S.-J. GUO AND S.-H. WANG
KODAI MATH. J.
36 (2013), 325–342

CROSSED PRODUCTS OF HOPF GROUP-COALGEBRAS

Shuang-jian Guo* and Shuan-hong Wang

Abstract

The main aim of this paper is to study Hopf group-crossed products and Hopf group-cleft extensions in the setting of Hopf group-coalgebras.

Introduction

In Hopf algebra theory, Hopf crossed products were introduced independently by Doi and Takeuchi [5] and Blattner, et al. [1] as a Hopf algebraic generalization of group crossed products. In particular, a Hopf crossed product is in fact always a Hopf cleft extension, provided the cocycle appeared in a Hopf crossed product is convolution invertible (see Blattner and Montgomery [2]).

Hopf group-algebras were introduced by Turaev in his work on homotopy quantum field theories (cf. Turaev [8]) as a generalization of ordinary Hopf algebras. It was proven in Caenepeel and De Lombaerde [3] that there exists a symmetric monoidal category, the so-called Turaev, in which the Hopf algebras are the same as Hopf group-coalgebras.

Apparently all notions that exist in classical and less classical Hopf algebra theory should have a group-version (see Virelizier [9], Wang [11, 12, 13], and Zunino [14, 15]). However, it is not easy to find a right way to do so because the notion of a Hopf group-coalgebra is not self-dual.

In this paper, it is studied that there exists an analogue of the crossed product for Hopf algebras in the setting of Hopf group-coalgebras. Furthermore, we can investigate group-cleft extensions and equivalences of group-crossed products.

The paper is organized as follows. In Section 1 the basic notions of group-coalgebras and Hopf group-coalgebras are recalled.

In Section 2, we introduce and study the notions of a group-crossed product and a group-cleft extension. In particular, we characterize group-crossed products by group-cleft extensions (see Theorem 2.10).

2010 *Mathematics Subject Classification*. 16T05; 16W30.

Key words and phrases. Hopf group-coalgebra; Hopf group-crossed product; Hopf group-cleft extensions; Twisted Drinfeld double.

*Corresponding author.

Received March 6, 2012; revised December 6, 2012.

In Section 3, we prove equivalences of group-crossed products for Hopf group-coalgebras (see Theorem 3.2). In Section 4, as an application of our new theory, we consider an example based on the Sweedler's 4-dimensional Hopf algebra (see Theorem 4.2, Theorem 4.3 and Theorem 4.6).

1. Preliminaries

Throughout this paper, we let π be a discrete group (with neutral element 1), k will be a fixed field, and the tensor product $\otimes = \otimes_k$ is always assumed to be over k. If U and V are k-vector spaces, $T_{U,V} : U \otimes V \to V \otimes U$ will denote the flip map defined by $T_{U,V}(u \otimes v) = v \otimes u$, for all $u \in U$ and $v \in V$.

π-coalgebra. Recall from Turaev [8] that a π-coalgebra is a family of k-spaces $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}$ (called a comultiplication) and a k-linear map $\varepsilon : C_1 \to k$ (called a counit), such that Δ is coassociative in the sense that,

$$\Delta_{\alpha,\beta} \otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}, \quad \forall \alpha, \beta, \gamma \in \pi$$

$$(id_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha})\Delta_{1,\alpha}, \quad \forall \alpha \in \pi.$$

We use the Sweedler's notation (see Virelizier [9]) for a comultiplication in the following way: for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, we write $\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}$.

Hopf π-coalgebra. Recall from Turaev [8] that a Hopf π-coalgebra is a π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)$ endowed with a family of k-linear maps $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ (called antipode) such that:
(1) each H_α is an algebra with multiplication m_α and unit element $1_\alpha \in H_\alpha$,
(2) $\varepsilon : H_1 \to k$ and $\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ are algebra maps, for all $\alpha, \beta \in \pi$,
(3) for each $\alpha \in \pi$, $m_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = 1_\alpha\varepsilon = m_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}$.

If a Hopf π-coalgebra H satisfies conditions (1) and (2), we call it a semi-Hopf π-coalgebra (see Wang [12]).

We also have the set of a π-group-like elements denoted by

$$G(C) = \left\{ c = (c_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} C_\alpha \mid \Delta_{\alpha,\beta}(c) = c_\alpha \otimes c_\beta, \varepsilon(c_1) = 1 \right\}.$$

Remark. (1) $(H_1; m_1; 1_1; \Delta_{1,1}; \varepsilon; S_1)$ is an ordinary Hopf algebra;
(2) The antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of H is said to be bijective if each S_α is bijective;
(3) The antipode of a Hopf π-coalgebra is anti-multiplicative and anti-comultiplicative, i.e., for all $\alpha, \beta \in \pi$, $a, b \in H_\alpha$

$$S_\alpha(ab) = S_\alpha(b)S_\alpha(a); \quad S_\alpha(1_\alpha) = 1_{\alpha^{-1}};$$

$$\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = T_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}; \quad \varepsilon S_1 = \varepsilon.$$

2. Group cleft extensions and existence of group crossed products

Definition 2.1. Let H be a Hopf π-coalgebra and A an algebra. We say that H acts weakly on A if there exists a family of maps: $H_\alpha \otimes A \to A$, $h \otimes a \mapsto h \rightharpoonup a$, $\forall \alpha \in \pi$, $h \in H_\alpha$, such that

(1) $1_\alpha \rightharpoonup a = a$, for any $a \in A$; $\alpha \in \pi$,

(2) $h \rightharpoonup (ab) = (h_{(1,\alpha)} \rightharpoonup a)(h_{(2,\beta)} \rightharpoonup b)$, for all $h \in H_{\alpha\beta}$, $a, b \in A$,

(3) $h \rightharpoonup 1_A = \varepsilon(h)1_A$, for every $h \in H_1$.

Furthermore, if A is an H_α module for each $\alpha \in \pi$ and satisfies (2) and (3), we call that A is a π-H-module algebra.

Let H be a Hopf π-coalgebra and A a family of algebras $A = \{A_\alpha, m_\alpha, 1_{A_\alpha}\}_{\alpha\in\pi}$. Let $\chi = \{\chi_\alpha : H_1 \otimes H_1 \to A_\alpha\}$ be a family of k-linear maps and that χ is an invertible map. Suppose that H acts weakly on each A_α with $\alpha \in \pi$. For any $\alpha \in \pi$, we define a multiplication on $A_\alpha \otimes H_\beta$ by

$$(a \otimes h)(b \otimes g) = a(h_{(1,1)} \rightharpoonup b)\chi_\alpha(h_{(2,1)}, g_{(1,1)}) \otimes h_{(3,\beta)}g_{(2,\beta)} \tag{2.1}$$

for all $a, b \in A_\alpha$ and $h, g \in H_\beta$ with $\beta \in \pi$.

Definition 2.2. For $\alpha \in \pi$, set $A_\alpha \#^\alpha_\chi H_\alpha = A_\alpha \otimes H_\alpha$ with the multiplication defined by Eq. (2.1). If the $A_\alpha \#^\alpha_\chi H_\alpha$ is an associative algebra with $1_{A_\alpha} \otimes 1_\alpha$ as identity element, we call the family of algebras $\{A_\alpha \#^\alpha_\chi H_\alpha\}_{\alpha\in\pi}$ as a Hopf π-crossed product and denote it by $A\#^\pi_\chi H$. And we call $(\rightharpoonup, \chi)$ the π-crossed system for $A\#^\pi_\chi H$. We denote $\{A_\alpha \#^\alpha_\chi H_1\}_{\alpha\in\pi}$ by $A\#^\pi_\chi H_1$ and denote $\{A_\alpha \#^\alpha_\chi H_\beta\}_{\alpha\in\pi}$ by $A\#^\pi_\chi H_\beta$ with some $\beta \in \pi$.

The proofs of the following two propositions are straightforward.

Proposition 2.3. *With the above notations. Then $A\#^\pi_\chi H$ is a Hopf π-crossed product if and only if the following conditions hold: for any $\alpha \in \pi$*

$$\chi_\alpha(1_1, h) = \varepsilon(h)1_{A_\alpha} = \chi_\alpha(h, 1_1), \tag{2.2}$$

$$(h_{(1,1)} \rightharpoonup (g_{(1,1)} \rightharpoonup a))\chi_\alpha(h_{(2,1)}, g_{(2,1)}) = \chi_\alpha(h_{(1,1)}, g_{(1,1)})(h_{(2,1)}g_{(2,1)} \rightharpoonup a), \tag{2.3}$$

$$\begin{aligned}\chi_\alpha(h_{(1,1)}, g_{(1,1)})\chi_\alpha(h_{(2,1)}g_{(2,1)}, k) \\ = (h_{(1,1)} \rightharpoonup \chi_\alpha(g_{(1,1)}, k_{(1,1)}))\chi_\alpha(h_{(2,1)}, g_{(2,1)}k_{(2,1)})\end{aligned} \tag{2.4}$$

for all $h, g, k \in H_1$ and $a \in A_\alpha$.

Proposition 2.4. *If $A\#^\pi_\chi H_1 = \{A_\alpha \#^\alpha_\chi H_1\}_{\alpha\in\pi}$ is a family of ordinary Hopf crossed algebras, then $A\#^\pi_\chi H$ is a Hopf π-crossed product and $A\#^\pi_\chi H_\beta = \{A_\alpha \#^\alpha_\chi H_\beta\}_{\alpha\in\pi}$ with some $\beta \in \pi$ is a family of associative algebras.*

Remark 2.5. (1) If we set $\pi = \{1\}$, then the Hopf π-crossed product is the ordinary Hopf crossed product.

