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PRINCIPAL TORUS BUNDLES OF LORENTZIAN $\mathcal{S}$-MANIFOLDS AND THE φ-NULL OSSERMAN CONDITION

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Abstract

In this short note we provide a few results about the projectability of the φ-null Osserman condition onto the classical and the null-Osserman condition, via semi-Riemannian submersions as projection maps of principal torus bundles induced by a Lorentzian $\mathcal{S}$-manifold.

1. Introduction

The Jacobi operator is, for several reasons, one of the most interesting objects induced by the curvature operator.

On a (semi-)Riemannian manifold (M, g), let us consider the unit spacelike $S^{+}(M)$ (resp. timelike $S^{-}(M)$) sphere bundle with fiber

$$S_p^{\pm}(M) = \{z \in T_pM \mid g_p(z, z) = \pm 1\},$$

and put $S(M) = \bigcup_{p \in M} S_p^{+}(M) \cup S_p^{-}(M)$.

For any $z \in S_p(M)$, $p \in M$, the *Jacobi operator with respect to* z is the endomorphism $R_z : z^{\perp} \to z^{\perp}$ such that $R_z(\cdot) = R_p(\cdot, z)z$ ([20]), where R is the $(1, 3)$-type curvature tensor on (M, g).

The Jacobi operator is obviously self-adjoint, hence a great deal of study has been carried out about the behaviour of its eigenvalues in the Riemannian case since R. Osserman proposed his Conjecture in [33] (see also [32]). Indeed, one easily sees that Riemannian space-forms are characterized by having Jacobi operators with exactly one constant eigenvalue corresponding to the sectional curvature. Those Riemannian manifolds whose Jacobi operators have eigenvalues independent both of the vector $z \in S_p(M)$ and of the point $p \in M$ are the *Osserman manifolds*. Any locally flat or locally rank-one symmetric space is an Osserman manifold, whilst the converse statement is known as the Osserman

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Conjecture. Several authors have dealt with this Conjecture, providing positive answers in many cases ([12], [13], [14], [28], [29], [30]).

One gets a different situation when considering the indefinite setting, where a semi-Riemannian manifold (M, g) is said to be *spacelike* (resp. *timelike*) *Osserman*, if the characteristic polynomial of R_z is independent of both $z \in S_p^+(M)$ (resp. $z \in S_p^-(M)$) and $p \in M$. It is known that (M, g) being spacelike Osserman is equivalent to (M, g) being timelike Osserman ([19], [20]). Several counterexamples to the Osserman Conjecture were found (see for example [5], [6], [21]) for non-Lorentzian semi-Riemannian manifolds.

Finally, in the Lorentzian setting a complete solution for the Osserman Conjecture was provided in a sequence of works by E. García-Río, D. N. Kupeli and M. E. Vázquez-Abal ([17], [18]), together with N. Blažić, N. Bokan and P. Gilkey ([4]). They proved that a Lorentzian manifold is Osserman if and only if it has constant sectional curvature (see also [20]).

A very fruitful, new Osserman-related condition for Lorentzian manifolds was defined in [18]. There, the authors introduced the Jacobi operator $\overline{R}_u$ with respect to a null (lightlike) vector u, and then they studied the so-called *null Osserman conditions* with respect to a unit timelike vector (see also [20]).

Here, we are concerned with an Osserman-related condition derived by the null Osserman condition, which we call the *φ-null Osserman condition*, introduced and studied by the first author in [7] for manifolds carrying Lorentzian globally framed f-structures. This condition appears to be a natural generalization of the null Osserman condition, to which it reduces when considering Lorentzian almost contact structures. This new definition was mainly motivated by the following considerations: although any Lorentzian Sasaki manifold $(M, \varphi, \xi, \eta, g)$ with constant φ-sectional curvature is globally null Osserman with respect to the timelike vector field ξ, there is no similar result when we consider Lorentzian $\mathscr{S}$-manifolds, which generalize Lorentzian Sasaki ones, and moreover, as we proved in [8], no Lorentzian $\mathscr{S}$-manifold can be neither null Osserman, nor Osserman. For more details about the φ-null Osserman condition we refer the reader to [7], where basic properties of such condition are studied, and to [8], where the study is furtherly developed and generalized. The main reference for the whole Osserman framework is [20].

