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ON MEROMORPHIC FUNCTIONS SHARING A ONE-POINT SET AND THREE TWO-POINT SETS CM

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Abstract

We show that if two meromorphic functions sharing a one-point set and three two-point sets CM, then one of them is a Möbius transform of the other.

1. Introduction

For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a finite set S in $\bar{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where the notations $f - \infty$ and $g - \infty$ mean $1/f$ and $1/g$, respectively. Also, if $f^{-1}(S) = g^{-1}(S)$, then we say that f and g share S IM (ignoring multiplicities). In particular if S is a one-point set $\{a\}$, then we say also that f and g share a CM or IM.

In [N], R. Nevanlinna showed the following theorem:

Theorem A. *Let f and g be two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ and let $a_1, \dots, a_4$ be four distinct points in $\bar{\boldsymbol{C}}$. If f and g share $a_1, \dots, a_4$ CM, then f is a Möbius transform of g, i.e., $f = (ag + b)/(cg + d)$ for some complex numbers a, b, c, d with $ad - bc \neq 0$, and there exists a permutation σ of $\{1, 2, 3, 4\}$ such that $a_{\sigma(3)}$ and $a_{\sigma(4)}$ are Picard exceptional values of f and g and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$.*

In [T] Tohge considered two meromorphic functions sharing 1, -1, ∞ and a two-point set containing none of them and Theorem 4 in [T] induces the following

Theorem B. *Let S_1, S_2, S_3 be one-point sets in $\bar{\boldsymbol{C}}$ and let S_4 be a two-point set in $\bar{\boldsymbol{C}}$. Assume that S_1, S_2, S_3, S_4 are pairwise disjoint. If two nonconstant*

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meromorphic functions f and g on $\mathbf{C}$ share S_1, S_2, S_3, S_4 CM, then f is a Möbius transform of g.

Also, Theorem 1.2 in [ST] and its proof induce

Theorem C. *Let S_1, S_2 be one-point sets in $\overline{\mathbf{C}}$ and let S_3, S_4 be two two-point sets in $\overline{\mathbf{C}}$. Assume that S_1, S_2, S_3, S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\mathbf{C}$ share S_1, S_2, S_3, S_4 CM, then f is a Möbius transform of g.*

In this paper we consider two meromorphic functions on $\mathbf{C}$ sharing a one-point set and three two-point sets in $\overline{\mathbf{C}}$ CM.

Theorem 1.1. *Let S_0 be a one-point set in $\overline{\mathbf{C}}$ and let S_1, S_2, S_3 be three two-point sets in $\overline{\mathbf{C}}$. Suppose that S_0, S_1, S_2 and S_3 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\mathbf{C}$ share $S_0, \dots, S_3$ CM, then f is a Möbius transform of g.*

We give a conjecure.

Conjecture. *Let $S_1, \dots, S_4$ be pairwise disjoint one-point or two-point sets in $\overline{\mathbf{C}}$. If two nonconstant meromorphic functions f and g share $S_1, \dots, S_4$ CM, then there exists a Möbius transformation T such that $f = T \circ g$.*

This conjecture is true for the cases that the number of one-point sets are four, three and two, and now Theorem 1.1 shows that it is true for the case that the number is one-point sets is one. The remaining problem is the case that the number of one-point sets is zero.

We assume that the reader is familiar with the standard notations and results of the value distribution theory (see, for example, [H]). In particular, we express by $S(r,f)$ quantities such that $\lim_{r\to\infty, r\notin E} S(r,f)/T(r,f) = 0$, where E is a subset of $(0,\infty)$ with finite linear measure and it is variable in each case.

We close the section by giving a generalization of Theorem A, which is a constant target version of Theorem 1 of [LY].

Lemma 1.2. *Let f and g be two nonconstant meromorphic functions on $\mathbf{C}$. Let $a_1, \dots, a_4$ be four distinct points in $\overline{\mathbf{C}}$ and let $b_1, \dots, b_4$ be four distinct points in $\overline{\mathbf{C}}$. If $f - a_j$ and $g - b_j$ share zero CM $(j = 1, \dots, 4)$, then f is a Möbius transform of g.*

2. Representations of rank N and some lemmas

In this section we introduce the definition of representations of rank N. Let G be a torsion-free abelian multiplicative group, and consider a q-tuple $A = (a_1, \dots, a_q)$ of elements a_i in G.