(2) If we take $\chi_\alpha(h,l) = \varepsilon(h)\varepsilon(l)1_{A_\alpha}$ with $\alpha \in \pi$, then the Hopf π-crossed product becomes the Hopf π-smash product (see Wang [11]).

(3) For some $1 \neq \beta \in \pi$, if $A\#_\chi^\pi H_\beta = \{A_\alpha \#_\chi^\alpha H_\beta\}_{\alpha\in\pi}$ is a family of associative algebras, then $A\#_\chi^\pi H$ is not necessarily a Hopf π-crossed product.

A concrete counterexample is presented as follows.

Let π and G be two finite groups, and $\phi : G \to \pi$ be a group homomorphism. Then ϕ induces a Hopf algebras morphism $F(\pi) \to F(G)$, given by $f \mapsto f \circ \phi$, whose image is central. Here $F(G) = \mathbf{C}^G$ and $F(\pi) = \mathbf{C}^\pi$ (where $\mathbf{C}$ is a complex domain) denote the Hopf algebras of complex-valued functions on G and π respectively. By Virelizier [10] this data yields to a Hopf π-coalgebra $H^\phi = \{H_\alpha^\phi\}_{\alpha\in\pi}$. Denote by $(e_g)_{g\in G}$ the standard basis of $F(G)$ given by $e_g(h) = \delta_{g,h}$. Then for any $\alpha, \beta \in \pi$, we have that

$$H_\alpha^\phi = \sum_{g\in\phi^{-1}(\alpha)} \mathbf{C}e_g, \quad \mu_\alpha(e_g \otimes e_h) = \delta_{g,h}e_g \quad \text{for any } g,h \in \phi^{-1}(\alpha),$$

$$1_\alpha = \sum_{g\in\phi^{-1}(\alpha)} e_g, \quad \varepsilon(e_g) = \delta_{g,1} \quad \text{for any } g \in \phi^{-1}(1),$$

$$\Delta_{\alpha,\beta}(e_g) = \sum_{hk=g} e_h \otimes e_k \quad \text{for any } g \in \phi^{-1}(\alpha\beta),\ h \in \phi^{-1}(\alpha),\ k \in \phi^{-1}(\beta),$$

$$S_\alpha(e_g) = e_{g^{-1}} \quad \text{for any } g \in \phi^{-1}(\alpha).$$

Given any k-algebra A, we can endow A with the action of H on A is trivial, the multiplication on $A_\alpha \otimes H_\beta$ is

$$(a \otimes e_h)(b \otimes e_g) = \sum_{h=g} ab\chi_\alpha(e_l, e_m) \otimes e_h,$$

for any $l, m \in \phi^{-1}(1)$, $h, g \in \phi^{-1}(\beta)$ and $a, b \in A_\alpha$. $A\#_\chi^\pi H$ is a Hopf π-crossed product if and only if the following conditions hold: for any $\alpha \in \pi$

$$\chi_\alpha(1_1, e_m) = \varepsilon(e_m)1_{A_\alpha} = \chi_\alpha(e_m, 1_1),$$
$$a\chi_\alpha(e_p, e_n) = \chi_\alpha(e_p, e_n)a,$$
$$\chi_\alpha(e_p, e_m)\chi_\alpha(e_l, e_n) = \chi_\alpha(e_m, e_p)\chi_\alpha(e_l, e_n),$$

for any $l, m, p, n \in \phi^{-1}(1)$ and $a \in A_\alpha$.

For some $1 \neq \beta \in \pi$, if $A\#_\chi^\pi H_\beta = \{A_\alpha \#_\chi^\alpha H_\beta\}_{\alpha\in\pi}$ is a family of associative algebras, for any $h, g, k \in \phi^{-1}(\beta)$ and $a, b, c \in A_\alpha$, then

$$\begin{aligned}[(a \otimes e_h)(b \otimes e_g)](c \otimes e_k) &= \sum_{h=g}[ab\chi_\alpha(e_l, e_m) \otimes e_h](c \otimes e_k)\\ &= \sum_{h=g=k} ab\chi_\alpha(e_l, e_m)c\chi_\alpha(e_p, e_n) \otimes e_h,\end{aligned}$$

for any $l, m, p, n \in \phi^{-1}(1)$, and

$$(a \otimes e_h)[(b \otimes e_g)(c \otimes e_k)] = \sum_{g=k} (a \otimes e_h)[bc\chi_\alpha(e_p, e_n) \otimes e_g]$$
$$= \sum_{h=g=k} abc\chi_\alpha(e_l, e_m)\chi_\alpha(e_p, e_n) \otimes e_h,$$

for any $l, m, p, n \in \phi^{-1}(1)$. Then

$$\sum ab\chi_\alpha(e_l, e_m)c\chi_\alpha(e_p, e_n) \otimes e_h = \sum abc\chi_\alpha(e_l, e_m)\chi_\alpha(e_p, e_n) \otimes e_h.$$

We let $a = b = 1_{A_\alpha}$ and $1_1 = \sum_{n \in \phi^{-1}(1)} e_n$, then

$$\sum \chi_\alpha(e_l, e_m)c \otimes e_h = \sum c\chi_\alpha(e_l, e_m) \otimes e_h. \tag{2.5}$$

From Eq. (2.5), we can't obtain that $\chi_\alpha(e_l, e_m)c = c\chi_\alpha(e_l, e_m)$, only take $\beta = 1$ and apply $id \otimes \varepsilon$ to the Eq. (2.5), we get $\chi_\alpha(e_l, e_m)c = c\chi_\alpha(e_l, e_m)$ for any $l, m \in \phi^{-1}(1)$. Then $A\#^\pi_\chi H$ is not necessarily a Hopf π-crossed product.

Definition 2.6. Let B be a family of algebras $B = \{B_\alpha, m_\alpha, 1_{B_\alpha}\}_{\alpha\in\pi}$ and $A \subset B = \{A_\alpha \subset B_\alpha\}_{\alpha\in\pi}$.

(1) We say that $A \subset B = \{A_\alpha \subset B_\alpha\}_{\alpha\in\pi}$ is a π-H-extension if B is a right π-H-comodule algebra (see Wang [11, 12]) with a family of k-linear maps $\rho = \{\rho_{\beta,\alpha} : B_{\beta\alpha} \to B_\beta \otimes H_\alpha\}_{\alpha,\beta\in\pi}$ and $B^{coH} = A$, where

$$B^{coH} = \left\{ b = (b_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} B_\alpha \mid \rho_{\alpha,\beta}(b_{\alpha\beta}) = b_\alpha \otimes 1_\beta \in B_\alpha \otimes H_\beta, \alpha, \beta \in \pi \right\},$$

(called a π-subalgebras of right π-coinvariants).

(2) A π-H-extension $A \subset B = \{A_\alpha \subset B_\alpha\}_{\alpha\in\pi}$ is a π-H-cleft if there exist a family of right π-H-comodule maps $\gamma = \{\gamma_\alpha : H_\alpha \to B_\alpha\}_{\alpha\in\pi}$ such that γ is convolution-invertible in the sense that there exist a family of maps $\gamma^{-1} = \{\gamma^{-1}_{\alpha^{-1}} : H_{\alpha^{-1}} \to B_\alpha\}_{\alpha\in\pi}$ satisfying

$$\gamma_\alpha(h_{(1,\alpha)})\gamma^{-1}_{\alpha^{-1}}(h_{(2,\alpha^{-1})}) = \gamma^{-1}_{\alpha^{-1}}(h_{(1,\alpha^{-1})})\gamma_\alpha(h_{(2,\alpha)}) = \varepsilon(h)1_{B_\alpha}$$

for all $h \in H_1$ and $\alpha \in \pi$.

Lemma 2.7. *Let $A \subset B = \{A_\alpha \subset B_\alpha\}_{\alpha\in\pi}$ be a π-H-cleft with a right π-H-comodule structure map: $\rho = \{\rho_{\beta\alpha} : B_{\beta\alpha} \to B_\beta \otimes H_\alpha\}_{\alpha,\beta\in\pi}$ via $b \to b_{(0,\beta)} \otimes b_{(1,\alpha)}$ for $\alpha, \beta \in \pi$ and with a π-H-cleft structure map: $\gamma = \{\gamma_\alpha : H_\alpha \to B_\alpha\}_{\alpha\in\pi}$ such that $\gamma_\alpha(1_\alpha) = 1_{B_\alpha}$ with $\alpha \in \pi$. Then we have*

(L1) $\rho_{\beta\alpha} \circ \gamma^{-1}_{(\beta\alpha)^{-1}} = (\gamma^{-1}_{\beta^{-1}} \otimes S_{\alpha^{-1}}) \circ T \circ \Delta_{\alpha^{-1},\beta^{-1}}$,

(L2) $b_{(0,\alpha)}\gamma^{-1}_{\alpha^{-1}}(b_{(1,\alpha^{-1})}) \in A = B^{coH}$ *for any* $b \in B_1$.