In this short note, we deal with the study of some relationships among the above three Osserman-related notions, providing a few results of equivalence, obtained by considering a natural structure of principal torus bundle arising from a Lorentzian $\mathscr{S}$-manifold, which involves semi-Riemannian submersions.

Indeed, from [3], a strong link between f-structures and Riemannian submersions is well-known. Namely, any compact and connected manifold endowed with a regular and normal $g.f.f$-structure is the total space of a principal torus bundle over a complex manifold, which, under suitable hypotheses, can be a Kähler manifold. Moreover, as also proved in [3], a compact, connected and regular Riemannian $\mathscr{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, with each ξ_α regular, projects itself onto a compact Kähler manifold and onto a compact and regular Sasakian manifold. These results have been extended to the semi-Riemannian case by the first author, together with A. M. Pastore, who in [10] proved that a compact, connected and regular indefinite (in particular, Lorentzian) $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ projects itself onto a compact (indefinite) Kähler manifold and onto a compact and regular indefinite (Lorentzian) Sasakian manifold, via semi-Riemannian submersions.

Based on the above, after recalling, in Section 2, some basic features of (almost) $\mathcal{S}$-manifolds, in Section 4 we carry on an investigation on the possibilities of projectability of the φ-null Osserman conditions via semi-Riemannian submersions with a Lorentzian $\mathcal{S}$-manifold as total space, and either a Lorentzian Sasakian manifold or a Kähler manifold as base space. Using some properties established in [8], which we briefly recall in Section 3, together with a few properties of semi-Riemannian submersions, and under an additional assumption on the eigenvectors of the Jacobi operators, we obtain equivalence results relating the φ-null Osserman condition with the classical and the null Osserman condition in the framework of principal torus bundles constructed on a given Lorentzian $\mathcal{S}$-manifold.

In what follows, all smooth manifolds are supposed to be connected, and all tensor fields and maps are assumed to be smooth. Moreover, according to [25], for the Riemannian curvature tensor of a semi-Riemannian manifold (M, g) we use the definition $R(X, Y, Z, W) = g(R(Z, W)Y, X) = g(([\nabla_Z, \nabla_W] - \nabla_{[Z,W]})Y, X)$ for any vector fields X, Y, Z, W on M.

Finally, for any $p \in M$ and any linearly independent vectors $x, y \in T_pM$ spanning a non-degenerate plane $\pi = \mathrm{span}(x, y)$, that is $g_p(x, x)g_p(y, y) - g_p(x, y)^2 \neq 0$, the sectional curvature of (M, g) at p with respect to π is, by definition, the real number

$$k_p(\pi) = k_p(x, y) = \frac{R_p(x, y, x, y)}{\Delta(\pi)},$$

where $\Delta(\pi) = g_p(x, x)g_p(y, y) - g_p(x, y)^2$.

2. Preliminaries

Let us recall some basic definitions and facts about (almost) $\mathcal{S}$-manifolds needed in the rest of the paper.

Framed f-manifolds were originally considered by H. Nakagawa in [26] and [27], based on the notion of f-structure, which was firstly introduced in 1963 by K. Yano ([36]) as a generalization of both (almost) contact and (almost) complex structures. Such structures were later studied and developed by S. I. Goldberg and K. Yano (see, for example, [22], [23]) and, in the subsequent years, by several authors ([1], [3], [11], [24], [35]).