Definition 2.1. Let N be a positive integer. We call integers μ_j representations of rank N of a_j if

$$\prod_{j=1}^{q} a_j^{\varepsilon_j} = \prod_{j=1}^{q} a_j^{\varepsilon_j'} \tag{2.1}$$

and

$$\sum_{j=1}^{q} \varepsilon_j \mu_j = \sum_{j=1}^{q} \varepsilon_j' \mu_j \tag{2.2}$$

are equivalent for any integers ε_j, ε_j' with $\sum_{j=1}^{q} |\varepsilon_j| \le N$ and $\sum_{j=1}^{q} |\varepsilon_j'| \le N$.

For the existence of representations of rank N, see [S].

For two entire functions α and β without zeros we say that they are equivalent if α/β is constant. Then we denote $\alpha \sim \beta$. This relation "equivalent" is an equivalence relation.

We introduce following Borel's Lemma, whose proof can be found, for example, on p. 186 of [La].

Lemma 2.2. *If entire functions $\alpha_0, \alpha_1, \dots, \alpha_n$ without zeros satisfy*

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 0,$$

then for each $j = 0, 1, \dots, n$ there exists some $k (\neq j)$ such that $\alpha_j \sim \alpha_k$, and the sum of all elements of each equivalence class in $\{\alpha_0, \dots, \alpha_n\}$ is zero.

Now we investigate the torsion-free abelian multiplicative group $G = \mathscr{E}/\mathscr{C}$, where $\mathscr{E}$ is the abelian group of entire functions without zeros and $\mathscr{C}$ is the subgroup of all non-zero constant functions. We represent by $[\alpha]$ the element of $\mathscr{E}/\mathscr{C}$ with the representative $\alpha \in \mathscr{E}$. Let $\alpha_1, \dots, \alpha_q$ be elements in $\mathscr{E}$. Take representations μ_j of rank N of $[\alpha_j]$. For $\alpha = \prod_{j=1}^{q} \alpha_j^{\varepsilon_j}$ we define its index $\mathrm{Ind}(\alpha)$ by $\sum_{j=1}^{q} \varepsilon_j \mu_j$. The indices depend only on $[\prod_{j=1}^{q} \alpha_j^{\varepsilon_j}]$ under the condition $\sum_{j=1}^{q} |\varepsilon_j| \le N$. Trivially $\mathrm{Ind}(1) = 0$, and hence $\mathrm{Ind}(\alpha) = 0$ and the constantness of α are equivalent, and $\mathrm{Ind}(\alpha) = \mathrm{Ind}(\alpha')$ is equivalent to that α/α' is constant, where $\alpha = \prod_{j=1}^{q} \alpha_j^{\varepsilon_j}$ and $\alpha' = \prod_{j=1}^{q} \alpha_j^{\varepsilon_j'}$ with $\sum_{j=1}^{q} |\varepsilon_j| \le N$ and $\sum_{j=1}^{q} |\varepsilon_j'| \le N$.

We use the following Lemma in the proof of Theorem 1.1 which is an application of Lemma 2.2 (for the proof see [ST, Lemma 2.3]).

Lemma 2.3. *Assume that there is a relation $\Psi(\alpha_1, \dots, \alpha_q) \equiv 0$ where $\Psi(X_1, \dots, X_q) \in \boldsymbol{C}[X_1, \dots, X_q]$ is a nonconstant polynomial of degree at most N of $X_1, \dots, X_q$. Then each term $aX_1^{\varepsilon_1} \cdots X_q^{\varepsilon_q}$ of $\Psi(X_1, \dots, X_q)$ has another term $bX_1^{\varepsilon_1'} \cdots X_q^{\varepsilon_q'}$ such that $\alpha_1^{\varepsilon_1} \cdots \alpha_q^{\varepsilon_q}$ and $\alpha_1^{\varepsilon_1'} \cdots \alpha_q^{\varepsilon_q'}$ have the same indices, where a and b are non-zero constants.*

3. A Lemma from the theory of general resultants

For the proof of Theorem 1.1 a result from the theorey of general resultants is represented in this section. We give it by proceeding as in [CLO, Chapter 3].