Proof. First observe that since ρ is an algebra map, $\rho_{\beta\alpha}\circ\gamma_{(\beta\alpha)^{-1}}$ is the inverse of $\rho_{\beta\alpha}\circ\gamma_{\beta\alpha} = (\gamma_\beta\otimes id)\circ\Delta_{\beta,\alpha}$. Let $\theta = (\gamma_\beta^{-1}\otimes S_{\alpha^{-1}})\circ T\circ\Delta_{\alpha^{-1},\beta^{-1}}$, for all $h\in H_1$. Then

$$\begin{aligned}
&[(\rho_{\beta\alpha}\circ\gamma_{(\beta\alpha)})*\theta](h)\\
&= (\gamma_\beta\otimes id)\circ\Delta_{\beta,\alpha}(h_{(1,\beta\alpha)})((\gamma_\beta^{-1}\otimes S_{\alpha^{-1}})\circ T\circ\Delta_{\alpha^{-1},\beta^{-1}})h_{(2,(\beta\alpha)^{-1})}\\
&= (\gamma_\beta(h_{(1,\beta)})\otimes h_{(2,\alpha)})(\gamma_{\beta^{-1}}^{-1}(h_{(4,\beta^{-1})})\otimes S_{\alpha^{-1}}(h_{(3,\alpha^{-1})}))\\
&= \varepsilon(h)1_{B_\beta}\otimes 1_\alpha.
\end{aligned}$$

Thus θ is a right inverse of $\rho_{\beta\alpha}\circ\gamma_{\beta\alpha}$, and so $\theta = \rho_{\beta\alpha}\circ\gamma^{-1}_{(\beta\alpha)^{-1}}$ by uniqueness of inverse.

As for (L2), we compute

$$\begin{aligned}
\rho_{\beta\alpha}(b_{(0,\beta\alpha)}\gamma^{-1}_{(\beta\alpha)^{-1}}(b_{(1,(\beta\alpha)^{-1})})) &= \rho_{\beta\alpha}(b_{(0,\beta\alpha)})\rho_{\beta\alpha}\gamma^{-1}_{(\beta\alpha)^{-1}}(b_{(1,(\beta\alpha)^{-1})})\\
&= b_{(0,\beta)}\gamma^{-1}_{\beta^{-1}}(b_{(3,\beta^{-1})})\otimes b_{(1,\alpha)})S_{\alpha^{-1}}(b_{(2,\alpha^{-1})}))\\
&= b_{(0,\beta)}\gamma^{-1}_{\beta^{-1}}(b_{(1,\beta^{-1})})\otimes 1_\alpha.
\end{aligned}$$

This finishes the proof. □

PROPOSITION 2.8. *Let $A\subset B = \{A_\alpha\subset B_\alpha\}_{\alpha\in\pi}$ be a π-H-cleft via $\gamma = \{\gamma_\alpha : H_\alpha\to B_\alpha\}_{\alpha\in\pi}$ such that $\gamma_\alpha(1_\alpha) = 1_{B_\alpha}$ with $\alpha\in\pi$. Then there is a Hopf π-crossed product with a weakly action of H on A given by*

$$h\rightharpoonup a = \gamma_\alpha(h_{(1,\alpha)})a\gamma^{-1}_{\alpha^{-1}}(h_{(2,\alpha^{-1})}),\quad \textit{for all } a\in A_\alpha,\ h\in H_1$$

and a family of convolution-invertible maps $\chi = \{\chi_\alpha : H_1\otimes H_1\to A_\alpha\}_{\alpha\in\pi}$ given by

$$\chi_\alpha(h,k) = \gamma_\alpha(h_{(1,\alpha)})\gamma_\alpha(k_{(1,\alpha)})\gamma^{-1}_{\alpha^{-1}}(h_{(2,\alpha^{-1})}k_{(2,\alpha^{-1})}),\quad \textit{for all } h,k\in H_1.$$

Furthermore, there is an algebra isomorphism $\Phi_\alpha : A_\alpha\#^\alpha_\chi H_\alpha\to B_\alpha$ given by $a_\alpha\otimes h_\alpha\mapsto a_\alpha\gamma_\alpha(h_\alpha)$ with $\alpha\in\pi$ such that $\Phi = \{\Phi_\alpha\}_{\alpha\in\pi}$ is both a left π-A-module and right π-H-comodule map, where right π-H-comodule structure map of $A_{\alpha\beta}\#^{\alpha\beta}_\chi H_{\alpha\beta}$ is given by $a_{\alpha\beta}\#h_{\alpha\beta}\to a_{\alpha\beta}\#h_{(1,\alpha)}\otimes h_{(2,\beta)}$.

Proof. First we compute, for $a = (a_\alpha)_{\alpha\in\pi}\in\prod_{\alpha\in\pi}A_\alpha$, $h\in H_1$,

$$\begin{aligned}
\rho_{\beta\alpha}(h\rightharpoonup a_{\beta\alpha}) &= \rho_{\beta\alpha}(\gamma_{\beta\alpha}(h_{(1,\beta\alpha)}))a\gamma^{-1}_{(\beta\alpha)^{-1}}(h_{(2,(\beta\alpha)^{-1})}))\\
&= (\rho_{\beta\alpha}\circ\gamma_{\beta\alpha}(h_{(1,\beta\alpha)}))\rho_{\beta\alpha}(a_{\beta\alpha})(\rho_{\beta\alpha}\circ\gamma^{-1}_{(\beta\alpha)^{-1}}(h_{(2,(\beta\alpha)^{-1})}))\\
&= (\gamma_\beta(h_{(1,\beta)})\otimes h_{(2,\alpha)})(a_\beta\otimes 1_\alpha)(\gamma_\beta^{-1}(h_{(4,\beta^{-1})})\otimes S_{\alpha^{-1}}(h_{(3,\alpha^{-1})}))\\
&= h\rightharpoonup a_\beta\otimes 1_\alpha\in A_\beta\otimes H_\alpha
\end{aligned}$$

and thus $h\rightharpoonup a\in A = B^{coH}$. Furthermore it is easy to see that Definition 2.1 (2) and (3) hold.

Similarly we can prove that $\chi = \{\chi_\alpha\}_{\alpha\in\pi}$ has values in A. In fact, for all $h, k \in H_1$,

$$\begin{aligned}\rho_{\beta\alpha}(\chi_{\beta\alpha}(h,k)) &= \rho_{\beta\alpha}\gamma_{\beta\alpha}(h_{(1,\beta\alpha)})\rho_{\beta\alpha}(\gamma_{\beta\alpha}(k_{(1,\beta\alpha)})\rho_{\beta\alpha}\gamma^{-1}_{(\beta\alpha)^{-1}}(h_{(2,(\beta\alpha)^{-1})}k_{(2,(\beta\alpha)^{-1})})\\ &= (\gamma_\beta(h_{1,\beta}) \otimes h_{(2,\alpha)})(\gamma_\beta(k_{1,\beta}) \otimes k_{(2,\alpha)})(\gamma^{-1}_{\beta^{-1}}(h_{(4,\beta^{-1})}k_{(4,\beta^{-1})})\\ &\quad \otimes S_{\alpha^{-1}}(h_{(3,\alpha^{-1})}k_{(3,\alpha^{-1})})\\ &= \gamma_\beta(h_{(1,\beta)})\gamma_\beta(k_{(1,\beta)})\gamma^{-1}_{\beta^{-1}}(h_{(2,\beta^{-1})}k_{(2,\beta^{-1})}) \otimes 1_\alpha\\ &= \chi_\beta(h,k) \otimes 1_\alpha.\end{aligned}$$

Now, for $\alpha \in \pi$, we define

$$\Psi_\alpha : B_\alpha \to A_\alpha\#^\alpha_\chi H_\alpha \quad \text{by } b \mapsto b_{(0,\alpha)}\gamma^{-1}_{\alpha^{-1}}(b_{(1,\alpha^{-1})})\#b_{(2,\alpha)}.$$

It is easy to show that Ψ_α is the inverse of Φ_α with $\alpha \in \pi$. Furthermore, Φ is an algebra map:

$$\begin{aligned}\Phi(a\#h)\Phi(b\#k) &= a\gamma_\alpha(h)b\gamma_\alpha(k)\\ &= a\gamma_\alpha(h_{(1,\alpha)})b\gamma^{-1}_{\alpha^{-1}}(h_{(2,\alpha^{-1})})\gamma_\alpha(h_{(3,\alpha)})\gamma_\alpha(k_{(1,\alpha)})\\ &\quad \gamma^{-1}_{\alpha^{-1}}(h_{(4,\alpha^{-1})}k_{(2,\alpha^{-1})})\gamma_\alpha(h_{(5,\alpha)}k_{(3,\alpha)})\\ &= a(h_{(1,1)}\cdot b)\chi(h_{(2,1)},k_{(1,1)})\gamma_\alpha(h_{(3,\alpha)}k_{(2,\alpha)})\\ &= \Phi((a\#h)(b\#k))\end{aligned}$$

for any $h, k \in H_\alpha$ and $a, b \in A_\alpha$. Therefore, we have $\{B_\alpha \cong A_\alpha\#^\alpha_\chi H_\alpha\}_{\alpha\in\pi}$.

Finally, it is easy to check that $\Phi = \{\Phi_\alpha\}_{\alpha\in\pi}$ is a left π-A-module map and is a right π-H-comodule map. □

Proposition 2.9. *Let $A\#^\pi_\chi H = \{A_\alpha\#^\alpha_\chi H_\alpha\}_{\alpha\in\pi}$ be a Hopf π-crossed product and define $\gamma = \{\gamma_\alpha : H_\alpha \to A_\alpha\#^\alpha_\chi H_\alpha\}_{\alpha\in\pi}$ by $\gamma_\alpha(h) = 1_{A_\alpha}\#h$ for $\alpha \in \pi$. Then $\gamma = \{\gamma_\alpha\}_{\alpha\in\pi}$ is a family of convolution invertible with inverse*

$$\gamma^{-1}_{\alpha^{-1}}(h) = \chi^{-1}_\alpha(S(h_{(2,1)}), h_{(3,1)})\#S_{\alpha^{-1}}(h_{(1,\alpha^{-1})}).$$

In particular $A \subset A\#^\pi_\chi H = \{A_\alpha \subset A_\alpha\#^\alpha_\chi H_\alpha\}_{\alpha\in\pi}$ is π-H-cleft.

