad Y_1(t) - E[Y_1^N(t) \mid \mathscr{F}^1] = o\left(\frac{1}{\sqrt{N}}\right), \quad (N \to \infty) \text{ uniformly in } t.$$

Hence we have

$$E[Z_1^N(t) \mid \mathscr{F}^1] = o(1), \quad (N \to \infty) \text{ uniformly in } t.$$

Set p^j, q^j for $j \geq 2$ by

$$p^j(t) = \sum_{k=1}^{K} \mathbf{1}(w_j = \tilde{w}_k) p_k(\tau^1(t), t), \quad q^j(t) = \sum_{k=1}^{K} \mathbf{1}(w_j = \tilde{w}_k) q_k(\tau^1(t), t).$$

Then it is easy to see that $p^j(t) = P(v_j((\tau^1(t), t]) = 0)$ and $q^j(t) = P(v_j((\tau^1(t), t]) \geq 1)$. We define W_j for $j \geq 2$ by

$$W_j(t) = \tilde{W}_j(t) - E[\tilde{W}_j(t) \mid \mathscr{F}^1],$$
$$\tilde{W}_j(t) = \mathbf{1}(v_j((\tau^1(t), t]) \geq 1).$$

Then we have

$$\text{(10)} \qquad \begin{aligned} &E[W_j(t) \mid \mathscr{F}^1] = 0, && \text{for all } t, \\ &E[W_j(s) W_j(t) \mid \mathscr{F}^1] = q^j(s) p^j(t), && \text{if } s \leq t \text{ and } \tau^1(s) = \tau^1(t), \\ &E[W_j(s) W_j(t) \mid \mathscr{F}^1] = 0, && \text{if } \tau^1(s) < \tau^1(t). \end{aligned}$$

Furthermore we have

$$\text{(11)} \qquad Y_1^N(t) - E[Y_1^N(t) \mid \mathscr{F}^1] = \begin{cases} \dfrac{1}{N}\displaystyle\sum_{j=2}^{N} \mathbf{1}(Y_j^N(0) > Y_1^N(0)) W_j(t) & \text{if } \tau^1(t) = 0, \\ \dfrac{1}{N}\displaystyle\sum_{j=2}^{N} W_j(t) & \text{if } \tau^1(t) > 0. \end{cases}$$

We define a σ-algebra $\mathscr{G}$ by $\mathscr{G} = \sigma(\mathscr{F}^1 \cup \sigma(t) \cup \sigma(W_j(t)))$ for some stopping time t. Then by the definition of W_j, we have

$$\text{(12)} \qquad E[(W_j(t+s) - W_j(t))^2 \mid \mathscr{G}] \leq 2\{E[\mathbf{1}(v_j([t, t+s)) \geq 1) \mid \mathscr{G}] + Cs\} \leq 4Cs$$

for $C = \sup_j \sup_s w_j(s)$, if $\tau^1(t) = \tau^1(t+s)$.

It is easy to compute the finite dimensional characteristic function of W_j conditioned on $\mathscr{F}^1$ as follows: Set $\mathbf{t} = (t_1^0, t_2^0, \ldots, t_{m^0}^0, t_1^1, t_2^1, \ldots, t_{m^1}^1, \ldots, t_1^n, t_2^n, \ldots, t_{m^n}^n)$ which satisfies $0 \le t_1^1 \le t_2^0 \le \cdots \le t_{m^0}^0 < \tau_1^1 \le t_1^1 \le t_2^1 \le \cdots \le t_{m^1}^1 < \tau_2^1 \le \cdots < \tau_n^1 \le t_1^n \le t_2^n \le \cdots \le t_{m^n}^n < \tau_{n+1}^1$ and $\mathbf{x} = (x_1^0, x_2^0, \ldots, x_{m^0}^0, x_1^1, x_2^1, \ldots, x_{m^1}^1, \ldots, x_1^n, x_2^n, \ldots, x_{m^n}^n) \in \mathbf{R}^M$ where $M = \sum_{k=0}^n m^k$. Then we have