A *globally framed f-structure* (briefly *g.f.f-structure*) on a manifold M is a non-vanishing $(1,1)$-type tensor field φ on M of constant rank satisfying the following conditions: $\varphi^3 + \varphi = 0$, and the subbundle $\ker(\varphi)$ is parallelizable. This is equivalent to the existence of s linearly independent vector fields ξ_α and

1-forms η^α ($\alpha \in \{1, \dots, s\}$), s being the dimension of $\ker(\varphi)$ at any point $p \in M$, such that

$$\varphi^2 = -I + \eta^\alpha \otimes \xi_\alpha \quad \text{and} \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta. \tag{2.1}$$

Each ξ_α is said to be a *characteristic vector field* of the structure, and a manifold M carrying a $g.f.f$-structure is denoted by $(M, \varphi, \xi_\alpha, \eta^\alpha)$, and called a *$g.f.f$-manifold*. When $s = 1$ (resp.: $s = 0$), we have an almost contact (resp.: almost complex) structure. From (2.1) one easily has $\varphi\xi_\alpha = 0$ and $\eta^\alpha \circ \varphi = 0$, for any $\alpha \in \{1, \dots, s\}$. Furthermore, $\mathrm{Im}(\varphi)$ is a distribution on M of even rank $r = 2n$ on which φ acts as an almost complex tensor field, and one has the splitting $TM = \mathrm{Im}(\varphi) \oplus \ker(\varphi)$, hence $\dim(M) = 2n + s$. A $g.f.f$-manifold is said to be *normal* if the $(1,2)$-type tensor field $N = [\varphi, \varphi] + 2d\eta^\alpha \otimes \xi_\alpha$ vanishes.

In [9], the authors study the properties of a $g.f.f$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha)$ endowed with a compatible indefinite metric, that is a semi-Riemannian metric g verifying

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$, where $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$. Such a manifold is said to be an *indefinite metric $g.f.f$-manifold* and denoted by $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$. From (2.2) one also has $g(X, \xi_\alpha) = \varepsilon_\alpha \eta^\alpha(X)$ and $g(X, \varphi Y) = -g(\varphi X, Y)$, for any $X, Y \in \Gamma(TM)$, and the splitting $TM = \mathrm{Im}(\varphi) \oplus \ker(\varphi)$ becomes orthogonal.

The *fundamental 2-form* Φ of an indefinite metric $g.f.f$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is defined by $\Phi(X, Y) = g(X, \varphi Y)$. If $\Phi = d\eta^\alpha$, for any $\alpha \in \{1, \dots, s\}$, the manifold is said to be an *indefinite almost $\mathcal{S}$-manifold*. Finally, a normal indefinite almost $\mathcal{S}$-manifold is, by definition, an *indefinite $\mathcal{S}$-manifold*. Such a manifold is characterized by the identity $(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2 X$, where $\bar{\xi} = \sum_{\alpha=1}^{s} \xi_\alpha$ and $\bar{\eta} = \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha$. It follows that $\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X$ and $\nabla_{\xi_\alpha} \xi_\beta = 0$, for any $\alpha, \beta \in \{1, \dots, s\}$, and each ξ_α is a Killing vector field.

For more details on (almost) $\mathcal{S}$-manifolds the reader is referred to [15] in the Riemannian case, and to [9] for the indefinite case.

3. Lorentzian $\mathcal{S}$-manifolds and the φ-null Osserman condition

The notion of φ-null Osserman condition is derived from that of null Osserman, which we briefly recall here, following [18] and [20].

Let (M, g) be a Lorentzian manifold and $p \in M$. If $u \in T_pM$ is a lightlike (or null) vector, that is $u \neq 0$ and $g_p(u, u) = 0$, then $\mathrm{span}(u) \subset u^\perp$. We can endow the quotient space $\bar{u}^\perp = u^\perp / \mathrm{span}(u)$, whose canonical projection is $\pi : u^\perp \to \bar{u}^\perp$, with a positive definite inner product $\bar{g}$ defined by $\bar{g}(\bar{x}, \bar{y}) = g_p(x, y)$, where $\pi(x) = \bar{x}$ and $\pi(y) = \bar{y}$, obtaining the Euclidean vector space $(\bar{u}^\perp, \bar{g})$.

