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LEAF-WISE INTERSECTIONS IN COISOTROPIC SUBMANIFOLDS

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Abstract

The leaf-wise intersection on a coisotropic submanifold of a symplectic manifold is a generalization of the Lagrangian intersection investigated by Weinstein. In a similar way as Weinstein's argument, we replace the leaf-wise intersections by zero points of some closed 1-form, and show the same result as Moser's on the existence of leaf-wise intersections under different conditions.

1. Introduction

In [7], Weinstein considered the existence problem of periodic orbits of Hamiltonian systems. He combined the problem with Lagrangian intersections, and reduced it to find zero points of some closed 1-form.

On the other hand, Moser generalized Lagrangian intersections to leaf-wise intersections on coisotropic submanifolds, and obtained the following theorem:

Theorem 1.1 (Moser [6]). *Let $(P, \omega = d\lambda)$ be a simply connected exact symplectic manifold and M be a compact coisotropic submanifold of P. If a symplectomorphism $\psi \in \mathrm{Symp}(P, \omega)$ is C^1-close to the identity $\mathrm{id}_P : P \to P$, then ψ has at least $\mathrm{cat}(M)$ leaf-wise intersections.*

Here, $\mathrm{cat}(M)$ is the Lusternik-Schnirelmann category of M (see Definition 3.6).

In this paper, using Theorem 3.4 below proved by Weinstein, we show the following:

Theorem 1.2 (Main theorem). *Let (P, ω) be a symplectic manifold and M be a coisotropic submanifold of P. If $\psi \in \mathrm{Symp}(P, \omega)$ is C^1-close to the identity $\mathrm{id}_P : P \to P$, then there exist a closed 1-form Γ on M and an embedding $G : M \to P \times P$ so that $\mathrm{pr}_1 \circ G(p)$ is a leaf-wise intersection of ψ for each $p \in \mathrm{Zero}(\Gamma)$.*

Here, $\mathrm{Zero}(\Gamma)$ is the set of zero points of Γ and $\mathrm{pr}_i : P \times P \to P$ is the projections to the i-th component.

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By this theorem and Theorem 3.7 below, we obtain the same conclusion as Moser's theorem under different conditions:

Corollary 1.3. *Let* (P, ω) *be a symplectic manifold and* M *be a coisotropic submanifold of* P. *Assume that* M *is compact and the first de Rham cohomology* $H^1(M, \mathbf{R})$ *vanishes. If a symplectomorphism* $\psi \in \mathrm{Symp}(P, \omega)$ *is* C^1*-close to the identity* $\mathrm{id}_P : P \to P$, *then* ψ *has at least* $\mathrm{cat}(M)$ *leaf-wise intersections.*

There are many results on leaf-wise intersections by other researchers. These results were given for more restricted class of coisotropic submanifolds under a weaker condition on ψ. In [4], Hofer proved the existence of leaf-wise intersections for restricted contact type hypersurfaces in $\mathbf{R}^{2n}$. He introduced the norm on the space of compactly supported Hamiltonian diffeomorphisms as follows. For a compactly supported Hamiltonian function H, the norm of H is defined by

$$\|H\| := \sup H - \inf H$$

and for a compactly supported Hamiltonian diffeomorphism ψ, the norm of ψ is defined by

$$\|\psi\| := \inf\{\|H\| \mid \psi = \phi_H^1\}$$

where ϕ_H^t is the flow of the Hamiltonian vector field X_H. He replaced the C^1-closeness assumption by the condition that the norm of ψ is smaller than a certain symplectic capacity. Recently, this result has been generalized by many researchers. For example, Ginzburg [3] generalizes the ambient space $\mathbf{R}^{2n}$ to subcritical Stein manifolds, and Albers and Frauenfelder [1] to generic symplectic manifolds. Moreover, Albers and Frauenfelder give a bound for the number of the leaf-wise intersections by the total Betti number using Rabinowitz Floer homology.

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2. Preliminaries

Let (P, ω) be a $2n$-dimensional symplectic manifold, namely, P is a smooth manifold and ω is a nondegenerate closed 2-form on P. For a submanifold M of P and a point p in M, we define $(T_pM)^\omega$ by

$$(T_pM)^\omega := \{v \in T_pP \mid \omega(v, w) = 0 \text{ for all } w \in T_pM\}.$$

Note that the dimension of $(T_pM)^\omega$ is equal to the codimension of M since ω is nondegenerate.