$$\varphi_{\mathbf{t}}^j(\mathbf{x}) := E\left[\exp\left(\sqrt{-1}\sum_{k=0}^n\sum_{i=1}^{m^k} W_j(t_i^k)x_i^k\right)\middle|\mathscr{F}^1\right] \tag{13}$$

$$= 1 - \frac{1}{2}\sum_{k=0}^n\sum_{i=1}^{m^k} q^j(t_i^k)p^j(t_i^k)(x_i^k)^2$$

$$- \sum_{k=0}^n\sum_{i=1}^{m^k-1}\sum_{l=i+1}^{m^k} q^j(t_i^k)p^j(t_l^k)x_i^k x_l^k + O(|\mathbf{x}|^3),$$

where $|\mathbf{x}| = \sqrt{\sum_{k=0}^n\sum_{i=1}^{m^k}(x_i^k)^2}$.

Since $\{\nu_j; j \ge 1\}$ is independent Poisson random measure, by the definition of Z_1^N and by using (9), (11), and (13), we conclude that $\Phi_{\mathbf{t}}^N$ the finite dimensional characteristic function of Z_1^N conditioned on $\mathscr{F}^1$ is given by

$$\Phi_{\mathbf{t}}^N(\mathbf{x}) := E\left[\exp\left(\sqrt{-1}\sum_{k=0}^n\sum_{i=1}^{m^k} Z_1^N(t_i^k)x_i^k\right)\middle|\mathscr{F}^1\right] \tag{14}$$

$$= E\Bigg[\exp\Bigg\{\sqrt{-1}\Bigg[\sum_{i=1}^{m^0}\Bigg\{\frac{1}{\sqrt{N}}\sum_{j=2}^N \mathbf{1}(Y_j^N(0) > Y_1^N(0))W_j(t_j^0)$$

$$- \sqrt{N}(Y_1(t_i^0) - E[Y_1^N(t_i^0)\,|\,\mathscr{F}_1])\Bigg\}x_i^0 + \sum_{k=1}^n\sum_{i=1}^{m^k}\Bigg\{\frac{1}{\sqrt{N}}\sum_{j=2}^N W_j(t_j^k)$$

$$- \sqrt{N}(Y_1(t_i^k) - E[Y_1^N(t_i^k)\,|\,\mathscr{F}_1])\Bigg\}x_i^k\Bigg]\Bigg\}\,\Bigg|\,\mathscr{F}^1\Bigg]$$

$$= \exp\Bigg\{-\frac{1}{2}\frac{1}{N}\sum_{j=2}^N \mathbf{1}(y_j^N \ge y_1^N)$$

$$\times\left(\sum_{i=1}^{m^0} q^j(t_i^0)p^j(t_i^0)(x_i^0)^2 - \sum_{i=1}^{m^0-1}\sum_{l=i+1}^{m^0} q^j(t_i^0)p^j(t_l^0)x_i^0x_l^0\right)$$

$$-\frac{1}{N}\sum_{j=2}^N\Bigg(\frac{1}{2}\sum_{k=1}^n\sum_{i=1}^{m^k} q^j(t_i^k)p^j(t_i^k)(x_i^k)^2$$

$$-\sum_{k=1}^n\sum_{i=1}^{m^k-1}\sum_{l=i+1}^{m^k} q^j(t_i^k)p^j(t_l^k)x_i^kx_l^k\Bigg) + o(1)\Bigg\}$$

$$= \exp\Bigg\{\sum_{l=1}^{K}\Bigg(-\frac{1}{2}\frac{1}{N}\sum_{j=2}^{N}\mathbf{1}(w_j = \tilde{w}_l)\mathbf{1}(y_j^N \geq y_1^N)$$

$$\times\Bigg(\sum_{i=1}^{m^0} q_l(t_i^0)p_l(t_i^0)(x_i^0)^2 - \sum_{i=1}^{m^0-1}\sum_{l=i+1}^{m^0} q_l(t_i^0)p_l(t_l^0)x_i^0 x_l^0\Bigg)$$

$$-\frac{1}{N}\sum_{j=2}^{N}\mathbf{1}(w_j = \tilde{w}_l)\Bigg(\frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m^k} q_l(t_i^k)p_l(t_i^k)(x_i^k)^2$$

$$-\sum_{k=1}^{n}\sum_{i=1}^{m^k-1}\sum_{l=i+1}^{m^k} q_l(t_i^k)p_l(t_l^k)x_i^k x_l^k\Bigg)\Bigg) + o(1)\Bigg\}.$$