Let d be a positive integer and let F_1, F_2, F_3 be three homogeneous polynomials of degree d of X, Y, Z. Denote their Jacobian determinant by J:

$$J = \begin{vmatrix} \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial Z} \\ \frac{\partial F_2}{\partial X} & \frac{\partial F_2}{\partial Y} & \frac{\partial F_2}{\partial Z} \\ \frac{\partial F_3}{\partial X} & \frac{\partial F_3}{\partial Y} & \frac{\partial F_3}{\partial Z} \end{vmatrix}.$$

Lemma 3.1. *All the partial derivatives* $\frac{\partial J}{\partial X}, \frac{\partial J}{\partial Y}, \frac{\partial J}{\partial Z}$ *are zero at each non-trivial common zero of* F_1, F_2, F_3.

Proof. By Euler's relation

$$XJ = \begin{vmatrix} X\frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial Z} \\ X\frac{\partial F_2}{\partial X} & \frac{\partial F_2}{\partial Y} & \frac{\partial F_2}{\partial Z} \\ X\frac{\partial F_3}{\partial X} & \frac{\partial F_3}{\partial Y} & \frac{\partial F_3}{\partial Z} \end{vmatrix} = d\begin{vmatrix} F_1 & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial Z} \\ F_2 & \frac{\partial F_2}{\partial Y} & \frac{\partial F_2}{\partial Z} \\ F_3 & \frac{\partial F_3}{\partial Y} & \frac{\partial F_3}{\partial Z} \end{vmatrix} \tag{3.1}$$

and

$$YJ = d\begin{vmatrix} \frac{\partial F_1}{\partial X} & F_1 & \frac{\partial F_1}{\partial Z} \\ \frac{\partial F_2}{\partial X} & F_2 & \frac{\partial F_2}{\partial Z} \\ \frac{\partial F_3}{\partial X} & F_3 & \frac{\partial F_3}{\partial Z} \end{vmatrix}, \quad ZJ = d\begin{vmatrix} \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & F_1 \\ \frac{\partial F_2}{\partial X} & \frac{\partial F_2}{\partial Y} & F_2 \\ \frac{\partial F_3}{\partial X} & \frac{\partial F_3}{\partial Y} & F_3 \end{vmatrix} \tag{3.2}$$

are obtained. Let $P = (X_0, Y_0, Z_0)$ be a non-trivial common zero of F_1, F_2 and F_3. Since $F_j(P) = 0$ for $j = 1, 2, 3$, all XJ, YJ and ZJ have zero at P by (3.1) and (3.2). Hence $J(P) = 0$ because at least one of X_0, Y_0, Z_0 is not zero. By differentiating (3.1) by X, Y, Z we get

$$J + X\frac{\partial J}{\partial X} = dJ + d\begin{vmatrix} F_1 & \frac{\partial^2 F_1}{\partial X \partial Y} & \frac{\partial F_1}{\partial Z} \\ F_2 & \frac{\partial^2 F_2}{\partial X \partial Y} & \frac{\partial F_2}{\partial Z} \\ F_3 & \frac{\partial^2 F_3}{\partial X \partial Y} & \frac{\partial F_3}{\partial Z} \end{vmatrix} + d\begin{vmatrix} F_1 & \frac{\partial F_1}{\partial Y} & \frac{\partial^2 F_1}{\partial X \partial Z} \\ F_2 & \frac{\partial F_2}{\partial Y} & \frac{\partial^2 F_2}{\partial X \partial Z} \\ F_3 & \frac{\partial F_3}{\partial Y} & \frac{\partial^2 F_3}{\partial X \partial Z} \end{vmatrix},$$