Proof. Let $\upsilon_{\alpha^{-1}}(h) = \chi^{-1}_\alpha(S(h_{(2,1)}), h_{(3,1)})\#S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})$. Then it is straightforward to verify that υ is a left inverse for γ, we have

$$\begin{aligned}\upsilon_{\alpha^{-1}}(h_{(1,\alpha^{-1})})\gamma_\alpha(h_{(2,\alpha)}) &= (\chi^{-1}_\alpha(S(h_{(2,1)}), h_{(3,1)})\#S_{\alpha^{-1}}(h_{(1,\alpha^{-1})}))(1_{A_\alpha}\#h_{(4,\alpha)})\\ &= \chi^{-1}_\alpha(S(h_{(3,1)}), h_{(4,1)})\chi_\alpha(S(h_{(2,1)}), h_{(5,1)})\#S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})h_{(6,\alpha)}\\ &= \varepsilon(S(h_{(2,1)}))\varepsilon(h_{(3,1)})\#S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})h_{(4,\alpha)}\\ &= \varepsilon(h)1_{A_\alpha}\#1_\alpha.\end{aligned}$$

To check that υ is a right inverse for γ is more complicated, by a computation similar to the above, we have

$$\gamma_\alpha(h_{(1,\alpha)})\upsilon_{\alpha^{-1}}(h_{(2,\alpha^{-1})}) = [h_{(1,1)} \rightharpoonup \chi_\alpha^{-1}(S(h_{(4,1)}), h_{(5,1)})]\chi_\alpha(h_{(2,1)}, S(h_{(3,1)}))\#1_\alpha \tag{2.6}$$

and hence that υ is a right inverse for γ if and only if

$$[h_{(1,1)} \rightharpoonup \chi_\alpha^{-1}(S(h_{(4,1)}), h_{(5,1)})]\chi_\alpha(h_{(2,1)}, S(h_{(3,1)})) = \varepsilon(h)1_{A_\alpha} \tag{2.7}$$

for all $h \in H_1$.

Since $\chi = \{\chi_\alpha : H_1 \otimes H_1 \to A_\alpha\}$ is invertible, Eq. (2.4) gives

$$h \rightharpoonup \chi_\alpha(g,k) = \chi_\alpha(h_{(1,1)}, g_{(1,1)})\chi_\alpha(h_{(2,1)}g_{(2,1)}, k_{(1,1)})\chi_\alpha^{-1}(h_{(3,1)}, g_{(3,1)}k_{(2,1)}) \tag{2.8}$$

for any $h, l, k \in H_1$. Letting $h \in H_1$ act on the identity

$$\chi_\alpha(g_{(1,1)}, k_{(1,1)})\chi_\alpha^{-1}(g_{(2,1)}, k_{(2,1)}) = \varepsilon(g)\varepsilon(k)1_{A_\alpha},$$

we have

$$[h_{(1,1)} \rightharpoonup \chi_\alpha(g_{(1,1)}, k_{(1,1)})][h_{(2,1)} \rightharpoonup \chi_\alpha^{-1}(g_{(2,1)}, k_{(2,1)})] = \varepsilon(h)\varepsilon(g)\varepsilon(k)1_{A_\alpha}. \tag{2.9}$$

Hence from Eq. (2.9) we obtain

$$h \rightharpoonup \chi_\alpha^{-1}(g,k) = \chi_\alpha(h_{(1,1)}, g_{(1,1)}k_{(1,1)})\chi_\alpha^{-1}(h_{(2,1)}g_{(2,1)}, k_{(2,1)})\chi_\alpha^{-1}(h_{(3,1)}, g_{(3,1)}). \tag{2.10}$$

We may now verify Eq. (2.8) using Eq. (2.11):

$$\begin{aligned}
&[h_{(1,1)} \rightharpoonup \chi_\alpha^{-1}(S(h_{(4,1)}), h_{(5,1)})]\chi_\alpha(h_{(2,1)}, S(h_{(3,1)})) \\
&\quad = \chi_\alpha(h_{(1,1)}, S(h_{(8,1)})h_{(9,1)})\chi_\alpha^{-1}(h_{(2,1)}S(h_{(7,1)}), h_{(10,1)}) \\
&\qquad \chi_\alpha^{-1}(h_{(3,1)}, S(h_{(6,1)}))\chi_\alpha(h_{(4,1)}, S(h_{(5,1)})) \\
&\quad = \chi_\alpha(h_{(1,1)}, S(h_{(6,1)})h_{(7,1)})\chi_\alpha^{-1}(h_{(2,1)}S(h_{(5,1)}), h_{(8,1)})\varepsilon(h_{(3,1)})\varepsilon(h_{(4,1)}) \\
&\quad = \chi_\alpha(h_{(1,1)}, S(h_{(4,1)})h_{(5,1)})\chi_\alpha^{-1}(h_{(2,1)}S(h_{(3,1)}), h_{(6,1)}) \\
&\quad = \varepsilon(h)1_{A_\alpha}.
\end{aligned}$$

By Proposition 2.8 and Proposition 2.9, we easily get the main result of this section as follows.

THEOREM 2.10. *With the above notations, a π-H-extension $A \subset B = \{A_\alpha \subset B_\alpha\}_{\alpha\in\pi}$ is a π-H-cleft if and only if $\{B_\alpha \cong A_\alpha\#_\chi^\alpha H_\alpha\}_{\alpha\in\pi}$.*

3. Equivalences of π-crossed products

In this section, we will show that an analogue of the result in Doi ([4], Lemma 2.1) still holds for the setting of Hopf group-coalgebras.

Let H be a Hopf π-coalgebra and A a family of algebras $A = \{A_\alpha, m_\alpha, 1_{A_\alpha}\}_{\alpha\in\pi}$ over k, and $\gamma = \{\gamma_\alpha : H_\alpha \to A_\alpha\}_{\alpha\in\pi}$ a family of convolution-invertible linear maps. Define $\chi^{\gamma_\alpha} = \{\chi_\alpha^{\gamma_\alpha} : H_1 \otimes H_1 \to A_\alpha\}_{\alpha\in\pi}$ and weakly action of H on A by

$$\chi^{\gamma_\alpha}(h,g) = \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup \gamma_\alpha(g_{(1,\alpha)}))\chi(h_{(3,1)}, g_{(2,1)})\gamma_{\alpha^{-1}}^{-1}(h_{(4,\alpha^{-1})}g_{(3,\alpha^{-1})})$$

and

$$h \rightharpoonup^{\gamma_\alpha} a = \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup a)\gamma_{\alpha^{-1}}^{-1}(h_{(3,\alpha^{-1})})$$

for any $h, g \in H_1$ and $a \in A_\alpha$ with $\alpha \in \pi$.

Lemma 3.1. *Let $A\#_\chi^\alpha H = \{A_\alpha \#_\chi^\alpha H_\alpha\}_{\alpha\in\pi}$ be a Hopf π-crossed algebra. Then $\chi^{\gamma_\alpha\mu_\beta} = (\chi^{\mu_\beta})^{\gamma_\alpha}$ and $\rightharpoonup^{\gamma_\alpha\mu_\beta} = (\rightharpoonup^{\mu_\beta})^{\gamma_\alpha}$ where $\gamma = \{\gamma_\alpha : H_\alpha \to A_\alpha\}_{\alpha\in\pi}$ and $\mu = \{\mu_\beta : H_\beta \to A_\beta\}_{\beta\in\pi}$ are a family of convolution-invertible linear maps.*

Proof. For any $h, g \in H_1$, $a \in A_{\alpha\beta}$ with $\alpha, \beta \in \pi$, we have

$$\begin{aligned}
\chi^{\gamma_\alpha\mu_\beta}(h,g) &= ((\gamma_\alpha\mu_\beta)_{\alpha\beta}(h_{(1,\alpha\beta)}))(h_{(2,1)} \rightharpoonup^{\gamma_\alpha\mu_\beta} (g_{(1,\alpha\beta)}))\\
&\quad \chi(h_{(3,1)}, g_{(2,1)})(\gamma_\alpha\mu_\beta)^{-1}_{(\alpha\beta)^{-1}}(h_{(4,(\alpha\beta)^{-1})}g_{(3,(\alpha\beta)^{-1})})\\
&= \gamma_\alpha(h_{(1,\alpha)})\mu_\beta(h_{(2,\beta)})(h_{(3,1)} \rightharpoonup \gamma_\alpha(g_{(1,\alpha)}))\\
&\quad (h_{(4,1)} \rightharpoonup \mu_\beta(g_{(2,\beta)})\chi(h_{(5,1)}, g_{(3,1)})\mu_{\beta^{-1}}^{-1}(h_{(6,\beta^{-1})}g_{(4,\beta^{-1})})\gamma_{\alpha^{-1}}^{-1}(h_{(7,\alpha^{-1})}g_{(5,\alpha^{-1})})\\
&= \gamma_\alpha(h_{(1,\alpha)})\mu_\beta(h_{(2,\beta)})(h_{(3,1)} \rightharpoonup \gamma_\alpha(g_{(1,\alpha)}))\mu_{\beta^{-1}}^{-1}(h_{(4,\beta^{-1})}\mu_\beta(h_{(5,\beta)})\\
&\quad (h_{(6,1)} \rightharpoonup \mu_\beta(g_{(2,\beta)})\chi(h_{(7,1)}, g_{(3,1)})\mu_{\beta^{-1}}^{-1}(h_{(8,\beta^{-1})}g_{(4,\beta^{-1})})\gamma_{\alpha^{-1}}^{-1}(h_{(9,\alpha^{-1})}g_{(5,\alpha^{-1})})\\
&= \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup^{\mu_\beta} \gamma_\alpha(g_{(1,\alpha)}))\chi^{\mu_\beta}(h_{(3,1)}, g_{(2,1)})\gamma_{\alpha^{-1}}^{-1}(h_{(4,\alpha^{-1})}g_{(3,\alpha^{-1})})\\
&= (\chi^{\mu_\beta})^{\gamma_\alpha}(h,g)
\end{aligned}$$

and thus $\chi^{\gamma_\alpha\mu_\beta} = (\chi^{\mu_\beta})^{\gamma_\alpha}$.

