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ON THE 3-RANK OF THE IDEAL CLASS GROUP OF QUADRATIC FIELDS

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Abstract

As for Scholz' inequalities $s \le r \le s+1$ with respect to the 3-rank of the ideal class group of quadratic fields, we give a criterion to be $r = s+1$. From this, we give a new family of imaginary quadratic fields whose ideal class groups have 3-rank at least two.

1. Introduction

Let d be a square-free positive integer and denote r and s the 3-ranks of the ideal class groups of the imaginary quadratic field $\mathbf{Q}(\sqrt{-d})$ and the real quadratic field $\mathbf{Q}(\sqrt{3d})$, respectively. Then by Scholz [6], we have inequalities

$$s \le r \le s+1. \tag{1.1}$$

This article has two goals. The first is to prove the following theorem which is a criterion for (1.1) to be $r = s+1$.

Theorem 1. *Let $d (\neq -1)$ be a square-free positive (resp. negative) integer with $3 \nmid d$. Let r and s denote the 3-ranks of the ideal class groups of $\mathbf{Q}(\sqrt{-d})$ and $\mathbf{Q}(\sqrt{3d})$ (resp. $\mathbf{Q}(\sqrt{3d})$ and $\mathbf{Q}(\sqrt{-d})$), respectively. Then the following are equivalent:*

(I) $r = s+1$ (*resp.* $r = s$);

(II) *There does not exist a cubic field K satisfying the following three conditions:*

- (II-1) *$K/\mathbf{Q}$ is not normal;*
- (II-2) *The galois closure $\bar{K}$ of $K/\mathbf{Q}$ contains $\mathbf{Q}(\sqrt{-d})$ and $\bar{K}/\mathbf{Q}(\sqrt{-d})$ is a cyclic cubic extension unramified outside 3;*
- (II-3) *$v_3(D_K) = 4$, that is, the discriminant D_K of K is exactly divisible by 3^4;*

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(III) *There exists a cubic field K satisfying the following three conditions*:
(III-1) $K/\mathbf{Q}$ *is not normal*;
(III-2) *The galois closure* $\overline{K}$ *of* $K/\mathbf{Q}$ *contains* $\mathbf{Q}(\sqrt{3d})$ *and* $\overline{K}/\mathbf{Q}(\sqrt{3d})$ *is a cyclic cubic extension unramified outside* 3;
(III-3) $v_3(D_K) = 3$, *that is, the discriminant* D_K *of* K *is exactly divisible by* 3^3;
(IV) *There does not exist a triple* $(u, v, m) \in \mathbf{Z}^3$ $(uvm \neq 0)$ *satisfying the following three conditions*:
(IV-1) $3v^2 d = u^2 - 4m^3$;
(IV-2) $(u, m) = 1$;
(IV-3) $m \equiv 1 \pmod 3$, $u^2 \equiv 1, 7 \pmod 9$;
(V) *There exists a triple* $(u, v, m) \in \mathbf{Z}^3$ $(uvm \neq 0)$ *satisfying the following three conditions*:
(V-1) $-v^2 d = u^2 - 4m^3$;
(V-2) $(u, m) = 1$;
(V-3) *One of the following six conditions holds*:
(a) $3 \mid m$, $u^2 \equiv 4, 7 \pmod 9$;
(b) $3 \nmid m$, $u \equiv 3, 6 \pmod 9$;
(c) $m \equiv 2 \pmod 3$, $u^2 \equiv 1, 4 \pmod 9$;
(d) $m \equiv 1 \pmod 9$, $u^2 \equiv 13, 22 \pmod{27}$;
(e) $m \equiv 4 \pmod 9$, $u^2 \equiv 4, 22 \pmod{27}$;
(f) $m \equiv 7 \pmod 9$, $u^2 \equiv 4, 13 \pmod{27}$.

This is proved by using some properties with respect to cubic polynomials which are stated in the next section.

The second goal is to prove the following theorem which gives a family of imaginary quadratic fields whose ideal class groups have 3-rank at least two.

THEOREM 2. *Let n and q be odd positive integers with $n \geq 3$ or $3 \mid q$. Then the* 3-*rank of the ideal class group of* $\mathbf{Q}(\sqrt{4 - 27^n q^{6n}})$ *is at least* 2.