$$X\frac{\partial J}{\partial Y} = d\begin{vmatrix} F_1 & \dfrac{\partial^2 F_1}{\partial Y^2} & \dfrac{\partial F_1}{\partial Z} \\ F_2 & \dfrac{\partial^2 F_2}{\partial Y^2} & \dfrac{\partial F_2}{\partial Z} \\ F_3 & \dfrac{\partial^2 F_3}{\partial Y^2} & \dfrac{\partial F_3}{\partial Z} \end{vmatrix} + d\begin{vmatrix} F_1 & \dfrac{\partial F_1}{\partial Y} & \dfrac{\partial^2 F_1}{\partial Y \partial Z} \\ F_2 & \dfrac{\partial F_2}{\partial Y} & \dfrac{\partial^2 F_2}{\partial Y \partial Z} \\ F_3 & \dfrac{\partial F_3}{\partial Z} & \dfrac{\partial^2 F_3}{\partial Y \partial Z} \end{vmatrix},$$

$$X\frac{\partial J}{\partial Z} = d\begin{vmatrix} F_1 & \dfrac{\partial^2 F_1}{\partial Y \partial Z} & \dfrac{\partial F_1}{\partial Z} \\ F_2 & \dfrac{\partial^2 F_2}{\partial Y \partial Z} & \dfrac{\partial F_2}{\partial Z} \\ F_3 & \dfrac{\partial^2 F_3}{\partial Y \partial Z} & \dfrac{\partial F_3}{\partial Z} \end{vmatrix} + d\begin{vmatrix} F_1 & \dfrac{\partial F_1}{\partial Y} & \dfrac{\partial^2 F_1}{\partial Z^2} \\ F_2 & \dfrac{\partial F_2}{\partial Y} & \dfrac{\partial^2 F_2}{\partial Z^2} \\ F_3 & \dfrac{\partial F_3}{\partial Y} & \dfrac{\partial^2 F_3}{\partial Z^2} \end{vmatrix}.$$

Hence $X\dfrac{\partial J}{\partial X}$, $X\dfrac{\partial J}{\partial Y}$, $X\dfrac{\partial J}{\partial Z}$ are all zero at P, and so are $Y\dfrac{\partial J}{\partial X}$, $Y\dfrac{\partial J}{\partial Y}$, $Y\dfrac{\partial J}{\partial Z}$, $Z\dfrac{\partial J}{\partial X}$, $Z\dfrac{\partial J}{\partial Y}$, $Z\dfrac{\partial J}{\partial Z}$ by the same way. As a conclusion all $\dfrac{\partial J}{\partial X}$, $\dfrac{\partial J}{\partial Y}$, $\dfrac{\partial J}{\partial Z}$ are zero at each common zero of F_1, F_2, F_3. □

COROLLARY 3.2. *Let*

$$P_j(z) = a_{j1}X^2 + a_{j2}Y^2 + a_{j3}Z^2 + a_{j4}XY + a_{j5}YZ + a_{j6}ZX \quad (j = 1, 2, 3)$$

be three quadratic homogeneous polynomials and let J be their Jacobian matrix. Suppose

$$\frac{\partial J}{\partial X} = a_{41}X^2 + a_{42}Y^2 + a_{43}Z^2 + a_{44}XY + a_{45}YZ + a_{46}ZX,$$

$$\frac{\partial J}{\partial Y} = a_{51}X^2 + a_{52}Y^2 + a_{53}Z^2 + a_{54}XY + a_{55}YZ + a_{56}ZX,$$

$$\frac{\partial J}{\partial Z} = a_{61}X^2 + a_{62}Y^2 + a_{63}Z^2 + a_{64}XY + a_{65}YZ + a_{66}ZX.$$

If there exists a non-trivial common zero of P_1, P_2 and P_3, then the determinant $|a_{jk}|_{1 \le j,k \le 6}$ is zero.

4. Proof of Theorem 1.1

Now we start the proof of Theorem 1.1. For the conclusion we may assume that $S_0 = \{\infty\}$, and we set

$$S_j = \{z; z^2 + a_j z + b_j = 0\} \quad (j = 1, 2, 3).$$

Put $P_j(z) = z^2 + a_j z + b_j$. By assumption we have $a_j^2 - 4b_j \neq 0$ and $R_{jk} := R(P_j, P_k) = (b_k - b_j)^2 - (a_k - a_j)(a_j b_k - a_k b_j) \neq 0$ $(j \neq k)$, where $R(P_j, P_k)$ is the resultant of P_j and P_k, and there exist entire functions α_j without zeros such that $P_j(f) = \alpha_j P_j(g)$ $(j = 1, 2, 3)$.