Proof of Lemma 3.1. Since $Z_1^N(\tau_n) = 0$ for all $N \geq 1$, $n \geq 0$, thanks to Aldous's tightness criterion and Chebyshev's inequality, it is enough to prove that

$$E[\{Z_1^N(t)\}^2 \,|\, \tau_n^1 \leq t < \tau_{n+1}^1] \leq C,$$
$$E[\{Z_1^N((t+\delta')-) - Z_1^N(t)\}^2 \,|\, \tau_n^1 \leq t < \tau_{n+1}^1] \leq C\delta,$$

for some constant C for all stopping time $t < T$ and $\delta > 0$ (see [2]). Here $t+\delta'$ designates $\min\{t+\delta', \tau_{n+1}^1\}$. Since $\{W_j; j \geq 2\}$ is independent, by (9), (10) and (11), we conclude that

$$E[\{Z_1^N(t)\}^2 \,|\, \tau_n^1 \leq t < \tau_{n+1}^1]$$
$$\leq 2\{E[(Y_1^N(t) - E[Y_1^N(t) \,|\, \mathscr{F}^1])^2 \,|\, \tau_n^1 \leq t < \tau_{n+1}^1] + o(1)\}$$
$$\leq 2\Bigg\{\frac{1}{N}\sum_{j=2}^{N} E[W_j(t)^2 \,|\, \tau_n^1 \leq t < \tau_{n+1}^1] + o(1)\Bigg\} \leq 2(1 + o(1)),$$

for all stopping time t.

By (12), we have

$$E[(W_j((t+\delta')-) - W_j(t))^2 \,|\, \tau_n^1 \leq t < \tau_{n+1}^1]$$
$$= E[E[(W_j((t+\delta')-) - W_j(t))^2 \,|\, \mathscr{G}] \,|\, \tau_n^1 \leq t < \tau_{n+1}^1] \leq 4C\delta.$$

since we have $t+\delta' = \min\{t+\delta', \tau_{n+1}^1\}$. □

Proof of Lemma 3.2. By the assumption of the Theorem 2.2, the definition of a, ρ, and (14), we conclude that Φ^N converges to

$$\prod_{j=1}^{K} \exp\left\{ -\frac{1}{2}\sum_{k=0}^{n}\sum_{i=1}^{m^k} \rho_{1,j}(t_i^k)(a_{1,j}(t_i^k))^2(x_i^k)^2 - \sum_{k=0}^{n}\sum_{i=0}^{m^k-1}\sum_{l=i+1}^{m^k} \rho_{1,j}(t_i^k)a_{1,j}(t_i^k)a_{1,j}(t_l^k)x_i^k x_l^k \right\}, \tag{15}$$

which is the product of characteristic functions of a Gaussian process with covariance $\Gamma(s,t) = a_{1,j}(s)a_{1,j}(t)\rho_{1,j}(s)$ for $s \le t$. By Proposition 3.3, Proposition 3.4, each term of (15) coincides with the finite dimensional characteristic function of $\Psi_{1,j}$ conditioned on $\mathscr{F}^1$. Hence we conclude that (15) coincides with the finite dimensional characteristic function of Z conditioned on $\mathscr{F}^1$. □

4. Proof of Lemma 2.4

In this section following identity plays a key role: By the definition of p_k, we have

$$p_k(s,t) = p_k(s,u)p_k(u,t) \tag{16}$$

for $s \le u \le t$, $1 \le k \le K$. We recall that $\Psi_{1,k}(t)$ defined by (4) has an expression

$$\begin{aligned}\Psi_{1,k}(t) &= \int_0^t a'_{1,k}(s)\Theta_{1,k}(s)\, ds + \int_0^t a_{1,k}(s)\sqrt{\rho'_{1,k}(s)}\, dB_k(s) - \int_0^t \Psi_{1,K}(s)v_1(ds)\\ &= \int_{\tau^1(t)}^t a'_{1,k}(s)\Theta_{1,k}(s)\, ds + \int_{\tau^1(t)}^t a_{1,k}(s)\, d\Theta_{1,k}(s).\end{aligned}$$