The *Jacobi operator with respect to* $\bar{u}$ is the endomorphism $\bar{R}_u : \bar{u}^{\perp} \to \bar{u}^{\perp}$ defined by $\bar{R}_u(\bar{x}) = \pi(R_p(x, u)u)$, for all $\bar{x} = \pi(x) \in \bar{u}^{\perp}$. It is easy to see that $\bar{R}_u$ is a self-adjoint endomorphism, hence it is diagonalizable.

If $z \in T_pM$ is a unit timelike vector, the *null congruence set* of z is defined to be the set $N(z) = \{u \in T_pM \mid g_p(u, u) = 0, g_p(u, z) = -1\}$. The elements of $N(z)$ are in one-to-one correspondence to those of the set $S(z) = \{x \in z^{\perp} \mid g_p(x, x) = 1\}$, called the *celestial sphere of* z, via the map $\psi : N(z) \to S(z)$ such that $\psi(u) = u - z$.

Definition 3.1 ([18, 20]). A Lorentzian manifold (M, g) is said to be *null Osserman with respect to* z, $z \in T_pM$ being a unit timelike vector, if the eigenvalues of $\bar{R}_u$ and their multiplicities are independent of $u \in N(z)$.

Following [7] and [8], we recall the basic facts related with the definition of the φ-null Osserman condition.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $g.f.f$-manifold, with $\dim(M) = 2n + s$, and $\alpha \in \{1, \dots, s\}$, $s \geqslant 1$. It is easy to see that one of the characteristic vector fields has to be timelike and, without loss of generality, we assume it is ξ_1. If $p \in M$, we define the *φ-celestial sphere* of $(\xi_1)_p$ to be the set $S_\varphi((\xi_1)_p) = S((\xi_1)_p) \cap \mathrm{Im}(\varphi_p)$, and the *$\varphi$-null congruence set* of $(\xi_1)_p$ to be $N_\varphi((\xi_1)_p) = \psi^{-1}(S_\varphi((\xi_1)_p))$.

Definition 3.2 ([7, 8]). A Lorentzian $g.f.f$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is said to be *φ-null Osserman with respect to* $(\xi_1)_p$, $p \in M$, if the eigenvalues of $\bar{R}_u$ and their multiplicities are independent of $u \in N_\varphi((\xi_1)_p)$.

Fix $p \in M$ and consider $u \in N_\varphi((\xi_1)_p)$. Since we can write $u = (\xi_1)_p + x$, with $x \in S_\varphi((\xi_1)_p)$, there is a natural one-to-one correspondence between the two kinds of Jacobi operator $R_x : x^{\perp} \to x^{\perp}$ and $\bar{R}_u : \bar{u}^{\perp} \to \bar{u}^{\perp}$. In [8] it is provided the relationship between these two operators with respect to the φ-null Osserman condition, which we summarize in the following proposition.

Proposition 3.3 ([8]). *Let* $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ *be a Lorentzian* $\mathcal{S}$*-manifold,* $\dim(M) = 2n + s$, $s \geqslant 1$. *For any* $p \in M$, M *is φ-null Osserman with respect to* $(\xi_1)_p$ *if and only if the eigenvalues of* R_x *with their multiplicities are independent of* $x \in S_\varphi((\xi_1)_p)$.

The above result enables us to write the definition of the φ-null Osserman condition in terms of operator R_x, $x \in S_\varphi((\xi_1)_p)$, instead of $\bar{R}_u$, $u \in N_\varphi((\xi_1)_p)$. It is clear that, in the case of a Lorentzian Sasaki manifold, the φ-null Osserman condition reduces to that of null Osserman one.

4. Principal torus bundles and the φ-null Osserman condition

From [3] it is known that under an assumption of regularity it is possible to relate metric $g.f.f$-manifolds both to almost complex and to almost contact metric manifolds via Riemannian submersions. The semi-Riemannian version of the results of [3] is provided in [10], where it is possible to find the following result.