Also,

$$\begin{aligned}
h(\rightharpoonup^{\mu_\beta})^{\gamma_\alpha} a &= \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup^{\mu_\beta} a)\gamma_{\alpha^{-1}}^{-1}(h_{(3,\alpha^{-1})})\\
&= \gamma_\alpha(h_{(1,\alpha)})\mu_\beta(h_{(2,1)(1,\beta)})(h_{(2,1)(2,1)} \rightharpoonup a)\mu_{\beta^{-1}}^{-1}(h_{(2,1)(3,\beta^{-1})})\gamma_{\alpha^{-1}}^{-1}(h_{(3,\alpha^{-1})})\\
&= \gamma_\alpha(h_{(1,\alpha)})\mu_\beta(h_{(2,\beta)}))(h_{(3,1)} \rightharpoonup a)\mu_{\beta^{-1}}^{-1}(h_{(4,\beta^{-1})})\gamma_{\alpha^{-1}}^{-1}(h_{(5,\alpha^{-1})})\\
&= h \rightharpoonup^{\gamma_\alpha\mu_\beta} a
\end{aligned}$$

and so $\rightharpoonup^{\gamma_\alpha\mu_\beta} = (\rightharpoonup^{\mu_\beta})^{\gamma_\alpha}$.

This completes the proof. □

Theorem 3.2. *Let H be a Hopf π-coalgebra and A a family of algebras $A = \{A_\alpha, m_\alpha, 1_{A_\alpha}\}_{\alpha\in\pi}$, and $\gamma = \{\gamma_\alpha : H_\alpha \to A_\alpha\}_{\alpha\in\pi}$ a family of convolution-invertible* *linear maps,* $\chi = \{\chi_\alpha : H_1 \otimes H_1 \to A_\alpha\}_{\alpha\in\pi}$ *is a family of k-linear maps. With the above notations* χ^{γ_α} *for any* $\alpha, \beta \in \pi$. *Then we have the following assertions*:

(1) *As algebras,* $A\#_{\chi}^{\alpha\beta}H \cong A\#_{\chi^{\gamma_\alpha}}^{\beta}H$;

(2) χ *satisfies Eq.* (2.2) *if and only if* χ^γ *satisfies Eq.* (2.2);

(3) $(\chi, \rightharpoonup)$ *satisfies Eq.* (2.3) *if and only if* $(\chi^\gamma, \rightharpoonup^\gamma)$ *satisfies Eq.* (2.3);

(4) *If* $(\chi, \rightharpoonup)$ *satisfies Eq.* (2.3), *then* $(\chi, \rightharpoonup)$ *satisfies Eq.* (2.4) *if and only if* $(\chi^\gamma, \rightharpoonup^\gamma)$ *satisfies Eq.* (2.4);

(5) $A\#_{\chi}^{\alpha\beta}H$ *is a Hopf* π*-crossed algebra if and only if* $A\#_{\chi^{\gamma_\alpha}}^{\beta}H$ *is a Hopf* π*-crossed algebras, and they are isomorphic.*

Proof. (1) Define $\Phi : A_\alpha\#_{\chi^{\gamma_\alpha}}^{\alpha\beta}H_{\alpha\beta} \mapsto A_\alpha\#_{\chi}^{\beta}H_\beta$ by $a\otimes h \to a\gamma_\alpha(h_{(1,\alpha)}) \otimes h_{(2,\beta)}$, For $a, b \in A_\alpha, h, g \in H_{\alpha\beta}$

$$\begin{aligned}
&\Phi((a\otimes h)(b\otimes g))\\
&\quad = \Phi(a(h_{(1,1)} \rightharpoonup^\gamma b)\chi^{\gamma_\alpha}(h_{(2,1)}, g_{(1,1)}) \otimes h_{(3,\alpha\beta)}g_{(2,\alpha\beta)})\\
&\quad = a(h_{(1,1)} \rightharpoonup^{\gamma_\alpha} b)\chi^{\gamma_\alpha}(h_{(2,1)}, g_{(1,1)})\gamma_\alpha(h_{(3,\alpha)}g_{(2,\alpha)}) \otimes h_{(4,\beta)}g_{(3,\beta)})\\
&\quad = a\gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup b)\gamma_{\alpha^{-1}}^{-1}h_{(3,\alpha^{-1})}\gamma_\alpha(h_{(4,\alpha)})h_{(5,1)} \rightharpoonup \chi(h_{(6,1)}, g_{(2,1)})\\
&\qquad \gamma_{\alpha^{-1}}^{-1}(h_{(7,\alpha^{-1})}g_{(3,\alpha^{-1})})\gamma_\alpha(h_{(8,\alpha)}g_{(4,\alpha)}) \otimes h_{(9,\beta)}g_{(5,\beta)}\\
&\quad = a\gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup b)(h_{(3,1)} \rightharpoonup \gamma_\alpha(g_{(1,\alpha)})\chi(h_{(4,1)}, g_{(2,1)}) \otimes h_{(5,\beta)}g_{(3,\beta)}\\
&\quad = \Phi(a\otimes h)\Phi(b\otimes g).
\end{aligned}$$

Clearly Φ is bijective, $\Phi^{-1}(a\otimes h) = \sum a\gamma_{\alpha^{-1}}^{-1}(h_{(1,\alpha^{-1})}) \otimes h_{(2,\alpha\beta)}$ $a, b \in A_\alpha$, $h \in H_\beta$, since

$$\begin{aligned}
\Phi\Psi(a\otimes h) &= \Phi(a\gamma_{\alpha^{-1}}^{-1}(h_{(1,\alpha^{-1})}) \otimes h_{(2,\alpha\beta)})\\
&= a\gamma_{\alpha^{-1}}^{-1}(h_{(1,\alpha^{-1})})\gamma_\alpha(h_{(2,\alpha\beta)(1,\alpha)}) \otimes h_{(2,\alpha\beta)(2,\beta)}\\
&= a\gamma_{\alpha^{-1}}^{-1}(h_{(1,1)(1,\alpha^{-1})})\gamma_\alpha(h_{(1,1)(2,\alpha)}) \otimes h_{(2,\beta)}\\
&= a\otimes h.
\end{aligned}$$

(2) Straightforward.

(3) If $(\chi, \rightharpoonup)$ satisfies Eq. (2.2), then

$$\begin{aligned}
&(h_{(1,1)} \rightharpoonup^{\gamma_\alpha} (g_{(1,1)} \rightharpoonup^{\gamma_\alpha} (a)))\chi^{\gamma_\alpha}(h_{(2,1)}, g_{(2,1)})\\
&\quad = \gamma_\alpha(h_{(1,1)(1,\alpha)})(h_{(1,1)(2,1)} \rightharpoonup (\gamma_\alpha(g_{(1,1)(1,\alpha)}))(g_{(1,1)(2,1)} \rightharpoonup a)\\
&\qquad \gamma_{\alpha^{-1}}^{-1}(g_{(1,1)(3,\alpha^{-1})})\gamma_{\alpha^{-1}}^{-1}(h_{(2,1)(3,\alpha^{-1})})\gamma_\alpha(h_{(2,1)(1,\alpha)})(h_{(2,1)(2,1)} \rightharpoonup \gamma_\alpha(g_{(2,1)(1,\alpha)})\\
&\qquad \chi(h_{(2,1)(3,1)}, g_{(2,1)(2,1)})\gamma_{\alpha^{-1}}^{-1}(h_{(2,1)(4,\alpha^{-1})}g_{(2,1)(3,\alpha^{-1})})\\
&\quad = \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup (\gamma_\alpha(g_{(1,\alpha)})))(g_{(2,1)} \rightharpoonup a)\gamma_{\alpha^{-1}}^{-1}(g_{(3,\alpha^{-1})})\gamma_{\alpha^{-1}}^{-1}(h_{(3,\alpha^{-1})})\\
&\qquad \gamma_\alpha(h_{(4,\alpha)})(h_{(5,1)} \rightharpoonup \gamma_\alpha(g_{(4,\alpha)})\chi(h_{(6,1)}, g_{(5,1)})\gamma_{\alpha^{-1}}^{-1}(h_{(7,\alpha^{-1})}g_{(6,\alpha^{-1})})
\end{aligned}$$

$$= \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup \gamma_\alpha(g_{(1,\alpha)}))(h_{(3,1)} \rightharpoonup (g_{(2,1)} \rightharpoonup a))$$
$$\chi(h_{(4,1)}, g_{(3,1)})\gamma^{-1}_{\alpha^{-1}}(h_{(5,\alpha^{-1})}g_{(4,\alpha^{-1})})$$
$$= \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup \gamma_\alpha(g_{(1,\alpha)}))\chi(h_{(3,1)}, g_{(2,1)})(h_{(4,1)}g_{(3,1)} \rightharpoonup a)$$
$$\gamma^{-1}_{\alpha^{-1}}(h_{(5,\alpha^{-1})}g_{(4,\alpha^{-1})})$$
$$= \chi^{\gamma_\alpha}(h_{(1,1)}, g_{(1,1)})(h_{(2,1)}g_{(2,1)} \rightharpoonup^{\gamma_\alpha} a).$$

Conversely, we get it from Lemma 3.1.