We deny the conclusion and assume

(NM) there exists no Möbius transformation T such that $f = T \circ g$.

Proposition 4.1. *Each α_j is not constant.*

Proof. Assume that α_j is a constant c for some $j = 1, 2, 3$, and we may assume that $j = 1$. Then

$$f^2 + a_1 f + b_1 = c(g^2 + a_1 g + b_1).$$

If $c = 1$, then this leads $f = g$ or $f + g + a_1 = 0$, which contradicts (NM). If $c \neq 1$, then $f(z)$ and $g(z)$ are different values except ξ_1, η_1 and ∞ for each $z \in \boldsymbol{C}$. So $f - \eta_j$ and $g - \xi_j$ share zero CM for $j = 2, 3$, and $f - \xi_j$ and $g - \eta_j$ share zero CM for $j = 2, 3$. By Lemma 1.2, f is a Möbius transform of g, which is a contradiction. □

Proposition 4.2. *Each α_j/α_k is not constant for $1 \leq j < k \leq 3$.*

Proof. Assume that α_j/α_k is a constant c for some distinct j and k, and we may assume that $\alpha_1/\alpha_2 = c$. Then

$$\frac{f^2 + a_1 f + b_1}{f^2 + a_2 f + b_2} = c\frac{g^2 + a_1 g + b_1}{g^2 + a_2 g + b_2}.$$

If $c = 1$, then we get

$$(g - f)\{(a_1 - a_2)fg + (b_1 - b_2)(f + g) + (a_2 b_1 - a_1 b_2)\} = 0,$$

which yields a contradiction to (NM) immediately. Hence $c \neq 1$, and this implies that $f - \eta_3$ and $g - \xi_3$ share zero CM and that $f - \xi_3$ and $g - \eta_3$ share zero CM. Also we see that f and g have no poles. Hence there exit entire functions β_1, β_2 without zeros such that

$$f - \xi_3 = \beta_1(g - \eta_3), \quad f - \eta_3 = \beta_2(g - \xi_3). \tag{4.1}$$

By simple calculation we have

$$f = \frac{(\eta_3 - \xi_3)\beta_1\beta_2 + \eta_3\beta_1 - \xi_3\beta_2}{\beta_1 - \beta_2}, \quad g = \frac{\eta_3\beta_1 - \xi_3\beta_2 - \xi_3 + \eta_3}{\beta_1 - \beta_2},$$

and by substituting these into $f^2 + a_1 f + b_1 = \alpha_1(g^2 + a_1 g + b_1)$ we obtain

$$\begin{aligned}
&(\eta_3 - \xi_3)^2\beta_1^2\beta_2^2 + P_1(\eta_3)\beta_1^2 + P_1(\xi_3)\beta_2^2 + (\eta_3 - \xi_3)(2\eta_3 + a_1)\beta_1^2\beta_2 \\
&\qquad - (\eta_3 - \xi_3)(2\xi_3 + a_1)\beta_1\beta_2^2 - \{2\xi_3\eta_3 + a_1(\xi_3 + \eta_3) + 2b_1\}\beta_1\beta_2 \\
&\quad = \alpha_1[P_1(\eta_3)\beta_1^2 + P_1(\xi_3)\beta_2^2 + (\eta_3 - \xi_3)^2 - \{2\xi_3\eta_3 + a_1(\xi_3 + \eta_3) + 2b_1\}\beta_1\beta_2 \\
&\qquad + (\eta_3 - \xi_3)(2\eta_3 + a_1)\beta_1 - (\eta_3 - \xi_3)(2\xi_3 + a_1)\beta_2].
\end{aligned}$$

Note that any of $(\eta_3-\xi_3)^2$, $P_1(\xi_3)$, $P_1(\eta_3)$ are not zero. Take representations μ, v_1, v_2 of rank 4 of $[\alpha_1]$, $[\beta_1]$, $[\beta_2]$. Since β_1, β_2 and β_1/β_2 are not constant by (NM), we have $v_1 \neq 0$, $v_2 \neq 0$ and $v_1 \neq v_2$. We may assume that $v_1 < v_2$, and it is enough to consider the two cases (I) $0 < v_1 < v_2$ and (II) $v_1 < 0 < v_2$, by replacing indices or representations in other cases.