THEOREM 4.1. *Let* $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ *be a compact, connected and regular indefinite* $\mathcal{S}$*-manifold, with* $\dim(M) = 2n + s$, $s \geqslant 2$. *Then, there exists a commutative diagram*

$$\begin{tikzcd} M \arrow[rr, "\tau"] \arrow[dr, "\pi"'] & & M' \arrow[dl, "\pi'"] \\ & N & \end{tikzcd}$$

where N *is a* $2n$*-dimensional compact Kähler manifold, either indefinite or not, and* M' *is a* $(2n+1)$*-dimensional compact and regular Sasakian manifold, indefinite or not. All the maps are semi-Riemannian submersions with totally geodesic fibres, and more precisely*:

- τ *is the projection of a principal* $\mathbf{T}^{s-1}$*-bundle over* M';
- π' *is the projection of a principal* $\mathbf{S}^1$*-bundle over* N;
- π *is the projection of a principal* $\mathbf{T}^s$*-bundle over* N.

where $\mathbf{T}^k$ *is the* k*-dimensional torus, for any* $k \in \mathbf{N}$, $k \geqslant 1$.

For the notion of regularity of a distribution and of a $g.f.f$-structure the reader is referred to [34] and [3]. The general idea of this result, as contained in [3], is to fibrate M by any $s - r$ of the vector fields ξ_α's, to obtain a principal $\mathbf{T}^{s-r}$-bundle over a $(2n+r)$-dimensional manifold M'. The remaining r characteristic vector fields are then projectable to M', inducing a $g.f.f$-structure on M' and preserving the regularity. Thus, M' can be fibrated again by its r characteristic vector fields, obtaining a principal $\mathbf{T}^r$-bundle over N, which finally produces a commutative diagram. In particular, if we fibrate a Lorentzian $\mathcal{S}$-manifold M by the $s-1$ spacelike characteristic vector fields, in Theorem 4.1 we obtain that N is a Kähler manifold and M' is a Lorentz Sasakian manifold.

We are going to find out some informations about the possibility of projecting the φ-null Osserman condition both onto the null Osserman condition and the classical Osserman condition, via the previous fibrations.

In general (see [16], [31]), given a C^∞-submersion $f : (M, g) \to (B, g')$ between semi-Riemannian manifolds, i.e. a map whose differential $(df)_p$ is surjective, for all $p \in M$, then $\mathcal{V} = (\ker(df)_p)_{p \in M}$ and $\mathcal{H} = (\ker(df)_p^\perp)_{p \in M}$ are, by definition, the *vertical* and the *horizontal* distributions of f. Such a map is said to be a *semi-Riemannian submersion* if each fibre $f^{-1}(p')$, $p' \in B$, is a (semi-)Riemannian submanifold of M and the restriction of $(df)_p$ to $\mathcal{H}_p$ is an isometry between $(\mathcal{H}_p, g_p)$ and $(T_{f(p)}B, g'_{f(p)})$, for all $p \in M$. A vector field U (resp. X) on M such that $U_p \in \mathcal{V}_p$ (resp. $X_p \in \mathcal{H}_p$) is called *vertical* (resp. *horizontal*). A vector field X on M such that there exists a vector field X' on B for which $f_* X = X'$ is said to be *projectable*, and any horizontal, projectable vector field on M is said to be *basic*. The vertical distribution is always integrable, with the fibres of f as leaves. Denoting by v and h the projections of TM onto $\mathcal{V}$ and $\mathcal{H}$, respectively, the *O'Neill tensors* of f are the $(1,2)$-type tensor fields T and A on M defined by:

$$T(X,Y) = T_X Y := v\nabla_{vX} hY + h\nabla_{vX} vY,$$
$$A(X,Y) = A_X Y := v\nabla_{hX} hY + h\nabla_{hX} vY.$$

They are both g-skew-symmmetric tensors, and they satisfy the following fundamental properties:

$$T_U W = T_W U \qquad U, W \in \mathcal{V}$$
$$A_X Y = -A_Y X = \frac{1}{2} v[X,Y] \quad X, Y \in \mathcal{H}$$

It follows that the horizontal distribution is integrable if and only if $A = 0$, and in this case the leaves are totally geodesic submanifolds of M. Furthermore, the fibres of f are totally geodesic semi-Riemannian submanifolds of M if and only if $T = 0$.