Let u be a stopping time such that $\tau^1(t) \le u \le t$. Then we have

$$\Psi_{1,k}(t) = \Psi_{1,k}(u) + \int_u^t a'_{1,k}(s)\Theta_{1,k}(s)\, ds + \int_u^t a_{1,k}(s)\, d\Theta_{1,k}(s).$$

By the definition of $a_{1,k}$, $\rho_{1,k}$ and p_k, we have

$$\begin{aligned}&\int_u^t a'_{1,k}(s)\Theta_{1,k}(s)\, ds\\ &\quad = -\int_u^t \tilde{w}_k(s)a_{1,k}(s)\Theta_{1,k}(u)\, ds - \int_u^t \tilde{w}_k(s)a_{1,k}(s)\left(\int_u^t \sqrt{\frac{\tilde{w}_k(v)}{p_k(\tau^1(t),v)}}\, dB_k(v)\right) ds.\end{aligned}$$

By (16), we have $a_{1,k}(s) = a_{1,k}(u)p_k(u,s)$. We also have $\frac{\partial}{\partial s}p_k(u,s) = -\tilde{w}_k(s)p_k(u,s)$ and $p_k(u,u) = 1$. Hence we have

$$\int_u^t \tilde{w}_k(s)a_{1,k}(s)\Theta_{1,k}(u)\, ds = \Psi_{1,k}(u)\int_u^t \tilde{w}_k(s)p_k(u,s)\, ds = \Psi_{1,k}(u)(p_k(u,t)-1).$$

We set $g_k(\tau^1(t))$ by

$$g_k(\tau^1(t)) = \begin{cases} f_k(y_1) & \text{if } \tau^1(t) = 0, \\ f_k(0) & \text{otherwise.} \end{cases}$$

Then by using (16) and the definition of $\rho'_k(s,t)$, we have

$$\begin{aligned} &\int_u^t \tilde{w}_k(s) a_{1,k}(s) \left(\int_u^t \sqrt{\frac{\tilde{w}_k(v)}{p_k(\tau^1(t), v)}}\, dB_k(v) \right) ds \\ &\quad = \sqrt{g_k(\tau^1(t)) p_k(\tau^1(t), u)} \int_u^t \tilde{w}_k(s) p_k(u,s) \left(\int_u^t \sqrt{\frac{\tilde{w}_k(v)}{p_k(u,v)}}\, dB_k(v) \right) ds \\ &\quad = \sqrt{g_k(\tau^1(t)) p_k(\tau^1(t), u)} \int_u^t \tilde{w}_k(s) p_k(u,s) \left(\int_u^t \sqrt{\rho'_k(u,v)}\, dB_k(v) \right) ds. \end{aligned}$$

Similarly we have

$$\begin{aligned} \int_u^t a_{1,k}(s)\, d\Theta_{1,k}(s) &= \sqrt{g_k(\tau^1(t))} \int_u^t p_k(\tau^1(t), s) \sqrt{\frac{\tilde{w}_k(s)}{p_k(\tau^1(t)), s)}}\, dB_k(s) \\ &= \sqrt{g_k(\tau^1(t)) p_k(\tau^1(t), u)} \int_u^t p_k(u,s) \sqrt{\rho'_k(u,s)}\, dB_k(s). \end{aligned}$$

Summarizing these identities, we have

$$\begin{aligned} \Psi_{1,k}(t) = \Psi_{1,k}(u) p_k(u,t) &+ \sqrt{g_k(\tau^1(t)) p_k(\tau^1(t), u)} \\ &\times \left\{ -\int_u^t \tilde{w}_k(s) p_k(u,s) \left(\int_u^t \sqrt{\rho'_k(u,v)}\, dB_k(v) \right) ds \right. \\ &\qquad \left. + \int_u^t p_k(u,s) \sqrt{\rho'_k(u,s)}\, dB_k(s) \right\}. \end{aligned}$$