(I) The case where $0 < v_1 < v_2$. Then the term with the maximal index in the lefthand side is only $\beta_1^2\beta_2^2$ and the term with the maximal index in the righthand side is only $\alpha_1\beta_2^2$. Note that any of their coefficients are not zero. Hence we have $(\eta_3-\xi_3)^2\beta_1^2\beta_2^2 = P_1(\xi_3)\alpha_1\beta_2^2$ by Lemma 2.2 and Lemma 2.3. By considering the terms with the minimal index in each side we get also $P_1(\eta_3)\beta_1^2 = (\eta_3-\xi_3)^2\alpha_1$. Therefore $\beta_1^2/\alpha_1 = P_1(\xi_3)/(\eta_3-\xi_3)^2 = (\eta_3-\xi_3)^2/P_1(\eta_3)$ and hence

$$\frac{(f-\xi_3)^2}{(g-\eta_3)^2}\cdot\frac{P_1(g)}{P_1(f)} = \frac{P_1(\xi_3)}{(\eta_3-\xi_3)^2} = \frac{(\eta_3-\xi_3)^2}{P_1(\eta_3)}. \tag{4.2}$$

If $f^{-1}(\eta_3) = g^{-1}(\xi_3)$ is empty, then $f-\eta_3$ and $g-\xi_3$ are entire functions without zeros. Deform the first equation of (4.1) as

$$(f-\eta_3)+(\eta_3-\xi_3) = \beta_1\{(g-\xi_3)+(\xi_3-\eta_3)\}.$$

Since f, g and β are not constant, by Lemma 2.2 we have

$$f-\eta_3 = \beta_1(\xi_3-\eta_3), \quad \eta_3-\xi_3 = \beta_1(g-\xi_3).$$

However we get a contradiction $(f-\eta_3)/(\eta_3-\xi_3) = (\xi_3-\eta_3)/(g-\xi_3)$ to (NM).

Hence there exists a point z such that $f(z)=\eta_3$, $g(z)=\xi_3$, so we get by (4.2) $P_1(\eta_3) = (\eta_3-\xi_3)^2 = P_1(\xi_3)$, which leads $a_1 = a_3$. We take the Möbius transformation $T_0(z) = -z-a_3$ and put $h = T_0\circ g = -g-a_3$. Then h and f share ∞, ξ_3, η_3 CM since $h^{-1}(\xi_3) = g^{-1}(\eta_3) = f^{-1}(\xi_3)$, $h^{-1}(\eta_3) = g^{-1}(\xi_3) = f^{-1}(\eta_3)$, and they share S_1 CM since $h^{-1}(S_1) = h^{-1}(\xi_1)\cup h^{-1}(\eta_1) = g^{-1}(\eta_1)\cup g^{-1}(\xi_1) = g^{-1}(S_1)$. Hence by Theorem B, there exists a Möbius transformation T such that $f = T\circ h$, so we get a contradiction to (NM).

(II) The case where $v_1 < 0 < v_2$. Then the term with the maximail index in the lefthand side is only β_2^2 and the term with the maximal index in the righthand side is only $\alpha_1\beta_2^2$. Hence $\beta_2^2 \sim \alpha_1\beta_2^2$, so α_1 is a constant, which contradicts Propostion 3.1.