LEMMA 4.2. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold, with* $\dim(M) = 2n + s$, $s \geqslant 1$. *Let $\pi : M \to N$ be a principal $\mathbf{T}^s$-bundle over a Kähler manifold, as in Theorem* 4.1. *We have*:

$$A_X Y = -g(X, \varphi Y)\bar{\xi}, \quad A_X \xi_\alpha = -\varepsilon_\alpha \varphi X, \tag{4.1}$$

for any $X, Y \in \mathrm{Im}(\varphi)$ and any $\alpha \in \{1, \dots, s\}$, where $\bar{\xi} = \sum_{\alpha=1}^s \xi_\alpha$.

Proof. By construction of π, we have $\mathcal{H}_p = \mathrm{Im}(\varphi_p)$ and $\mathcal{V}_p = \mathrm{span}((\xi_1)_p, \dots, (\xi_s)_p)$ for any $p \in M$. Thus, since $\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X$, by direct calculation we get:

$$A_X Y = v(\nabla_X Y) = \sum_{\alpha=1}^s \varepsilon_\alpha g(\nabla_X Y, \xi_\alpha)\xi_\alpha = -\sum_{\alpha=1}^s \varepsilon_\alpha g(Y, \nabla_X \xi_\alpha)\xi_\alpha$$
$$= \sum_{\alpha=1}^s g(Y, \varphi X)\xi_\alpha = -g(X, \varphi Y)\bar{\xi},$$

for all $X, Y \in \mathcal{H}$. Analogously, we have $A_X \xi_\alpha = h(\nabla_X \xi_\alpha) = -\varepsilon_\alpha \varphi X$ for all $X \in \mathcal{H}$ and $\alpha \in \{1, \dots, s\}$. □

For a semi-Riemannian submersion $f : (M, g) \to (B, g')$, let us denote by R^* the $(1,3)$-type $\mathcal{H}$-valued tensor field on M such that, if $X, Y, Z \in \Gamma(TM)$ are basic vector fields f-related to $X', Y', Z' \in \Gamma(TB)$, then $R^*(X,Y)Z$ is the unique basic vector field f-related to $R'(X', Y')Z'$. Thus, for any $x \in \mathcal{H}_p$, one can consider the self-adjoint endomorphism $R^*_x : x^\perp \cap \mathcal{H}_p \to x^\perp \cap \mathcal{H}_p$ such that $R^*_x(y) = R^*_p(y, x)x$.

Lemma 4.3. *Let $f:(M,g)\to(B,g')$ be a semi-Riemannian submersion. For any orthogonal vectors $x,y\in\mathscr{H}_p$ one has*

$$R_x^*(y)=h_pR_x(y)-3A_xA_x(y). \tag{4.2}$$

Proof. From standard formulas on the curvature tensors of a submersion (see [16], pag. 13), we have

$$\begin{aligned} g_p(R_x^*(y),z) &= R_p^*(x,y,x,z)\\ &= R_p(x,y,x,z)+2g_p(A_x(y),A_x(z))-g_p(A_y(x),A_x(z))\\ &= g_p(h_pR_x(y),z)-3g_p(A_xA_x(y),z)\end{aligned}$$

for any $z\in x^\perp\cap\mathscr{H}_p$, which yields (4.2). □

Proposition 4.4. *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ be a Lorentzian $\mathscr{S}$-manifold, with $\dim(M)=2n+s$, $s\geqslant 1$. Let $\pi:M\to N$ be a principal $\mathbf{T}^s$-bundle over a Kähler manifold (N,J,G), as in Theorem 4.1. Let $p\in M$, and suppose that, for any $x\in S_\varphi((\xi_1)_p)$, φx is an eigenvector of R_x. Then, M is φ-null Osserman with respect to $(\xi_1)_p$ if and only if N is Osserman at $p'=\pi(p)$.*