This is an expansion of the following:

THEOREM 3 ([4, Theorem 3]). *For an odd integer $n \geq 3$, the* 3-*rank of the ideal class group of* $\mathbf{Q}(\sqrt{4 - 27^n})$ *is at least* 2.

The idea of the proof of Theorem 2 is very simple. Put $k := \mathbf{Q}(\sqrt{4 - 27^n q^{6n}})$, $k' := \mathbf{Q}(\sqrt{-3(4 - 27^n q^{6n})})$ and denote r and s the 3-ranks of the ideal class groups of k and k', respectively. Under the situation of Theorem 2, we prove

$$r = s + 1 \tag{1.2}$$

and

$$s \geq 1. \tag{1.3}$$

Then we obtain $r \geq 2$. By using Theorem 1, equation (1.2) will be proved. In order to prove inequality (1.3), it is sufficient to show that the class number of k' is divisible by 3. We will give a cubic polynomial whose splitting field over $\mathbf{Q}$ is an unramified cyclic cubic extension of k'.

Remark 1.1. The condition that $4 - 27^n q^{6n}$ is square-free is not necessary. It is hard to determine when it holds.

2. Cubic polynomials

In this section, we introduce some properties of cubic polynomials which are used for the proof of our theorem.

Let $d(\neq -1)$ be a square-free integer with $3 \nmid d$ and put $k := \mathbf{Q}(\sqrt{-d})$, $k' := \mathbf{Q}(\sqrt{3d})$. Define the subset R_k (resp. R_k') of the integer ring $\mathcal{O}_k$ of k (resp. $\mathcal{O}_{k'}$ of k') by

$$R_k := \{\gamma \in \mathcal{O}_k \mid N(\gamma) \in \mathbf{Z}^3\}$$

$$(\text{resp. } R_{k'} := \{\gamma \in \mathcal{O}_{k'} \mid N(\gamma) \in \mathbf{Z}^3\}).$$

Moreover, for $\gamma \in R_k$ (resp. $\gamma \in R_{k'}$) with

$$\gamma = \frac{u + v\sqrt{-d}}{2}, \quad N(\gamma) = m^3 \quad (u, v, m \in \mathbf{Z})$$

$$(\text{resp. } \gamma = \frac{u + v\sqrt{3d}}{2}, \ N(\gamma) = m^3 \ (u, v, m \in \mathbf{Z})),$$

define the polynomial f_γ by

$$f_\gamma(X) := X^3 - 3mX - u$$

and denote by $\mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$ the minimal splitting field of f_γ over $\mathbf{Q}$. For the irreducibility of f_γ, the following holds:

PROPOSITION 2.1 ([1, Lemma 1]). *For $\gamma \in R_k$ (resp. $\gamma \in R_{k'}$), f_γ is irreducible over $\mathbf{Q}$ if and only if γ is not a cube in $\mathcal{O}_k$ (resp. in $\mathcal{O}_{k'}$).*

Next, define the subset R_k' of R_k (resp. $R_{k'}'$ of $R_{k'}$) by

$$R_k' := \{\gamma \in R_k \mid (N(\gamma), \mathrm{Tr}(\gamma)) = 1, \gamma \notin \mathcal{O}_k^3\}$$

$$(\text{resp. } R_{k'}' := \{\gamma \in R_{k'} \mid (N(\gamma), \mathrm{Tr}(\gamma)) = 1, \gamma \notin \mathcal{O}_{k'}^3\}).$$

For $\gamma \in R_k'$ (resp. $\gamma \in R_{k'}'$), f_γ is irreducible over $\mathbf{Q}$ by Proposition 2.1, and $\mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$ is an S_3-field containing k' (resp. k) because the discriminant $\mathrm{disc}(f_\gamma)$ of f_γ is

$$\mathrm{disc}(f_\gamma) = 4(3m)^3 - 27u^2 = 27v^2 d = 3d \times (3v)^2$$

$$(\text{resp. } \mathrm{disc}(f_\gamma) = 4(3m)^3 - 27u^2 = -81v^2 d = -d \times (9v)^2).$$

Moreover, the following holds:

PROPOSITION 2.2 ([2, Proposition 6.5]). (1) *For* $\gamma \in R'_k$, $\mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$ *is a cyclic cubic extension of* k' *unramified outside* 3 *and contains a cubic field* K *with* $v_3(D_K) = 1$ *or* 3. *Conversely, let* L *be an* S_3*-field containing* k' *and a cubic field* K *with* $v_3(D_K) = 1$ *or* 3 *which is a cyclic cubic extension of* k' *unramified outside* 3. *Then there exists* $\gamma \in R'_k$ *so that* $L = \mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$.