(4) If $(\chi, \rightharpoonup)$ satisfies Eq. (2.3) and Eq. (2.4), then for $h, g, m \in H_1$

$$(h_{(1,1)} \rightharpoonup^{\gamma_\alpha} \chi^{\gamma_\alpha}(g_{(1,1)}, m_{(1,1)}))\chi^{\gamma_\alpha}(h_{(2,1)}, g_{(2,1)}m_{(2,1)})$$
$$= \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup [(\gamma_\alpha g_{(1,\alpha)}(g_{(2,1)} \rightharpoonup^{\gamma_\alpha} (m_{(1,\alpha)})))\chi(g_{(3,1)}, m_{(2,1)})$$
$$\gamma^{-1}_{\alpha^{-1}}(g_{(4,\alpha^{-1})}m_{(3,\alpha^{-1})})])\gamma^{-1}_{\alpha^{-1}}(h_{(3,\alpha^{-1})})\gamma_\alpha(h_{(4,\alpha)})(h_{(5,1)} \rightharpoonup \gamma_\alpha(g_{(5,\alpha)}m_{(4,\alpha)}))$$
$$\chi(h_{(6,1)}, g_{(6,1)}m_{(5,1)})\gamma^{-1}_{\alpha^{-1}}(h_{(7,\alpha^{-1})}g_{(7,\alpha^{-1})}m_{(6,\alpha^{-1})})$$
$$= \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup \gamma_\alpha(g_{(1,\alpha)}))(h_{(3,1)} \rightharpoonup (g_{(2,1)} \rightharpoonup \gamma_\alpha(m_{(1,\alpha)})))$$
$$(h_{(4,1)} \rightharpoonup \chi(g_{(3,1)}m_{(2,1)}))\chi(h_{(5,1)}, g_{(4,1)}m_{(3,1)})\gamma^{-1}_{\alpha^{-1}}(h_{(6,\alpha^{-1})}g_{(5,\alpha^{-1})}m_{(4,\alpha^{-1})})$$
$$= \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup \gamma_\alpha(g_{(1,\alpha)}))(h_{(3,1)} \rightharpoonup (g_{(2,1)} \rightharpoonup \gamma_\alpha(m_{(1,\alpha)})))$$
$$\chi(h_{(4,1)}, g_{(3,1)})\chi(h_{(5,1)}, g_{(4,1)}m_{(2,1)})\gamma^{-1}_{\alpha^{-1}}(h_{(6,\alpha^{-1})}g_{(5,\alpha^{-1})}m_{(4,\alpha^{-1})})$$
$$= \gamma_\alpha(h_{(1,\alpha)})(h_{(2,1)} \rightharpoonup \gamma_\alpha(g_{(1,\alpha)}))\chi(h_{(3,1)}g_{(2,1)})(h_{(4,1)}g_{(3,1)} \rightharpoonup \gamma_\alpha(m_{(1,\alpha)}))$$
$$\chi(h_{(5,1)}g_{(4,1)}, m_{(2,1)})\gamma^{-1}_{\alpha^{-1}}(h_{(6,\alpha^{-1})}g_{(5,\alpha^{-1})}m_{(4,\alpha^{-1})})$$
$$= \chi^{\gamma_\alpha}(h_{(1,1)}, g_{(1,1)})\chi(h_{(2,1)}g_{(2,1)}, m).$$

Conversely, we get it from Lemma 3.1.

(5) Clearly. □

4. An example

In this section we will give applications of our theory. Recall that the definition of H_4. As a k-algebra, H_4 is generated by two symbols X and Y which satisfies the relations $X^2 = 1$, $Y^2 = 0$ and $XY + YX = 0$. The coalgebra structure on H_4 is determined by

$$\Delta(X) = X \otimes X, \quad \text{and} \quad \Delta(Y) = 1 \otimes Y + Y \otimes X.$$

Consequently, H_4 has the basis 1 (identity), $X, Y, Z = XY$, and $\Delta(Z) = X \otimes Z + Z \otimes 1$. The antipode of H_4 is given by $S(X) = X$, $S(Y) = Z$, $S(Z) = -Y$.

We now consider the dual H_4^* of H_4. We have $H_4 \cong H_4^*$ (as Hopf algebras) via

$$1 \mapsto 1^* + X^*, \quad X \mapsto 1^* + X^*, \quad Y \mapsto Y^* + Z^*, \quad Z \mapsto Y^* - Z^*,$$

here $\{I^*, X^*, Y^*, Z^*\}$ denote the dual basis of $\{1, X, Y, Z\}$, then we let $T = 1^* + X^*$, $P' = Y^* + Z^*$, $Q' = Y^* - Z^*$, we get another basis $\{1, T, P', Q' = TP'\}$ of H_4^*.

In what follows, let $\pi = \mathbf{C}^*$.

For $\alpha \in \pi$, define two group homomorphisms $\Phi : \pi \to Aut(H_4)$ and $\Phi^* : \pi \to Aut(H_4^*)$ as

$$\Phi_\alpha(1) = 1, \quad \Phi_\alpha(X) = X, \quad \Phi_\alpha(Y) = \alpha Y, \quad \Phi_\alpha(Z) = \alpha Z,$$
$$\Phi_\alpha^*(1) = 1, \quad \Phi_\alpha^*(T) = T, \quad \Phi_\alpha^*(P') = \alpha^{-1} P', \quad \Phi_\alpha^*(Q') = \alpha^{-1} Q'.$$

Now, it follows from Wang [13] that, we have the Turaev $\delta(\pi)$-coalgebra

$$D(H_4, H_4^*, \langle\, , \rangle; \Phi, \Phi^*) = \{D(H_4, H_4^*, \langle\, , \rangle; \Phi, \Phi^*, (\alpha, \beta))\}_{(\alpha, \beta) \in \delta(\pi)},$$

which is denoted by $\mathscr{D}(\theta)$. Then the Turaev $\delta(\pi)$-coalgebra structure on $\mathscr{D}(\theta)$ is given, for all $\alpha, \beta, \lambda, \gamma \in \pi$, by the following (a)–(d):

(a) The multiplication relations:

$$X^2 = 1, \quad Y^2 = 0, \quad XY + YX = 0, \quad T^2 = 1, \quad P'^2 = 0, \quad TP' + P'T = 0,$$
$$XT = TX, \quad XP' + P'X = 0, \quad TY + YT = 0, \quad YP' = P'Y.$$

(b) The $\delta(\pi)$-comultiplication and counit relations:

$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(X) = X \otimes X, \qquad \varepsilon(X) = 1;$$
$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(Y) = Y \otimes 1 + X \otimes Y, \qquad \varepsilon(Y) = 0;$$
$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(Z) = Z \otimes X + 1 \otimes Z, \qquad \varepsilon(Z) = 0;$$
$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(T) = T \otimes T, \qquad \varepsilon(T) = 1;$$
$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(P) = \gamma P' \otimes T + 1 \otimes \alpha P', \qquad \varepsilon(P') = 0;$$
$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(Q') = \gamma Q' \otimes 1 + T \otimes \alpha Q', \qquad \varepsilon(Q') = 0.$$

(c) The antipode relations:

$$S_{(\alpha,\beta)}(X) = X, \quad S_{(\alpha,\beta)}(Y) = -Z, \quad S_{(\alpha,\beta)}(Z) = Y,$$
$$S_{(\alpha,\beta)}(T) = T, \quad S_{(\alpha,\beta)}(P') = \alpha\beta Q', \quad S_{(\alpha,\beta)}(Q') = -\alpha\beta P'.$$

(d) The crossing relations:

$$\varphi_{(\alpha,\beta)}^{(\lambda,\gamma)}(X) = X, \quad \varphi_{(\alpha,\beta)}^{(\lambda,\gamma)}(Y) = \alpha\beta^{-1} Y, \quad \varphi_{(\alpha,\beta)}^{(\lambda,\gamma)}(Z) = \alpha\beta^{-1} Z,$$
$$\varphi_{(\alpha,\beta)}^{(\lambda,\gamma)}(T) = T, \quad \varphi_{(\alpha,\beta)}^{(\lambda,\gamma)}(P') = \alpha^{-1}\beta^{-1} P', \quad \varphi_{(\alpha,\beta)}^{(\lambda,\gamma)}(Q') = \alpha^{-1}\beta Q'.$$

Furthermore, π is a subgroup of $\delta(\pi)$ and $(\alpha,\beta)^{-1} = (\beta^{-1}\alpha^{-1}\beta,\beta^{-1})$, then, for all $X, Y, Z, T, P, Q \in D_{(1,1)}$

$$\Delta_{(\alpha,\beta),(\alpha,\beta)^{-1}}(X) = X \otimes X, \qquad \varepsilon(X) = 1;$$
$$\Delta_{(\alpha,\beta),(\alpha,\beta)^{-1}}(Y) = Y \otimes 1 + X \otimes Y, \qquad \varepsilon(Y) = 0;$$
$$\Delta_{(\alpha,\beta),(\alpha,\beta)^{-1}}(Z) = Z \otimes X + 1 \otimes Z, \qquad \varepsilon(Z) = 0;$$
$$\Delta_{(\alpha,\beta),(\alpha,\beta)^{-1}}(T) = T \otimes T, \qquad \varepsilon(T) = 1;$$
$$\Delta_{(\alpha,\beta),(\alpha,\beta)^{-1}}(P) = \beta^{-1}P' \otimes T + 1 \otimes \alpha P', \qquad \varepsilon(P') = 0;$$
$$\Delta_{(\alpha,\beta),(\alpha,\beta)^{-1}}(Q') = \beta^{-1}Q' \otimes 1 + T \otimes \alpha Q', \qquad \varepsilon(Q') = 0.$$

We assume $\alpha = \beta^{-1}$ and $P = \alpha P'$, then $Q = \alpha Q'$.