Substituting $\tau^{\mathbf{L}(t)}$ for u and comparing this and (6), we have only to prove following identity:

$$\sum_{j=1}^{S_l(t)} a^{j,k}(\tau^{\mathbf{L}}(t)) - a^{j-1,k}(\tau^{\mathbf{L}}(t)) = g_k(\tau^1(t)) p_k(\tau^1(t), u).$$

It is trivial since

$$\sum_{j=1}^{S_1(t)} a^{j,k}(\tau^{\mathbf{L}}(t)) - a^{j,k}(\tau^{\mathbf{L}}(t)) = a^{1,k}(\tau^{\mathbf{L}}(t)) = g_k(\tau^1(t)) p_k(\tau^1(t), u). \qquad \square$$

5. Proof of Theorem 2.5

Outline of the proof of Theorem 2.5 is the same as that of Theorem 2.2. Namely, we have only to show the tightness and the convergence of the finite dimensional characteristic function.

Since the proof of the tightness is the same as that in Lemma 3.1, we omit the proof.

In order to see the characteristic function, we define W_j^i for $1 \le i \le L$, $j \ge 1$ with $i \ne j$ by

$$W_j^i(t) = \tilde{W}_j^i(t) - E[\tilde{W}_j^i(t) \mid \mathscr{F}^{\mathbf{L}}], \quad \tilde{W}_j^i(t) = \mathbf{1}(\nu_j(\tau^i(t), t] \ge 1).$$

Since $|W_j^i| \le 1$, we have

$$Z_i^N = \begin{cases} \dfrac{1}{\sqrt{N}} \displaystyle\sum_{j=L+1}^{N} \mathbf{1}(Y_j^N(0) > Y_i^N(0)) W_j^i(t) + o(1) & \text{if } \tau^i(t) = 0, \\ \dfrac{1}{\sqrt{N}} \displaystyle\sum_{j=L+1}^{N} W_j^i(t) + o(1) & \text{if } \tau^i(t) > 0, \end{cases}$$

for $1 \le i \le L$ (cf. (11)). Since ν_j are independent Poisson measures, $\{W_j;\ j \ge L+1\}$ with $W_j = (W_j^1, W_j^2, \dots, W_j^L)$ are independent random vectors. Hence finite dimensional distribution of Z^N converges to some Gaussian random variables. Therefore we only observe the limit of the covariance.

By the definition of $W_j^i(t)$, for $1 \le i, k \le L$, $j, l \ge L+1$, $s \le t$, we have

$$E[W_j^i(t) \mid \mathscr{F}^{\mathbf{L}}] = 0$$

$$E[W_j^i(t) W_l^k(s) \mid \mathscr{F}^{\mathbf{L}}] = \begin{cases} q^j(\tau^k(s), s) p^j(\tau^i(t), t) & \text{if } j = l, \text{ and } \tau^i(t) \le \tau^k(s), \\ q^j(\tau^i(t), s) p^j(\tau^k(s), t) & \text{if } j = l, \text{ and } \tau^k(s) \le \tau^i(t) < s, \\ 0 & \text{if } j \ne l, \text{ or } s \le \tau^i(t). \end{cases}$$

Since $p^j(a, c) = p^j(a, b) p^j(b, c)$ for $a \le b \le c$, we have

$$\begin{aligned} &E[W_j^i(t) W_l^k(s) \mid \mathscr{F}^{\mathbf{L}}] \\ &\quad = \begin{cases} q^j(\tau^k(s), s) p^j(\tau^i(t), \tau^k(s)) p^j(\tau^k(s), t) & \text{if } j = l, \text{ and } \tau^i(t) \le \tau^k(s), \\ q^j(\tau^i(t), s) p^j(\tau^k(s), \tau^i(t)) p^j(\tau^i(t), t) & \text{if } j = l, \text{ and } \tau^k(s) \le \tau^i(t) < s, \\ 0 & \text{if } j \ne l, \text{ or } s \le \tau^i(t). \end{cases} \end{aligned}$$

We set $g_k(\tau^i(t))$ by

$$g_k(\tau^i(t)) = \begin{cases} f_k(y_i) & \text{if } \tau^i(t) = 0, \\ f_k(0) & \text{otherwise.} \end{cases}$$