(2) *For* $\gamma \in R'_{k'}$, $\mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$ *is a cyclic cubic extension of* k *unramified outside* 3 *and contains a cubic field* K *with* $v_3(D_K) = 0$ *or* 4. *Conversely, let* L *be an* S_3*-field containing* k *and a cubic field* K *with* $v_3(D_K) = 0$ *or* 4 *which is a cyclic cubic extension of* k *unramified outside* 3. *Then there exists* $\gamma \in R'_{k'}$ *so that* $L = \mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$.

In the final of this section, we state a theorem with respect to the ramification of the prime 3.

PROPOSITION 2.3 ([5, Theorem 1, Theorem 2]). *Suppose that the cubic polynomial*

$$h(X) = X^3 - aX - b, \quad a, b \in \mathbf{Z}$$

is irreducible over $\mathbf{Q}$, *and that either* $v_3(a) < 2$ *or* $v_3(b) < 3$ *holds. Let* θ *be a root of* $h(X) = 0$, *and put* $K = \mathbf{Q}(\theta)$. *Then the following holds*:

(1) *The prime* 3 *is totally ramified in* $K/\mathbf{Q}$ *if and only if one of the following three conditions holds*:
   (i) $1 \le v_3(b) \le v_3(a)$;
   (ii) $3 \mid a,\ 3 \nmid b,\ a \not\equiv 3 \pmod 9,\ b^2 \not\equiv a + 1 \pmod 9$;
   (iii) $a \equiv 3 \pmod 9,\ 3 \nmid b,\ b^2 \not\equiv a + 1 \pmod{27}$.

(2) *The condition* $v_3(D_K) = 3$ *holds if and only if one of the following three conditions holds*:
   (iv) $v_3(a) = v_3(b) = 1$;
   (v) $3 \mid a,\ 3 \nmid b,\ a \not\equiv 3 \pmod 9,\ b^2 \not\equiv a + 1 \pmod 9$;
   (vi) $a \equiv 3 \pmod 9,\ b^2 \equiv 4 \pmod 9,\ b^2 \not\equiv a + 1 \pmod{27}$.

(3) *The condition* $v_3(D_K) = 4$ *holds if and only if one of the following two conditions holds*:
   (vii) $v_3(a) = v_3(b) = 2$;
   (viii) $a \equiv 3 \pmod 9,\ 3 \nmid b,\ b^2 \not\equiv 4 \pmod 9$.

3. Proofs of Theorems

3.1. Proof of Theorem 1. The equivalently of (I), (II) and (III) immediately follows from [2, Theorem 7.1]. We will prove that (II) $\Leftrightarrow$ (IV) and (III) $\Leftrightarrow$ (V).

(IV) $\Rightarrow$ (II): Suppose that there exists a cubic field satisfying the conditions (II-1), (II-2) and (II-3). Then by Proposition 2.2 (2), there exists an element $\gamma \in R'_{\mathbf{Q}(\sqrt{3d})}$ such that f_γ satisfies one of the conditions (vii) and (viii) of Proposition 2.3 (3). Express

$$\gamma = \frac{u + v\sqrt{3d}}{2}, \quad N(\gamma) = m^3 \quad (u, v, m \in \mathbf{Z},\ (u, m) = 1).$$

It is clear that both of (IV-1) and (IV-2) hold. Noting that

$$f_\gamma(X) = X^3 - 3mX - u,$$

we have

$$f_\gamma \text{ satisfies (vii)} \Leftrightarrow 3 \,||\, m,\ \ 3^2 \,||\, u,$$

$$f_\gamma \text{ satisfies (viii)} \Leftrightarrow 3 \nmid u,\ \ 3m \equiv 3 \pmod 9,\ \ u^2 \not\equiv 4 \pmod 9.$$

By $(u, m) = 1$, we see that the condition (IV-3) holds.