Definition 2.1. A submanifold M is said to be *Lagrangian* if $(T_pM)^\omega = T_pM$ holds for each $p \in M$, and is said to be *coisotropic* if $(T_pM)^\omega \subset T_pM$ holds for each $p \in M$.

Definition 2.2. A diffeomorphism $\psi : P \to P$ is called *symplectomorphism* if $\psi^*\omega = \omega$ holds. We denote the set of symplectomorphisms by $\mathrm{Symp}(P, \omega)$.

Definition 2.3. Let $H : P \to \mathbf{R}$ be a smooth function. The *Hamiltonian vector field* X_H is defined by

$$i(X_H)\omega = dH.$$

We denote the flow of X_H by ϕ_H^t.

Let $\mathscr{L}(P, \omega)$ be the set of all Lagrangian submanifolds of (P, ω). Now, we introduce the C^1-topology on $\mathscr{L}(P, \omega)$.

First, we recall the definition of C^1-topology on $C^\infty(M, N)$ for manifolds M and N. Embed M and N in Euclidean spaces and consider the induced metrics on M and N. For each $f \in C^\infty(M, N)$ and $\delta \in C^0(M, \mathbf{R}_{>0})$, define $W(f, \delta)$ by

$$W(f, \delta) := \Big\{ g \in C^\infty(M, N) \mid d(f(x), g(x)) < \delta(x) \ \ (x \in M), \\ \frac{\|df_x(v) - dg_x(v)\|}{\|v\|} < \delta(x) \ \ (x \in M,\ v \in T_xM \backslash \{0\}) \Big\}.$$

The C^1-topology on $C^\infty(M, N)$ is the topology generated by $\{W(f, \delta)\}_{f, \delta}$.

Next, we define the C^1-topology on $\mathscr{L}(P, \omega)$. Let M be a manifold such that $\dim M = (1/2) \dim P$. For a closed subset $A \subset M$ and a C^1-open subset $\mathscr{A} \subset C^\infty(A, P)$, define $\mathscr{N}_{A, \mathscr{A}}$ by

$$\mathscr{N}_{A, \mathscr{A}} := \{ L \in \mathscr{L}(P, \omega) \mid f(A) \subset L \ \text{for some} \ f \in \mathscr{A} \}.$$

The C^1-topology on $\mathscr{L}(P, \omega)$ is the topology generated by $\{\mathscr{N}_{A, \mathscr{A}}\}_{A, \mathscr{A}}$.

3. Weinstein's theorem on Lagrangian intersections

Definition 3.1. Let M and N be submanifolds of P. Then M and N *intersect cleanly* if $\Sigma := M \cap N$ is a submanifold of P satisfying

$$T_x\Sigma = T_xM \cap T_xN$$

for each $x \in \Sigma$. In this case, Σ is called a *clean intersection* of M and N.

Remark 3.2. When $\Sigma = M \cap N$ is a submanifold of P, we always have the inclusion $T_x\Sigma \subset T_xM \cap T_xN$.

Example 3.3. (1) Let M and N be submanifolds of P. If M and N intersect transversely, then they intersect cleanly.

(2) Let P be a vector space. Then any linear subspaces M and N of P intersect cleanly.

(3) Define $P = \mathbf{R}^3$, $M = \{(x, y, z) \in P \mid y = x^2\}$, and $N = \{(x, y, z) \in P \mid y = 2x^2\}$. Then M and N do not intersect cleanly since $T_p(M \cap N) = T_pM \cap T_pN$ does not hold for every $p \in M \cap N$.

THEOREM 3.4 (Weinstein [7]). *Let L_1 and L_2 be Lagrangian submanifolds of a symplectic manifold (P, ω) and assume that they intersect cleanly. Then there exists a C^1-neighborhood $\mathcal{N}_1 \times \mathcal{N}_2$ of $(L_1, L_2) \in \mathcal{L}(P, \omega) \times \mathcal{L}(P, \omega)$ such that for each $(L_1', L_2') \in \mathcal{N}_1 \times \mathcal{N}_2$ there exists a closed 1-form Γ on $\Sigma := L_1 \cap L_2$ and an embedding $G : \Sigma \to P$ which satisfy $G(p) \in L_1' \cap L_2'$ for each $p \in \mathrm{Zero}(\Gamma)$.*

This theorem is based on the following example:

Example 3.5. Consider the cotangent bundle $P := T^*M$ of a manifold M. We have the natural symplectic structure ω_M on T^*M locally written by

$$\omega_M = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

Here, $(x_1, \dots, x_n)$ is a local coordinate on M and $(y_1, \dots, y_n)$ is the fiber coordinate with respect to $(\partial/\partial x_1, \dots, \partial/\partial x_n)$.