Proof. Suppose first that $s\geqslant 2$. Fix $p'\in N$, with $p'=\pi(p)$, $p\in M$, and let $x'\in T_{p'}N$ a unit vector, and $y',z'\in x'^\perp$. Let $x\in S_\varphi((\xi_1)_p)$, $V=x^\perp\cap\operatorname{Im}(\varphi_p)$ and $y,z\in V$ such that $x'=(d\pi)_p(x)$, $y'=(d\pi)_p(y)$ and $z'=(d\pi)_p(z)$. Then

$$g_p(R_x^*(y),z)=G_{p'}((d\pi)_p(R_x^*(y)),(d\pi)_p(z))=G_{p'}(R'_{x'}(y'),z'),$$

which implies that the Jacobi operators $R_x^*:V\to V$ and $R'_{x'}:x'^\perp\to x'^\perp$ have the same characteristic polynomial. Using (4.1) one has $A_xA_x(y)=-(s-2)g_p(y,\varphi x)\varphi x$ and since R_x leaves the subspace V invariant, (4.2) gives

$$R_x^*(y)=R_x(y)+3(s-2)g_p(y,\varphi x)\varphi x$$

for any $y\in V$. Observe that if φx is an eigenvector of R_x, we have

$$g_p(R_x(y),\varphi x)\varphi x=g_p(y,R_x(\varphi x))\varphi x=R_x(g_p(y,\varphi x)\varphi x),$$

that is the endomorphism of V such that $y\mapsto g_p(y,\varphi x)\varphi x$ commutes with R_x. This implies they are simultaneously diagonalizable, and if λ_i, $i\in\{1,\dots,r\}$ are the eigenvalues of R_x, counted with multiplicities, with λ_1 relative to φx, then $\lambda_1+3(s-2)$, λ_j, $j\in\{2,\dots,r\}$ are the eigenvalues of R_x^*. By Proposition 3.3 we obtain our statement.

If $s=1$ then the proof goes through as above, except for the fact that one has $A_xA_x(y)=g_p(y,\varphi x)\varphi x$. □

Remark 4.5. It is clear, from the previous proof, that in case $s=2$ the hypothesis of φx being an eigenvector of R_x can be dropped without affecting the result. Furthermore, in case $s=1$, the statement is relative to the projection π' of the commutative diagram in the Theorem 4.1.

LEMMA 4.6. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold, with* $\dim(M) = 2n+s$, $s \geqslant 2$. *Let $\tau : M \to M'$ be a principal $\mathbf{T}^{s-1}$-bundle over a Lorentz Sasakian manifold, as in Theorem* 4.1. *We have*:

$$A_X Y = -g(X, \varphi Y) \sum_{\alpha=2}^{s} \xi_\alpha, \quad A_X \xi_\alpha = -\varphi X, \tag{4.3}$$

for any $X, Y \in \operatorname{Im}(\varphi) \oplus \operatorname{span}(\xi_1)$ and any $\alpha \in \{2, \dots, s\}$.

Proof. By construction of τ, we have the splitting $\mathcal{H}_p = \operatorname{Im}(\varphi_p) \oplus \operatorname{span}(\xi_1)$ and $\mathcal{V}_p = \operatorname{span}((\xi_2)_p, \dots, (\xi_s)_p)$ for any $p \in M$. Proceeding along the same lines as the proof of Lemma 4.2, we get (4.3). □

PROPOSITION 4.7. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold, with* $\dim(M) = 2n+s$, $s \geqslant 2$. *Let $\tau : M \to M'$ be a principal $\mathbf{T}^{s-1}$-bundle over a Lorentz Sasakian manifold M' with structure $(\varphi', \xi', \eta', g')$ as in Theorem* 4.1. *Let $p \in M$, and suppose that, for any $x \in S_\varphi((\xi_1)_p)$, φx is an eigenvector of R_x. Then, M is φ-null Osserman with respect to $(\xi_1)_p$ if and only if M' is null Osserman with respect to $\xi'_{p'}$, $p' = \tau(p)$.*

Proof. One can follow the same proof of Proposition 4.4 where, using (4.3), one has $A_x A_x(y) = -(s-1) g_p(y, \varphi x) \varphi x$. □

Propositions 4.4 and 4.7 can be summarized as follows.