We complete the proof. □

Now, put

$$F_j = X^2 - \alpha_j Y^2 + b_j(1-\alpha_j)Z^2 + a_j XZ - a_j\alpha_j YZ \quad (j=1,2,3).$$

Then

$$\frac{\partial F_j}{\partial X} = 2X + a_j Z,$$
$$\frac{\partial F_j}{\partial Y} = -2\alpha_j Y - a_j\alpha_j Z,$$
$$\frac{\partial F_j}{\partial Z} = 2b_j(1-\alpha_j)Z + a_j X - a_j\alpha_j Y,$$

and the Jacobian matrix

$$J = -\begin{vmatrix} 2X + a_1 Z & 2\alpha_1 Y + a_1\alpha_1 Z & 2b_1(1-\alpha_1)Z + a_1 X - a_1\alpha_1 Y \\ 2X + a_2 Z & 2\alpha_2 Y + a_2\alpha_2 Z & 2b_2(1-\alpha_2)Z + a_2 X - a_2\alpha_2 Y \\ 2X + a_3 Z & 2\alpha_3 Y + a_3\alpha_3 Z & 2b_3(1-\alpha_3)Z + a_3 X - a_3\alpha_3 Y \end{vmatrix}$$
$$= -8D_1 XYZ - 4D_2 X^2 Y + 4D_3 XY^2 - 4D_4 XZ^2$$
$$- 2D_5 X^2 Z - 4D_6 YZ^2 + 2D_7 Y^2 Z - 2D_8 Z^3,$$

where

$$D_1 = \begin{vmatrix} 1 & \alpha_1 & b_1(1-\alpha_1) \\ 1 & \alpha_2 & b_2(1-\alpha_2) \\ 1 & \alpha_3 & b_3(1-\alpha_3) \end{vmatrix}$$
$$= (b_1 - b_2)\alpha_1\alpha_2 + (b_2 - b_3)\alpha_2\alpha_3 + (b_3 - b_1)\alpha_3\alpha_1$$
$$+ (b_2 - b_3)\alpha_1 + (b_3 - b_1)\alpha_2 + (b_1 - b_2)\alpha_3,$$

$$D_2 = \begin{vmatrix} 1 & \alpha_1 & a_1 \\ 1 & \alpha_2 & a_2 \\ 1 & \alpha_3 & a_3 \end{vmatrix} = (a_2 - a_3)\alpha_1 + (a_3 - a_1)\alpha_2 + (a_1 - a_2)\alpha_3,$$

$$D_3 = \begin{vmatrix} 1 & \alpha_1 & a_1\alpha_1 \\ 1 & \alpha_2 & a_2\alpha_2 \\ 1 & \alpha_3 & a_3\alpha_3 \end{vmatrix} = (a_2 - a_1)\alpha_1\alpha_2 + (a_3 - a_2)\alpha_2\alpha_3 + (a_1 - a_3)\alpha_3\alpha_1,$$

$$D_4 = \begin{vmatrix} 1 & a_1\alpha_1 & b_1(1-\alpha_1) \\ 1 & a_2\alpha_2 & b_2(1-\alpha_2) \\ 1 & a_3\alpha_3 & b_3(1-\alpha_3) \end{vmatrix}$$
$$= (a_2 b_1 - a_1 b_2)\alpha_1\alpha_2 + (a_3 b_2 - a_2 b_3)\alpha_2\alpha_3 + (a_1 b_3 - a_3 b_1)\alpha_3\alpha_1$$
$$+ a_1(b_2 - b_3)\alpha_1 + a_2(b_3 - b_1)\alpha_2 + a_3(b_1 - b_2)\alpha_3,$$

$$D_5 = \begin{vmatrix} 1 & a_1\alpha_1 & a_1 \\ 1 & a_2\alpha_2 & a_2 \\ 1 & a_3\alpha_3 & a_3 \end{vmatrix} = a_1(a_2 - a_3)\alpha_1 + a_2(a_3 - a_1)\alpha_2 + a_3(a_1 - a_2)\alpha_3,$$

$$D_6 = \begin{vmatrix} a_1 & \alpha_1 & b_1(1-\alpha_1) \\ a_2 & \alpha_2 & b_2(1-\alpha_2) \\ a_3 & \alpha_3 & b_3(1-\alpha_3) \end{vmatrix}$$

$$= a_3(b_1 - b_2)\alpha_1\alpha_2 + a_1(b_2 - b_3)\alpha_2\alpha_3 + a_2(b_3 - b_1)\alpha_3\alpha_1$$

$$+ (a_3b_2 - a_2b_3)\alpha_1 + (a_1b_3 - a_3b_1)\alpha_2 + (a_2b_1 - a_1b_2)\alpha_3,$$