(II) $\Rightarrow$ (IV): Suppose that there exists a triple $(u, v, m) \in \mathbf{Z}^3$ $(uvm \neq 0)$ satisfying the conditions (IV-1), (IV-2), (IV-3). Put $\gamma := (u + v\sqrt{3d})/2$. By the condition (IV-1), we have

$$m^3 = \frac{u^2 - 3v^2 d}{4},$$

and so $N(\gamma) \in \mathbf{Z}^3$. Moreover, $(N(\gamma), \mathrm{Tr}(\gamma)) = 1$ follows from the condition (IV-2). To prove $\gamma \notin \mathcal{O}^3_{\mathbf{Q}(\sqrt{3d})}$, assume on the contrary that $\gamma \in \mathcal{O}^3_{\mathbf{Q}(\sqrt{3d})}$. Then we can express

$$\frac{u + v\sqrt{3d}}{2} = \left(\frac{a + b\sqrt{3d}}{2}\right)^3 \quad (a, b \in \mathbf{Z}). \tag{3.1}$$

On the one hand, we have

$$4u = a^3 + 9ab^2 d \tag{3.2}$$

by comparing the traces of both sides of (3.1). On the other hand, we have

$$m^3 = \left(\frac{a^2 - 3b^2 d}{4}\right)^3$$

by taking the norm of both sides of (3.1). Then we have

$$4m = a^2 - 3b^2 d.$$

From this together with (3.2), we obtain the relation

$$u = a^3 - 3am.$$

Then by the condition (IV-3), we have

$$a^3 - a \equiv 1, 4, 5, 8 \pmod 9.$$

However, no rational integer satisfies this congruence. Therefore we get a contradiction, and hence we have $\gamma \notin \mathcal{O}^3_{\mathbf{Q}(\sqrt{3d})}$. Thus we obtain $\gamma \in R'_{\mathbf{Q}(\sqrt{3d})}$. Then by Proposition 2.2 (2), the cubic field K contained in $\mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$ satisfies (II-1) and (II-2). Moreover, it follows from the condition (IV-3) that f_γ satisfies (viii). By Proposition 2.3 (3), therefore, the cubic fields contained in $\mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$ satisfy the condition (II-3).

(III) $\Rightarrow$ (V): Suppose that there exists a cubic field satisfying the conditions (III-1), (III-2) and (III-3). Then by Proposition 2.2 (1), there exists an element $\gamma \in R'_{\mathbf{Q}(\sqrt{-d})}$ such that f_γ satisfies one of the conditions (iv), (v) and (vi) of Proposition 2.3 (2). Express

$$\gamma = \frac{u + v\sqrt{-d}}{2}, \quad N(\gamma) = m^3 \quad (u, v, m \in \mathbf{Z},\ (u, m) = 1).$$

It is clear that both of (V-1) and (V-2) hold. Noting that

$$f_\gamma(X) = X^3 - 3mX - u,$$

we have

$$f_\gamma \text{ satisfies (iv)} \Leftrightarrow 3 \,\|\, u,\ \ 3 \nmid m,$$

$$f_\gamma \text{ satisfies (v)} \Leftrightarrow 3 \nmid u,\ \ 3m \not\equiv 3 \pmod 9,\ \ u^2 \not\equiv 3m + 1 \pmod 9,$$

$$f_\gamma \text{ satisfies (vi)} \Leftrightarrow 3m \equiv 3 \pmod 9,\ \ u^2 \equiv 4 \pmod 9,\ \ u^2 \not\equiv 3m + 1 \pmod{27}.$$

From these, we get the conditions (a)$\sim$(f) in (V-3).

(V) $\Rightarrow$ (III): Suppose that there exists a triple $(u, v, m) \in \mathbf{Z}^3$ $(uvm \neq 0)$ satisfying the conditions (V-1), (V-2), (V-3). Put $\gamma := (u + v\sqrt{-d})/2$. By the condition (V-1), we have

$$m^3 = \frac{u^2 + v^2 d}{4},$$

and so $N(\gamma) \in \mathbf{Z}^3$. Moreover, $(N(\gamma), \mathrm{Tr}(\gamma)) = 1$ follows from the condition (V-2). To prove $\gamma \notin \mathcal{O}^3_{\mathbf{Q}(\sqrt{-d})}$, we assume on the contrary that $\gamma \in \mathcal{O}^3_{\mathbf{Q}(\sqrt{-d})}$. Then we can write

$$\frac{u + v\sqrt{-d}}{2} = \left(\frac{a + b\sqrt{-d}}{2}\right)^3 \quad (a, b \in \mathbf{Z}). \tag{3.3}$$