Let α be a 1-form on M. Then $\alpha(M) \subset T^*M$ is a Lagrangian submanifold if and only if α is a closed form. Such a Lagrangian submanifold is said to be horizontal. Let $L_1 = \alpha_1(M)$ and $L_2 = \alpha_2(M)$ be horizontal Lagrangian submanifolds. Define a closed 1-form Γ on M and an embedding $G : M \to T^*M$ by

$$\Gamma := \alpha_2 - \alpha_1,$$

$$G := \frac{1}{2}(\alpha_1 + \alpha_2).$$

Then, it is easy to see that $G(p) \in L_1 \cap L_2$ holds for each $p \in \mathrm{Zero}(\Gamma)$.

Now, we estimate the number of elements of Lagrangian intersections.

DEFINITION 3.6. Let M be a topological space. The *Lusternik-Schnirelmann category* (or simply *LS category*) of M is the least number of contractible open sets which cover M. We denote the LS category by $\mathrm{cat}(M)$. If we cannot cover M by finite contractible open sets, then we define $\mathrm{cat}(M) = +\infty$.

THEOREM 3.7 (Lusternik-Schnirelmann [5]). *Let M be a compact manifold. Then, any smooth function $f \in C^\infty(M)$ has at least* $\mathrm{cat}(M)$ *critical points.*

Thus we obtain the following from Theorem 3.4:

Corollary 3.8 (Weinstein [7]). *Let L_1 and L_2 be Lagrangian submanifolds of (P,ω) intersecting cleanly. Assume that $\Sigma := L_1 \cap L_2$ is compact and the first de Rham cohomology $H^1(\Sigma, \mathbf{R})$ vanishes. Then there exists a C^1-neighborhood $\mathcal{N}_1 \times \mathcal{N}_2$ of $(L_1, L_2) \in \mathcal{L}(P,\omega) \times \mathcal{L}(P,\omega)$ so that the set $L_1' \cap L_2'$ has at least* $\mathrm{cat}(\Sigma)$ *points for each $(L_1', L_2') \in \mathcal{N}_1 \times \mathcal{N}_2$.*

Remark 3.9. This corollary is obvious when $\dim(L_1' \cap L_2') \geq 1$.

4. Leaf-wise intersections

In this section, we introduce leaf-wise intersections and give some examples. Let M be a coisotropic submanifold of P whose codimension is r. $(TM)^{\omega}$ defines a distribution on M of rank r which satisfies the following property.

Lemma 4.1. *For a coisotropic submanifold M, $(TM)^{\omega}$ is a completely integrable distribution on M.*

Proof. Let V_1 and V_2 be vector fields on M which is tangent to the distribution $(TM)^{\omega}$. It is sufficient to show that the vector field $[V_1, V_2]$ also belongs to $(TM)^{\omega}$ by the Frobenius Theorem. For any vector field W on M, we have

$$\omega([V_1, V_2], W) = -d\omega(V_1, V_2, W) + V_1\omega(V_2, W) - V_2\omega(V_1, W) \\ + W\omega(V_1, V_2) + \omega([V_1, W], V_2) - \omega([V_2, W], V_1).$$

The first term vanishes since ω is closed. We remark here that $\omega(\tilde{V}, \tilde{W}) = 0$ for any vector fields $\tilde{V} \in TM$ and $\tilde{W} \in (TM)^{\omega}$. Therefore all the other terms vanish and then we obtain $[V_1, V_2] \in (TM)^{\omega}$. □

By this lemma, $(TM)^{\omega}$ defines a foliation on M. We call this foliation the characteristic foliation. We denote the leaf of $(TM)^{\omega}$ through p by L_p. The dimension of the leaf L_p is equal to the rank r of $(TM)^{\omega}$.

Definition 4.2. Let M be a coisotropic submanifold of P. Then $p \in M$ is called a *leaf-wise intersection* of $\psi \in \mathrm{Symp}(P,\omega)$ if $\psi(p) \in L_p$.