Then for $1 \le i, k \le L$, $s \le t$, we have

$$E[Z_i^N(t)Z_k^N(s)\,|\,\mathscr{F}^{\mathbf{L}}] = \frac{1}{N}\sum_{j=L+1}^{N}\sum_{l=L+1}^{N} E[W_j^i(t)W_l^k(s)\,|\,\mathscr{F}^{\mathbf{L}}]$$

$$\to \begin{cases} \displaystyle\sum_{\kappa=1}^{K} g_\kappa(\tau^i(t))p_\kappa(\tau^i(t),\tau^k(s))q_\kappa(\tau^k(s),s)p_\kappa(\tau^k(s),t) \\ \qquad \text{if } \tau^i(t)\le\tau^k(s) \text{ and } g_\kappa(\tau^i(t))\le g_\kappa(\tau^k(s)), \\ \displaystyle\sum_{\kappa=1}^{K} g_\kappa(\tau^k(s))p_\kappa(\tau^k(s),\tau^i(t))q_\kappa(\tau^i(t),s)p_\kappa(\tau^i(t),t) \\ \qquad \text{if } \tau^k(s)\le\tau^i(t)<s \text{ and } g_\kappa(\tau^k(s))\le g_\kappa(\tau^i(t)), \\ 0, \quad \text{if } s\le\tau^i(t), \end{cases}$$

as $N\to\infty$. Note that the condition $g_\kappa(\tau^i(t))\le g_\kappa(\tau^k(s))$ (and $g_\kappa(\tau^k(s))\le g_\kappa(\tau^i(t))$) makes sense iff $\tau^i(t)=\tau^k(s)=0$, since if $\tau^i(t)<\tau^k(s)$, then we have $g_\kappa(\tau^i(t))\le g_\kappa(\tau^k(s))$ (if $\tau^k(s)<\tau^i(t)$, then we have $g_\kappa(\tau^k(s))\le g_\kappa(\tau^i(t))$).

We set $\Upsilon_{j,k}(t)$ for $1\le j\le L$, $1\le k\le K$ by

$$\Upsilon_{j,k}(t) = -\int_{\tau^{\mathbf{L}}(t)}^{t} \tilde{w}_k(s)p_k(\tau^{\mathbf{L}}(t),s)\Xi_{j,k}(s)\,ds + \int_{\tau^{\mathbf{L}}(t)}^{t} p_k(\tau^{\mathbf{L}}(t),s)\,d\Xi_{j,k}(s).$$

Then by the definition of Ξ, we have

$$\text{(17)}\quad E[\Upsilon_{j,k}(t)\,|\,\mathscr{F}^{\mathbf{L}}]=0,$$

$$E[\Upsilon_{j,k}(t)\Upsilon_{l,m}(s)\,|\,\mathscr{F}^{\mathbf{L}}] = \begin{cases} p_k(\tau^{\mathbf{L}}(t),t)q_k(\tau^{\mathbf{L}}(s),s) & \text{if } j=l,\ k=m, \text{ and } \tau^{\mathbf{L}}(s)=\tau^{\mathbf{L}}(t), \\ 0 & \text{otherwise,} \end{cases}$$

for $s\le t$.

We define $\{\tau_n^{\mathbf{L}};n\ge 0\}$ by $\{\tau_n^{\mathbf{L}};n\ge 0\}=\bigcup_{l=1}^{L}\{\tau_n^l;n\ge 0\}$ with $\tau_n^{\mathbf{L}}<\tau_{n+1}^{\mathbf{L}}$ for $n\ge 0$. Suppose that $\tau^l(t)=\tau_n^{\mathbf{L}}$ and $\tau^{\mathbf{L}}(t)=\tau_{n+m}^{\mathbf{L}}$ for some $n\ge 0$, $m\ge 1$. By the definition of $\Psi_{l,k}(t)$, we have