Lemma 4.1. *Let V be any π-coalgebra and W an algebra, let $f : V \to W$ be a convolution-invertible linear map. Assume that $a \in P_{gh}(V) = \{a = (a_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} V_\alpha \mid \Delta(a) = g_\alpha \otimes a_\beta + a_\alpha \otimes h_\beta\}$ where $g, h \in G(V)$. Then $f(g)$ and $f(h)$ is in $U(W)$, the set of units of W, and $f^{-1}(a) = -f(g)^{-1}f(a)f(h)^{-1}$, here f^{-1} denotes the convolution-inverse of f.*

Proof. Straightforward. □

Theorem 4.2. *Let $C \subset B$ be any π-$\mathscr{D}(\theta)$-extension. Then B is a π-$\mathscr{D}(\theta)$-cleft if and only if there exist elements x, y, t, p in B where $x^2 = 1$, $xy + yx = 0$ with $x, t \in U(B)$ such that*

$$\rho(x) = x \otimes X, \quad \rho(y) = x \otimes Y + y \otimes 1;$$
$$\rho(t) = t \otimes T, \quad \rho(p) = p \otimes T + 1 \otimes P.$$

If this is the case, we have

(1) *The map $\phi : D_{(\alpha,\beta)} \to B$ defined by*

$$\phi(1) = 1, \quad \phi(X) = x, \quad \phi(Y) = y, \quad \phi(Z) = xy,$$
$$\phi(T) = t, \quad \phi(P) = p, \quad \phi(Q) = tp,$$

is a section. The inverse ϕ^{-1} is given by

$$\phi^{-1}(X) = x^{-1}, \quad \phi^{-1}(Y) = -x^{-1}y, \quad \phi^{-1}(Z) = y,$$
$$\phi^{-1}(1) = 1, \quad \phi^{-1}(T) = t^{-1}, \quad \phi^{-1}(P) = -pt^{-1}, \quad \phi^{-1}(Q) = -p.$$

(2) *B is a free left C-module with basis $\{1, x, y, z = xy, t, p, q = tp\}$.*

(3) *If we let $s = y^2$, $w = t^2$, $u = p^2$, $v = tp + pt$, then $w \in U(C)$, $s, u, v \in C$.*

(4) *The π-crossed system corresponding to ϕ as in the part (1) is given by the following: $X, Y, Z, T, P, Q \in D_{(1,1)}$*

$$\begin{cases} X \rightharpoonup c = xcx^{-1} = F(c), \quad Y \rightharpoonup c = x[x^{-1}y, c] = D(c), \\ Z \rightharpoonup c = [xy, c]x^{-1} = -FD(c), \quad T \rightharpoonup c = tct^{-1} = G(c), \\ P \rightharpoonup c = [p, c]t^{-1} = E(c), \quad Q \rightharpoonup c = t[p, c] = GE(c)w, \end{cases}$$

$$\begin{cases} \chi(1,1) = \chi(1,X) = \chi(X,1) = \chi(X,X) = 1, \\ \chi(1,Y) = \chi(1,Z) = \chi(X,Y) = \chi(X,Z) = 0, \\ \chi(Y,1) = \chi(Y,X) = \chi(Z,1) = \chi(Z,X) = 0, \\ \chi(Y,Y) = s = -\chi(Z,Z), \quad \chi(Z,Y) = F(s) = -\chi(Y,Z) \end{cases}$$

and

$$\begin{cases} \chi(1,1) = \chi(1,T) = \chi(T,1) = 1, \quad \chi(T,T) = \alpha, \\ \chi(1,P) = \chi(1,Q) = \chi(T,P) = \chi(T,Q) = 0, \\ \chi(P,1) = \chi(Q,1) = 0, \quad \chi(P,T) = v, \quad \chi(Q,T) = G(v), \\ \chi(Q,P) = G(u) = -\chi(P,Q), \quad \chi(P,P) = u, \quad \chi(Q,Q) = -wu. \end{cases}$$

Proof. (1) Assume $C \subset B$ be any π-$\mathscr{D}(\theta)$-extension. Choose a section $\psi : D_{(w^{-1},u^{-1})} \to B$. Set $x = \psi(X)$, $t = \psi(T)$, $y = \psi(Y)$ and $p = \psi(P)$. Then clearly $x, t \in U(B)$ and $\rho(x) = x \otimes X$, $\rho(y) = x \otimes Y + y \otimes 1$, $\rho(t) = t \otimes T$, $\rho(p) = p \otimes T + 1 \otimes P$. Conversely, from such elements x, y, p, t in B, define a new map $\phi : D_{(\alpha,\beta)} \to B$ by $\phi(1) = 1$, $\phi(X) = x$, $\phi(Y) = y$, $\phi(Z) = xy$, $\phi(T) = t$, $\phi(P) = p$, $\phi(Q) = tp$, Then ϕ is also a section, since we have

$$\begin{aligned} \rho\phi(Z) &= \rho(xy) = \rho(x)\rho(y) = (x \otimes X)(y \otimes 1 + x \otimes Y) \\ &= xy \otimes X + 1 \otimes Z = (\phi \otimes 1)\Delta(Z). \end{aligned}$$

The inverse ϕ^{-1} is immediately obtained from Lemma 4.1.

(2) Follows from the normal basis property for π-cleft extensions.

(3) Since $\rho(w) = \rho(t^2) = (t \otimes T)^2 = t^2 \otimes 1 = w \otimes 1$, we have $w \in C$, The fact that w is in $U(C)$ follows from $x \in U(B)$ and $\rho(t^{-2}) = t^{-2} \otimes 1$, Now we compute

$$\rho(y^2) = (y \otimes 1 + x \otimes Y)^2 = x^2 \otimes Y^2 + (xy + yx) \otimes Y + y^2 \otimes 1 = y^2 \otimes 1,$$

$$\rho(p^2) = (1 \otimes P + p \otimes T)^2 = 1 \otimes P^2 + p \otimes (TP + PT) + p^2 \otimes T^2 = p^2 \otimes 1,$$

$$\begin{aligned} \rho(tp + pt) &= (t \otimes T)(1 \otimes P + p \otimes T) + (1 \otimes P + p \otimes T)(t \otimes T) \\ &= t \otimes TP + tp \otimes T^2 + t \otimes PT + pt \otimes T^2 \\ &= t \otimes (TP + PT) + (tp + pt) \otimes T^2 \\ &= (tp + pt) \otimes 1. \end{aligned}$$

This proves $s, u, v \in C$.

(4) We compute the weak action: for $X, Y, Z, T, P, Q \in D_{(1,1)}$,

$$X \rightharpoonup c = \phi(X)c\phi^{-1}(X) = xcx^{-1},$$

$$Y \rightharpoonup c = \phi(Y)c\phi^{-1}(1) + \phi(X)c\phi^{-1}(Y) = yc - xcx^{-1}y,$$

$$Z \rightharpoonup c = \phi(Z)c\phi^{-1}(X) + \phi(1)c\phi^{-1}(Z) = xycx^{-1} + cy,$$
$$T \rightharpoonup c = \phi(T)c\phi^{-1}(T) = tct^{-1},$$
$$P \rightharpoonup c = \phi(1)c\phi^{-1}(P) + \phi(P)c\phi^{-1}(T) = c(-pt^{-1}) + pct^{-1},$$
$$Q \rightharpoonup c = \phi(T)c\phi^{-1}(Q) + \phi(Q)c\phi^{-1}(1) = tc(-p) + tpc.$$

The trace map $tr : B \to C$ is as follows:

$$tr(1) = tr(x) = tr(t) = 1, \quad tr(y) = tr(z) = tr(p) = tr(q) = 0.$$

For example,

$$\begin{aligned}\chi(Q, T) = \chi(TP, T) &= tr(\phi(TP)\phi(T)) = tr(tpt)\\ &= tr(tv) - tr(t^2p)\\ &= tr(G(v)t) - tr(wp)\\ &= G(v)\ tr(t) - w\ tr(y)\\ &= G(v).\end{aligned}$$

This completes the proof. □

THEOREM 4.3. *Let C be an algebra. Given $F, D, G, E \in End_k(C)$ and $w \in U(C)$, $s, u, v \in C$. Define $\rightharpoonup : \mathscr{D}(\theta)_{(\alpha,\beta)} \otimes C \to C$ and $\chi : \mathscr{D}(\theta)_{(1,1)} \otimes \mathscr{D}(\theta)_{(1,1)} \to C$ by*

$$X \rightharpoonup c = F(c), \quad Y \rightharpoonup c = D(c), \quad Z \rightharpoonup c = -FD(c);$$
$$1 \rightharpoonup c = c, \quad T \rightharpoonup c = G(c), \quad P \rightharpoonup c = E(c), \quad Z \rightharpoonup c = GE(c)w.$$

and χ as in 4.2(4), *Then $(\rightharpoonup, \chi)$ is π-crossed system for $\mathscr{D}(\theta)$ over C if and only if the following conditions hold*:

(1) *F and G are algebra maps*;
(2) $D(cc') = D(c)c' + F(c)D(c')$, $E(cc') = cE(c') + E(c)G(c')$ $(c, c' \in C)$;
(3) $F^2(c) = c$, $G^2(c)w = wc$ $(c \in C)$;
(4) $DF(c) = -FD(c)$, $(GE(c) + EG(c))w = vc - G(c)v$ $(c \in C)$;
(5) $D^2(c) + cs = sc$, $E^2w = [u, c]$ $(c \in C)$;
(6) $G(w) = w$;
(7) $D(s) = 0$, $E(u) = 0$;
(8) $E(w) = 0$;
(9) $F(s) = s$, $E(v) = u - G(u)$.