Example 4.3. (1) $M = P$ is a coisotropic submanifold of P with $r = 0$. If $p \in M$ is a leaf-wise intersection of $\psi \in \mathrm{Symp}(P,\omega)$, then the point p is a fixed point of ψ since the characteristic leaf is given by $L_p = \{p\}$.

(2) Let M be a connected Lagrangian submanifold of P. Then M is a coisotropic submanifold of P with $r = n$ and the characteristic leaf L_p is nothing but M. If $p \in M$ is a leaf-wise intersection of $\psi \in \mathrm{Symp}(P,\omega)$, then the point p belongs to $M \cap \psi^{-1}(M)$. Note that $\psi^{-1}(M)$ is also a Lagrangian submanifold of P since ψ^{-1} preserves the symplectic structure ω. Thus the leaf-wise intersections of ψ are the Lagrangian intersections of M and $\psi^{-1}(M)$.

(3) Let $H_1, \dots, H_r \in C^\infty(P)$ be Poisson commuting functions and assume that $dH_1, \dots, dH_r$ are linearly independent. Then, a regular level set $M = H^{-1}(c)$ of $H = (H_1, \dots, H_r)$ is a coisotropic submanifold and the characteristic leaf is given by

$$L_p = \{\phi_{H_1}^{t_1} \circ \cdots \circ \phi_{H_r}^{t_r}(p) \mid t_1, \dots, t_r \in \mathbf{R}\}.$$

5. Proof of the main theorem

For a leaf-wise intersection $p \in M$, the pair $(p, \psi(p)) \in M \times M$ satisfies $\psi(p) \in L_p$. Then, we consider $\tilde{\mathscr{M}} := \{(p,q) \in M \times M \mid q \in L_p\}$. In general, $\tilde{\mathscr{M}}$ is not an embedded submanifold but immersed submanifold in $M \times M$. So we define $\mathscr{M}$ by the set of pairs $(p,q) \in \tilde{\mathscr{M}}$ with p and q being connected by a path in L_p of length less than $\varepsilon(p)$. Here ε is a suitable positive continuous function on M such that $\mathscr{M}$ is an embedded submanifold in $M \times M$. Note that the dimension of $\mathscr{M}$ is $2n$.

First, we show that $\mathscr{M}$ is a Lagrangian submanifold of $(P \times P, \omega_\times)$, where $\omega_\times$ is defined by $\omega_\times = \mathrm{pr}_1^* \omega - \mathrm{pr}_2^* \omega$. It is sufficient to show $\omega_\times = 0$ on $T_{(p,q)}\mathscr{M}$ at each points $(p,q) \in \mathscr{M}$. From now on, we fix a point $(p,q) \in \mathscr{M}$. Choose a diffeomorphism $\phi : M \to M$ which satisfies $\phi(p) = q$ and preserves the leaves. Note that it is sufficient to define ϕ in a neighborhood of p.

Claim 1. *$\phi^*\omega|_M = \omega|_M$ holds on some neighborhood $M \cap U$ of p.*

We choose a family of leaf-preserving diffeomorpfisms $\phi_s : M \to M$ $(0 \le s \le 1)$ which satisfies $\phi_0 = \mathrm{id}_M$ and $\phi_1 = \phi$. Define a vector field W_s by

$$W_s := \frac{d}{ds}\phi_s$$

then W_s belongs to $(TM)^\omega$ because every ϕ_s preserves leaves. On the other hand, since ω is a closed 2-form, there is a local 1-form α on some neighborhood U of p such that $\omega = d\alpha$. We abbreviate $\alpha_M = \alpha|_{(M \cap U)}$, then

$$\begin{aligned}
\phi^*\alpha_M - \alpha_M &= \int_0^1 \left(\frac{d}{ds}\phi_s^*\alpha_M\right) ds \\
&= \int_0^1 (\phi_s^* \mathscr{L}_{W_s}\alpha_M)\, ds \\
&= \int_0^1 (\phi_s^*(i(W_s)d\alpha_M) + d\phi_s^*(i(W_s)\alpha_M))\, ds \\
&= d\int_0^1 \phi_s^*(i(W_s)\alpha_M)\, ds
\end{aligned}$$

holds and therefore we obtain $\phi^*\omega|_M = \omega|_M$ on $M \cap U$.