THEOREM 4.8. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a compact, connected and regular Lorentzian $\mathcal{S}$-manifold, with* $\dim(M) = 2n+s$, $s \geqslant 2$. *Consider the commutative diagram of principal torus bundles*

$$\begin{tikzcd} M \arrow[rr, "\tau"] \arrow[dr, "\pi"'] & & M' \arrow[dl, "\pi'"] \\ & N & \end{tikzcd}$$

where N is a $2n$-dimensional compact Kähler manifold and M' is a $(2n+1)$-dimensional compact and regular Lorentz Sasakian manifold, with unit timelike characteristic vector field $\xi' = \tau_*(\xi_1)$. Let $p \in M$, and suppose that φx is an eigenvector of R_x for any $x \in S_\varphi((\xi_1)_p)$. The following three statements are equivalent.

(a) *M is φ-null Osserman with respect to $(\xi_1)_p$;*
(b) *N is Osserman at $q = \pi(p)$;*
(c) *M' is null Osserman with respect to $\xi'_{p'}$, $p' = \tau(p)$.*

Remark 4.9. It is clear that the three Osserman-type conditions in the above theorem can be also considered either pointwise or globally. Moreover, if we use the pointwise conditions, from the equivalence $(a) \Leftrightarrow (b)$ it follows that N is Einstein at each point and the connectedness implies that it is a Kähler-Einstein manifold.

Remark 4.10. In case $\tau : M \to M'$ is a principal $\mathbf{T}^{s-1}$-bundle from a Lorentzian $\mathscr{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ with $\dim(M) = 2n + s$, $s \geqslant 2$, over a Sasakian manifold M' with structure $(\varphi', \xi', \eta', g')$ as in Theorem 4.1, we could ask about the Osserman condition on M'. Let us suppose M' pointwise Osserman, since it is odd-dimensional, it has constant sectional curvature c ([12, 20]). Being $k(X', \xi') = 1$, for any $X' \in \mathrm{Im}(\varphi')$, then $c = 1$ and M' is locally isometric to the sphere $\mathbf{S}^{2n+1}$ with its standard Sasakian structure (see [2], p. 114). By construction of the bundle projection τ, we can suppose that $\mathscr{H}_p = \mathrm{Im}(\varphi_p) \oplus \mathrm{span}((\xi_s)_p)$ and $\mathscr{V}_p = \mathrm{span}((\xi_1)_p, \dots, (\xi_{s-1})_p)$. Hence, with calculations similar to those of Lemma 4.2, one has $A_X Y = g(Y, \varphi X) \sum_{\alpha=1}^{s-1} \xi_\alpha$. By standard formulas on sectional curvatures of the total and the base spaces of a semi-Riemannian submersion (see [16], p. 14) we have

$$k(x, \varphi x) = k'(x', \varphi' x') - 3g(A_x \varphi x, A_x \varphi x) = 1 - 3(s - 3), \quad x \in \mathrm{Im}(\varphi_p),$$

which gives a necessary condition on the φ-sectional curvature of M for M' to be an Osserman Sasakian manifold.

Remark 4.11. Analogously, in case $\tau : M \to M'$ is a principal $\mathbf{T}^{s-1}$-bundle from a Lorentzian $\mathscr{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, with $\dim(M) = 2n + s$, $s \geqslant 2$, over a Lorentz Sasakian manifold M' with structure $(\varphi', \xi', \eta', g')$ as in Theorem 4.1, we could ask again about the Osserman condition on M'. It is known that any connected Lorentzian Osserman manifold is a space-form ([20]), and since $k(X', \xi') = -1$, for any $X' \in \mathrm{Im}(\varphi')$, M' has constant sectional curvature $c = -1$. As in the previous calculations, using (4.3), we have

$$k(x, \varphi x) = k'(x', \varphi' x') - 3g(A_x \varphi x, A_x \varphi x) = -1 - 3(s - 1), \quad x \in \mathrm{Im}(\varphi_p),$$

which is a necessary condition on the φ-sectional curvature of M for M' to be a Lorentzian Osserman manifold.

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