$$\begin{aligned}\Psi_{l,k}(t) &= \Psi_{l,k}(\tau_{n+m}^{\mathbf{L}})p_k(\tau_{n+m}^{\mathbf{L}},t) + \sum_{j=1}^{S_l(t)} \sqrt{a^{j,k}(\tau_{n+m}^{\mathbf{L}})-a^{j-1,k}(\tau_{n+m}^{\mathbf{L}})}\,\Upsilon_{j,k}(t) \\ &= \Bigg\{\Psi_{l,k}(\tau_{n+m-1}^{\mathbf{L}})p_k(\tau_{n+m-1}^{\mathbf{L}},\tau_{n+m}^{\mathbf{L}}-) \\ &\quad + \sum_{j=1}^{S_l(\tau_{n+m}^{\mathbf{L}}-)} \sqrt{a^{j,k}(\tau_{n+m-1}^{\mathbf{L}})-a^{j-1,k}(\tau_{n+m-1}^{\mathbf{L}})}\,\Upsilon_{j,k}(\tau_{n+m}^{\mathbf{L}}-)\Bigg\} p_k(\tau_{n+m}^{\mathbf{L}},t) \\ &\quad + \sum_{j=1}^{S_l(t)} \sqrt{a^{j,k}(\tau_{n+m}^{\mathbf{L}})-a^{j-1,k}(\tau_{n+m}^{\mathbf{L}})}\,\Upsilon_{j,k}(t)\end{aligned}$$

$$= \Psi_{l,k}(\tau^{\mathbf{L}}_{n+m-1}) p_k(\tau^{\mathbf{L}}_{n+m-1}, t)$$

$$+ \sum_{j=1}^{S_l(\tau^{\mathbf{L}}_{n+m}-)} \sqrt{a^{j,k}(\tau^{\mathbf{L}}_{n+m-1}) - a^{j-1,k}(\tau^{\mathbf{L}}_{n+m-1})}\Upsilon_{j,k}(\tau^{\mathbf{L}}_{n+m}-) p_k(\tau^{\mathbf{L}}_{n+m}, t)$$

$$+ \sum_{j=1}^{S_l(\tau^{\mathbf{L}}_{n+m+1}-)} \sqrt{a^{j,k}(\tau^{\mathbf{L}}_{n+m}) - a^{j-1,k}(\tau^{\mathbf{L}}_{n+m})}\Upsilon_{j,k}(t).$$

Inductively, we have

$$\Psi_{l,k}(t) = \Psi_{l,k}(\tau^{\mathbf{L}}_{n}) p_k(\tau^{\mathbf{L}}_{n}, t) \tag{18}$$

$$+ \sum_{u=0}^{m} \sum_{j=1}^{S_l(\tau^{\mathbf{L}}_{n+m-u+1}-)} \sqrt{a^{j,k}(\tau^{\mathbf{L}}_{n+m-u}) - a^{j-1,k}(\tau^{\mathbf{L}}_{n+m-u})}$$

$$\times \Upsilon_{j,k}(\tau^{\mathbf{L}}_{n+m-u+1}-) p_k(\tau^{\mathbf{L}}_{n+m-u+1}, t)$$

$$+ \sum_{j=1}^{S_l(\tau^{\mathbf{L}}_{n+m+1}-)} \sqrt{a^{j,k}(\tau^{\mathbf{L}}_{n+m}) - a^{j-1,k}(\tau^{\mathbf{L}}_{n+m})}\Upsilon_{j,k}(t),$$

$$= \sum_{u=0}^{m} \sum_{j=1}^{S_l(\tau^{\mathbf{L}}_{n+m-u+1}-)} \sqrt{a^{j,k}(\tau^{\mathbf{L}}_{n+m-u}) - a^{j-1,k}(\tau^{\mathbf{L}}_{n+m-u})}$$

$$\times \Upsilon_{j,k}(\tau^{\mathbf{L}}_{n+m-u+1}-) p_k(\tau^{\mathbf{L}}_{n+m-u+1}, t)$$

$$+ \sum_{j=1}^{S_l(\tau^{\mathbf{L}}_{n+m+1}-)} \sqrt{a^{j,k}(\tau^{\mathbf{L}}_{n+m}) - a^{j-1,k}(\tau^{\mathbf{L}}_{n+m})}\Upsilon_{j,k}(t),$$

since $\Psi_{l,k}(\tau^{\mathbf{L}}_{n}) = 0$, by our assumption.