Claim 2. *For any* $\zeta = (\xi, \eta) \in T_{(p,q)}\mathscr{M}$, $\eta - d\phi(\xi) \in (T_q M)^{\omega}$ *holds.*

In fact, choose a curve $\gamma(t) = (p(t), q(t))$ on $\mathscr{M}$ such that

$$\gamma(0) = (p, q), \quad \dot{\gamma}(0) = \zeta.$$

Since $\gamma(t)$ is a curve on $\mathscr{M}$, $q(t)$ lies on the leaf $L_{p(t)}$ for each t. On the other hand, since ϕ preserves the leaves, $\phi(p(t))$ also lies on the leaf $L_{p(t)}$. Then $q(t)$ and $\phi(p(t))$ always lie on the same leaf. Therefore, we obtain

$$\eta - d\phi(\xi) = \frac{dq}{dt}(0) - \frac{d(\phi \circ p)}{dt}(0) \in T_q L_p = (T_q M)^{\omega}.$$

Claim 3. $\mathscr{M}$ *is a Lagrangian submanifold of the symplectic manifold* $(P \times P, \omega_{\times})$.

For $\zeta = (\xi, \eta)$, $\tilde{\zeta} = (\tilde{\xi}, \tilde{\eta}) \in T_{(p,q)}\mathscr{M}$,

$$\begin{aligned} \omega_{\times}(\zeta, \tilde{\zeta}) &= \omega|_M(\xi, \tilde{\xi}) - \omega|_M(\eta, \tilde{\eta}) \\ &= \omega|_M(\xi, \tilde{\xi}) - \omega|_M(d\phi(\xi), d\phi(\tilde{\xi})) \\ &= \omega|_M(\xi, \tilde{\xi}) - \phi^*\omega|_M(\xi, \tilde{\xi}) \\ &= 0. \end{aligned}$$

Thus $\omega_{\times} = 0$ holds on $T_{(p,q)}\mathscr{M}$, and we can see that $\mathscr{M}$ is a Lagrangian submanifold of the symplectic manifold $(P \times P, \omega_{\times})$.

Claim 4. *Two Lagrangian submanifolds* $\mathscr{M}$ *and* $\Delta_P := \{(p, p) \mid p \in P\}$ *intersect cleanly along* $\mathscr{M} \cap \Delta_P = \Delta_M$.

It is sufficient to show

$$T_{(p,p)}\Delta_M \supset T_{(p,p)}\mathscr{M} \cap T_{(p,p)}\Delta_P$$

holds at each points $(p, p) \in \Delta_M$ (see Remark 3.2). For any vector $\zeta = (\xi, \eta) \in T_{(p,p)}\mathscr{M} \cap T_{(p,p)}\Delta_P$, ξ is equal to η since $\zeta = (\xi, \eta)$ is tangent to Δ_P. In addition, ξ belongs to $T_p M$ since $\zeta = (\xi, \xi)$ is tangent to $\mathscr{M} \subset M \times M$. Therefore $\zeta = (\xi, \xi) \in T_{(p,p)}\Delta_M$ which implies the claim.

If $\psi \in \mathrm{Symp}(P, \omega)$ is C^1-close to the identity $\mathrm{id}_P : P \to P$, then Δ_P and $\mathrm{Graph}(\psi)$ are also C^1-close to each other. Applying Theorem 3.4 to $(L_1, L_2) = (\mathscr{M}, \Delta_P)$ and $(L_1', L_2') = (\mathscr{M}, \mathrm{Graph}(\psi))$, we know that there exists a closed 1-form Γ on M and an embedding $G : M \to P \times P$ such that $G(p) \in \mathscr{M} \cap \mathrm{Graph}(\psi)$ for all $p \in \mathrm{Zero}(\Gamma)$. On the other hand, we obtain

$$\mathscr{M} \cap \mathrm{Graph}(\psi) \subset \{(p, \psi(p)) \in M \times M \mid \psi(p) \in L_p\}.$$

Thus $\psi(\mathrm{pr}_1 \circ G(p)) \in L_{\mathrm{pr}_1 \circ G(p)}$ follows for each $p \in \mathrm{Zero}(\Gamma)$. Therefore $\mathrm{pr}_1 \circ G(p)$ is a leaf-wise intersection of ψ for each $p \in \mathrm{Zero}(\Gamma)$.

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