On the one hand, we have

$$4u = a^3 - 3ab^2 d \tag{3.4}$$

by comparing the traces of both sides of (3.3). On the other hand, we have

$$m^3 = \left(\frac{a^2 + b^2 d}{4}\right)^3$$

by taking the norm of both sides of (3.3). Then we have

$$4m = a^2 + b^2 d.$$

From this together with (3.4), we obtain the relation

$$u = a^3 - 3am. \tag{3.5}$$

Here we assume that m and u satisfy the condition (a) of (V-3). Then by (3.5), we have

$$a^3 \equiv 2, 4, 5, 7 \pmod 9.$$

However, no rational integer satisfies this congruence. Therefore we get a contradiction, and hence we have $\gamma \notin \mathcal{O}^3_{\mathbf{Q}(\sqrt{-d})}$. Similarly, if m and u satisfy the conditions (b)~(f) in (V-3), we can get a contradiction from (3.5), and hence we have $\gamma \notin \mathcal{O}^3_{\mathbf{Q}(\sqrt{-d})}$. Thus we obtain $\gamma \in R'_{\mathbf{Q}(\sqrt{-d})}$. Then by Proposition 2.2 (1), the cubic field K contained in $\mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$ satisfies (III-1) and (III-2).

Moreover, we easily verify that

$$m \text{ and } u \text{ satisfy (b)} \Rightarrow f_\gamma \text{ satisfies (iv)},$$
$$m \text{ and } u \text{ satisfy (a) or (c)} \Rightarrow f_\gamma \text{ satisfies (v)},$$
$$m \text{ and } u \text{ satisfy (d) or (e) or (f)} \Rightarrow f_\gamma \text{ satisfies (vi)}.$$

By Proposition 2.3 (2), therefore, the cubic fields contained in $\mathrm{Spl}_{\mathbf{Q}}(f_\gamma)$ satisfy the condition (III-3). This completes the proof of Theorem 1.

3.2. Proof of Theorem 2. First, we prove (1.2). Express

$$4 - 27^n q^{6n} = -v'^2 d,$$

where d is a square-free positive integer. Put $u = 4$, $v = 2v'$, $m = 3^n q^{2n}$; we can verify that u, v and m satisfy the following conditions of Theorem 1:

(V-1) $u^2 - 4m^3 = 2^2(4 - 27^n q^{6n}) = -v^2 d$,

(V-2) $(u, m) = 1$,

(V-3) (a) $3 \mid m$, $u^2 \equiv 7 \pmod 9$.

Since d is positive, therefore, (1.2) follows from Theorem 1.

Next, we prove (1.3). Define the element $\alpha \in \mathcal{O}_k$ by

$$\alpha := \frac{3^{(n+1)/2} q^n (3^n q^{2n} - 2) + \sqrt{4 - 27^n q^{6n}}}{2}.$$

Then we have

$$N(\alpha) = (3^n q^{2n} - 1)^3,$$
$$\mathrm{Tr}(\alpha) = 3^{(n+1)/2} q^n (3^n q^{2n} - 2),$$

and hence

$$(N(\alpha), \mathrm{Tr}(\alpha)) = 1,$$
$$N(\alpha) \in \mathbf{Z}^3.$$

To prove $\alpha \notin \mathcal{O}_k^3$, let us apply Proposition 2.1 to

$$f_\alpha(X) = X^3 - 3(3^n q^{2n} - 1)X - 3^{(n+1)/2} q^n (3^n q^{2n} - 2).$$

By putting $t := 3^{(n-1)/2} q^n$, we get

$$f_\alpha(X) = X^3 - 3(3t^2 - 1)X - 3t(3t^2 - 2).$$

Now we assume that f_α is reducible over $\mathbf{Q}$. Then there exists a rational number x such that

$$x^3 - 3(3t^2 - 1)x - 3t(3t^2 - 2) = 0. \tag{3.6}$$

Here we take a change of variables by

$$x = 3y - 2l,$$
$$t = l - y$$

and substitute them into (3.6). Then we have

$$9y^3 - 9ly^2 + l^3 + 3y = 0.$$

Multiplying both side of this by $3^3/y^3$ and putting $p := -3l/y$, $s := 3^2/y$, we have