By the definition of q_k, we have $q_k(s,u)p_k(u,t) + q_k(u,t) = q_k(s,t)$ for $s \le u \le t$. Suppose that $s < t$, $\tau^j(t) = \tau^{\mathbf{L}}_n$, $\tau^l(s) = \tau^{\mathbf{L}}_{n+m}$, $\tau^{\mathbf{L}}(s) = \tau^{\mathbf{L}}_{n+m+v}$ and $\tau^{\mathbf{L}}(t) = \tau^{\mathbf{L}}_{n+m+v+z}$ for some $n, z \ge 0$, $m, v \ge 1$. In this situation, we have $S_j(\tau^{\mathbf{L}}_{n+m+v-u+1}-) < S_l(\tau^{\mathbf{L}}_{n+m+v-u+1}-)$, for $0 \le u \le v$. By using these results, (17) and (18), we have

$$E[\Psi_{j,k}(t)\Psi_{l,k}(s) \mid \mathscr{F}^{\mathbf{L}}]$$

$$= \sum_{u=0}^{v} \sum_{w=1}^{S_j(\tau^{\mathbf{L}}_{n+m+v-u+1}-)} \{a^{w,k}(\tau^{\mathbf{L}}_{n+m+v-u}) - a^{w-1,k}(\tau^{\mathbf{L}}_{n+m-u})\} p_k(\tau^{\mathbf{L}}_{n+m+v-u}, \tau^{\mathbf{L}}_{n+m+v-u+1})$$

$$\times q_k(\tau^{\mathbf{L}}_{n+m+v-u}, \tau^{\mathbf{L}}_{n+m+v-u+1}) p_k(\tau^{\mathbf{L}}_{n+m+v-u+1}, t) p_k(\tau^{\mathbf{L}}_{n+m+v-u+1}, s)$$

$$= g_k(\tau^j(t))p_k(\tau^k(t),\tau^l(s))\Bigg\{\sum_{u=0}^{v-1} p_k(\tau^l(s),\tau^{\mathbf{L}}_{n+m+v-u})p_k(\tau^{\mathbf{L}}_{n+m+v-u},\tau^{\mathbf{L}}_{n+m+v-u+1})$$

$$\times q_k(\tau^{\mathbf{L}}_{n+m+v-u},\tau^{\mathbf{L}}_{n+m+v-u+1})p_k(\tau^{\mathbf{L}}_{n+m+v-u+1},t)p_k(\tau^{\mathbf{L}}_{n+m+v-u+1},s)$$

$$+ p_k(\tau^l(s),\tau^{\mathbf{L}}_{n+m+v})p_k(\tau^{\mathbf{L}}_{n+m+v},\tau^{\mathbf{L}}_{n+m+v+1})q_k(\tau^{\mathbf{L}}_{n+m+v},s)p_k(\tau^{\mathbf{L}}_{n+m+v-u+1},t)\Bigg\}$$

$$= g_k(\tau^j(t))p_k(\tau^j(t),\tau^l(s))p_k(\tau^l(s),t)q_k(\tau^l(s),s).$$

Similarly, we suppose that $s \le t$ and $\tau^l(s) \le \tau^j(t) \le s$, then we have

$$E[\Psi_{j,k}(t)\Psi_{l,k}(s) \mid \mathscr{F}^{\mathbf{L}}] = \begin{cases} g_k(\tau^j(t))p_k(\tau^l(s),\tau^j(t))p_k(\tau^j(t),t)q_k(\tau^j(t),s) \\ \qquad \text{if } \tau^l(s) = \tau^j(t) = 0 \text{ and } g_k(\tau^j(t)) < g_k(\tau^l(s)), \\ g_k(\tau^l(s))p_k(\tau^l(s),\tau^j(t))p_k(\tau^j(t),t)q_k(\tau^j(t),s) \\ \qquad \text{otherwise.} \end{cases}$$

Hence we have done. □

Acknowledgment. The author would like to thank the anonymous referee for his/her careful reading of the paper.

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