Proof. $\Rightarrow$) it is easy to see that χ is invertible and χ^{-1} is given by

$$\begin{cases}\chi^{-1}(X, X) = 1, \quad \chi^{-1}(X, Y) = \chi^{-1}(X, Z) = \chi^{-1}(Y, X) = \chi^{-1}(Z, X) = 0,\\ \chi^{-1}(Y, Y) = -s = -\chi^{-1}(Z, Z), \quad \chi^{-1}(Y, Z) = F(s) = -\chi^{-1}(Z, Y),\end{cases}$$

and

$$\begin{cases} \chi^{-1}(T,T) = w^{-1}, \quad \chi^{-1}(T,P) = \chi^{-1}(T,Q) = 0, \\ \chi^{-1}(P,T) = -vw^{-1}, \quad \chi^{-1}(Q,T) = -w^{-1}G(v), \\ \chi^{-1}(P,P) = uw^{-1}, \quad \chi^{-1}(Q,Q) = u, \quad \chi^{-1}(P,Q) = G(u) = -\chi^{-1}(Q,P). \end{cases}$$

Condition (1) comes from a measuring condition for $X \rightharpoonup$ and $T \rightharpoonup$ (resp. $Y \rightharpoonup$ and $P \rightharpoonup$). (3), (4) and (5) come from Eq. (2.3) for $(h,g) = (X,X)$ and (T,T), (Y,X) and (P,T), (Y,Y) and (P,P). (6), (7), (8) and (9) come from Eq. (2.4) for $(h,g,m) = (X,X,X)$ and (T,T,T), (Y,Y,Y) and (P,P,P), (Y,X,X) and (P,T,T), (Y,Y,X) and (P,P,T).

$\Leftarrow$) It is enough to check the conditions Eq. (2.3) and Eq. (2.4) for k-basis X, Y, Z, T, P, Q, but it follows by simple and long calculation. □

We consider the algebra:

$A(s,w,u,v) = k\langle x,y,t,p \mid x^2 = 1, y^2 = s, xy + yx = 0, t^2 = w, p^2 = u, tp + pt = v\rangle.$

Definition 4.4. Let C be an algebra. A 8-tuple (F,D,G,E,s,w,u,v) where $F,D,G,E \in End_k(C)$ and $w \in U(C)$, $s,u,v \in C$, is called a π-$\mathscr{D}(\theta)$-cleft datum over C, if the above (1)–(9) are satisfied. We obtain the π-crossed product

$$A(F,D,G,E,s,w,u,v \mid C) = C\#^{\pi}_{\chi}\mathscr{D}(\theta).$$

Observe that if we let $x = 1 \otimes X$, $y = 1 \otimes Y$, $t = 1 \otimes T$, $p = 1 \otimes P \in A(F,D,G,E,s,w,u,v \mid C)$, then $\{1,x,y,z = xy,t,p,q = tp\}$ forms a left C-basis and the following relations hold:

$$x^2 = (1 \otimes X)(1 \otimes X) = (X \rightharpoonup 1)\chi(X,X) \otimes X^2 = 1 \otimes 1 = 1,$$

$$y^2 = s, \quad xy + yx = 0,$$

$$t^2 = (1 \otimes T)(1 \otimes T) = (T \rightharpoonup 1)\chi(T,T) \otimes T^2 = w \otimes 1 = w,$$

$$p^2 = u, \quad tp + pt = v,$$

$$xc = F(c)x, \quad yc = F(c)y + D(c), \quad zc = cz - FD(c) \quad (c \in C),$$

$$tc = G(c)x, \quad yc = cy + E(c)x, \quad zc = G(c)z + GE(c)w \quad (c \in C).$$

By Theorem 4.2 and Theorem 4.3, we have

Corollary 4.5. *Any π-$\mathscr{D}(\theta)$-cleft extension $C \subset B$ is isomorphic with $A(F,D,G,E,s,w,u,v \mid C)$ for some π-$\mathscr{D}(\theta)$-cleft datum over C.*

We next consider when two $\mathscr{D}(\theta)$ π-cleft extensions over C are isomorphic. Let $(\rightharpoonup,\chi)$ and $(\rightharpoonup',\chi')$ be group crossed systems of π-coalgebra $\mathscr{D}(\theta)$ over C, and $C\#^{\pi}_{\chi}\mathscr{D}(\theta)$, $C\#^{\pi}_{\chi'}\mathscr{D}(\theta)$ be the corresponding π-crossed products, and when they are isomorphism as $\mathscr{D}(\theta)$ extension satisfying

$$\chi'(h,g) = \gamma_{(\alpha,\beta)}(h_1)(h_2 \rightharpoonup \gamma_{(\alpha,\beta)}(g_1)\chi(h_3,g_2)\gamma^{-1}_{(\alpha,\beta)^{-1}}(h_4 g_3)$$
$$h \rightharpoonup' a = \gamma_{(\alpha,\beta)}(h_1)(h_2 \rightharpoonup a)\gamma^{-1}_{(\alpha,\beta)^{-1}}(h_3)$$

for $h, g \in D_{(1,1)}$, $a \in C$, where $\gamma_{(\alpha,\beta)} \in Hom(D_{(\alpha,\beta)}, C)$ are convolution-invertible linear maps.

THEOREM 4.6. *Let* (F, D, G, E, s, w, u, v) *and* $(F', D', G', E', s', w', u', v')$ *be* π-$\mathscr{D}(\theta)$*-cleft data over an arbitrary algebra* C*. Then one has that* $A(F, D, G, E, s, w, u, v \mid C) \cong A(F', D', G', E', s', w', u', v' \mid C)$ *as* π-$\mathscr{D}(\theta)$*-extension if and only if there exist elements* m, m', n, n' *in* C *with* $m, n \in U(C)$ *such that for all* $c \in C$

(1) $F'(c) = mF(c)m^{-1}$, $G'(c) = nG(c)n^{-1}$;
(2) $D'(c) + F'(c)m' = D(c)c + mD(c)$, $G'(c) = \{(n'G + E)(c) - cn'\}n^{-1}$;
(3) $mF(m) = 1$, $w' = nG(n)w$;
(4) $s' = m'^2 + mD(c) + mF(m)s$, $u' = u + n'v + (n'G + E)(n')w$;
(5) $-mF(m') = m'm + mD(m)$, $v' = nv + \{(n'G + E)(n) + nG(n')\}w$.

Proof. Straightforward. □

Acknowledgement. The authors thank the referee for helpful comments on the manuscripts of this paper. The work was partially supported by the NSF of Jiangsu Province (No. BK2012736) and the Research and Innovation Project for College Graduates of Jiangsu Province (No: cxlx_0094).

REFERENCES

[1] R. BLATTNER, M. COHEN AND S. MONTGOMERY, Crossed product and inner actions of Hopf algebras, Trans. Amer. Math. Soc. **298** (1986), 671–711.

[2] R. BLATTNER AND S. MONTGOMERY, Crossed product and Galois extensions of Hopf algebras, Pacific Journal of Math. **137** (1989), 37–53.

[3] S. CAENEPEEL AND M. DE LOMBAERDE, A categorical approach to Turaev's Hopf group-coalgebras, Comm. Algebra **34** (2006), 2631–2657.

[4] Y. DOI, Equivalent crossed product for a Hopf algebra, Comm. Algebra **17** (1989), 3053–3085.

[5] Y. DOI AND M. TAKEUCHI, Cleft comodule algebras for a bialgebra, Comm. Algebra **14** (1986), 801–817.

[6] Y. DOI AND M. TAKEUCHI, Quaternion algebras and Hopf crossed products, Comm. Algebra **23** (1995), 3291–3325.

[7] S. MONTGOMERY, Hopf algebras and their actions on rings, Conference Board of the Mathematical Sciences regional conference series in math. **82**, Amer. Math. Soc. Providence, RI, 1993.

[8] V. TURAEV, Homotopy field theory in dimension 3 and crossed group categories, 2000, preprint, GT/0005291.

[9] A. VIRELIZIER, Hopf group-coalgebras, J. Pure and Applied Algebra **171** (2002), 75–122.

[10] A. VIRELIZIER, Involutory Hopf group-coalgebras and flat bundles over 3-manifolds, Fund. Math. **188** (2005), 241–270.

[11] S. H. WANG, Group twisted smash products and Doi-Hopf modules for T-coalgebras, Comm. Algebra **32** (2004), 3417–3436.

[12] S. H. Wang, Morita contexts, π-Galois extensions for Hopf π-coalgebras, Comm. Algebra **34** (2006), 521–546.
[13] S. H. Wang, Turaev group coalgebras and twisted Drinfeld double, Indiana Univ. Math. J. **58** (2009), 1395–1417.
[14] M. Zunino, Double construction for crossed Hopf coalgebra, J. Algebra **278** (2004), 43–75.
[15] M. Zunino, Yetter-Drinfeld modules for crossed structures, J. Pure and Applied Al gebra **193** (2004), 313–343.

Shuang-jian Guo
School of Mathematics and Statistics
Guizhou University of Finance and Economics
Guiyang 550025
P.R. China
E-mail: shuangjguo@gmail.com

Shuan-hong Wang
Department of Mathematics
Southeast University Nanjing 210096
P.R. China
E-mail: shuanhwang2002@yahoo.com