$$s^2 = p^3 - 3^4 p - 3^5.$$

This is a contradiction because the elliptic curve

$$Y^2 = X^3 - 3^4 X - 3^5$$

has no solution in $\mathbf{Q}$. Hence f_α is irreducible over $\mathbf{Q}$. Then by Proposition 2.1, we have $\alpha \notin \mathcal{O}_k^3$. Hence, we get $\alpha \in R_k'$. By Proposition 2.2 (1), therefore, $\mathrm{Spl}_{\mathbf{Q}}(f_\alpha)$ is a cyclic cubic extension of k' unramified outside 3. Now we recall the assumption "$n \geq 3$ or $3 \mid q$". Under this assumption, f_α does not satisfy any of the conditions (i), (ii), (iii) of Proposition 2.3 (1). Then the prime divisor of 3 in k' is unramified in $\mathrm{Spl}_{\mathbf{Q}}(f_\alpha)$. Therefore $\mathrm{Spl}_{\mathbf{Q}}(f_\alpha)$ is an unramified cyclic cubic extension of k', and hence we obtain (1.3). Theorem 2 is now proved.

4. Some known results

In this section, we give an alternative proof of some known results by using Theorem 1. For two quadratic fields $\mathbf{Q}(\sqrt{D})$ and $\mathbf{Q}(\sqrt{-3D})$, let r denote the 3-rank of the ideal class group of the imaginary quadratic field, and let s denote the one of the other.

Theorem 4 ([3, Theorem 1]). *Let a and b be rational integers with $3 \nmid b$ and put $D := -4a^3 + 9b^2$. Suppose that D is square-free. Then we have*

$$r = \begin{cases} s, & \text{if } D > 0, \\ s + 1, & \text{if } D < 0. \end{cases} \tag{4.1}$$

Proof. Since D is square-free, we have $3 \nmid a$ and hence we also have $3 \nmid D$. Put $u = 3b$, $v = 1$, $m = a$ and $d = -D$. Then (u, v, m) satisfies (V-1), (V-2) and (b) of (V-3) in Theorem 1. Thus we get (4.1). □

THEOREM 5 ([7, Theorem 2, Theorem 4]). *Let A and B be positive integers and put $D := A^6 + 4B^6$. Suppose that D is square-free. Then we have*

$$r = \begin{cases} s, & \text{if } 3 \nmid B, \\ s+1, & \text{if } 3 \mid B. \end{cases}$$

Proof. We easily have $3 \nmid D$ and $D > 0$. Put $d = -D$. In the case $3 \nmid B$ and $3 \mid A$, it is easily verified that $(u, v, m) = (4B^3, 2, -A^2)$ satisfies (V-1), (V-2) and (a) of (V-3). Then we have $r = s$. In the case $3 \nmid B$ and $3 \nmid A$, $(u, v, m) = (A^3, 1, -B^2)$ satisfies (V-1), (V-2) and (c) of (V-3). Then we also have $r = s$. In the case $3 \mid B$, we put $u = A^6 + 6A^4B^2 + 6A^2B^4 - 2B^6$, $v = A(A^2 + 2B^2)$ and $m = (A^2 + B^2)^2$. Noting that $3 \nmid A$, we see that (u, v, m) satisfies (IV-1), (IV-2) and (IV-3). Therefore we have $r = s + 1$. □

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REFERENCES

[1] Y. KISHI, A criterion for a certain type of imaginary quadratic fields to have 3-ranks of the ideal class groups greater than one, Proc. Japan Acad. **74** (1998), 93–97.

[2] Y. KISHI, A constructive approach to Spiegelung relations between 3-ranks of absolute ideal class groups and congruent ones modulo $(3)^2$ in quadratic fields, J. Number Theory **83** (2000), 1–49.

[3] Y. KISHI, A note on the 3-rank of quadratic fields, Arch. Math. (Basel) **81** (2003), 520–523.

[4] Y. KISHI, On the ideal class group of certain quadratic fields, Glasgow Math. J. **52** (2010), 575–581.

[5] P. LLORENTE AND E. NART, Effective determination of the decomposition of the rational primes in a cubic field, Proc. Amer. Math. Soc. **87** (1983), 579–585.

[6] A. SCHOLZ, Über die Beziehung der Klassenzahl quadratischer Körper zueinander, J. Reine Angew. Math. **166** (1932), 201–203.

[7] D. SHANKS AND P. WEINBERGER, A quadratic field of prime discriminant requiring three generators for its class group, and related theory, Acta Arith. **21** (1972), 71–87.

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