$$D_7 = \begin{vmatrix} a_1 & \alpha_1 & a_1\alpha_1 \\ a_2 & \alpha_2 & a_2\alpha_2 \\ a_3 & \alpha_3 & a_3\alpha_3 \end{vmatrix} = a_3(a_2 - a_1)\alpha_1\alpha_2 + a_1(a_3 - a_2)\alpha_2\alpha_3 + a_2(a_1 - a_3)\alpha_3\alpha_1,$$

$$D_8 = \begin{vmatrix} a_1 & a_1\alpha_1 & b_1(1-\alpha_1) \\ a_2 & a_2\alpha_2 & b_2(1-\alpha_2) \\ a_3 & a_3\alpha_3 & b_3(1-\alpha_3) \end{vmatrix}$$

$$= a_3(a_2b_1 - a_1b_2)\alpha_1\alpha_2 + a_1(a_3b_2 - a_2b_3)\alpha_2\alpha_3 + a_2(a_1b_3 - a_3b_1)\alpha_3\alpha_1$$

$$+ a_1(a_3b_2 - a_2b_3)\alpha_1 + a_2(a_1b_3 - a_3b_1)\alpha_2 + a_3(a_2b_1 - a_1b_2)\alpha_3.$$

Moreover, for later, we put

$$D_0 = \begin{vmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2) + (a_3b_1 - a_1b_3) + (a_1b_2 - a_2b_1),$$

$$D_9 = \begin{vmatrix} 1 & b_1(1-\alpha_1) & a_1 \\ 1 & b_2(1-\alpha_2) & a_2 \\ 1 & b_3(1-\alpha_3) & a_3 \end{vmatrix} = (a_3 - a_2)b_1\alpha_1 + (a_1 - a_3)b_2\alpha_2 + (a_2 - a_1)b_3\alpha_3 - D_0,$$

$$D_{10} = \begin{vmatrix} \alpha_1 & b_1(1-\alpha_1) & a_1\alpha_1 \\ \alpha_2 & b_2(1-\alpha_2) & a_2\alpha_2 \\ \alpha_3 & b_3(1-\alpha_3) & a_3\alpha_3 \end{vmatrix}$$

$$= D_0\alpha_1\alpha_2\alpha_3 + (a_1 - a_2)b_3\alpha_1\alpha_2 + (a_2 - a_3)b_1\alpha_2\alpha_3 + (a_3 - a_1)b_2\alpha_3\alpha_1.$$

So we have

$$\frac{\partial J}{\partial X} = -8D_1YZ - 8D_2XY + 4D_3Y^2 - 4D_4Z^2 - 4D_5XZ,$$

$$\frac{\partial J}{\partial Y} = -8D_1XZ - 4D_2X^2 + 8D_3XY - 4D_6Z^2 + 4D_7YZ,$$

$$\frac{\partial J}{\partial Z} = -8D_1XY - 8D_4XZ - 2D_5X^2 - 8D_6YZ + 2D_7Y^2 - 6D_8Z^2.$$

Set

$$R^* = \frac{1}{64}\begin{vmatrix} 1 & -\alpha_1 & b_1(1-\alpha_1) & 0 & a_1 & -a_1\alpha_1 \\ 1 & -\alpha_2 & b_2(1-\alpha_2) & 0 & a_2 & -a_2\alpha_2 \\ 1 & -\alpha_3 & b_3(1-\alpha_3) & 0 & a_3 & -a_3\alpha_3 \\ 0 & 4D_3 & -4D_4 & -8D_2 & -4D_5 & -8D_1 \\ -4D_2 & 0 & -4D_6 & 8D_3 & -8D_1 & 4D_7 \\ -2D_5 & 2D_7 & -6D_8 & -8D_1 & -8D_4 & -8D_6 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha_1 & b_1(1-\alpha_1) & 0 & a_1 & a_1\alpha_1 \\ 1 & \alpha_2 & b_2(1-\alpha_2) & 0 & a_2 & a_2\alpha_2 \\ 1 & \alpha_3 & b_3(1-\alpha_3) & 0 & a_3 & a_3\alpha_3 \\ 0 & D_3 & D_4 & -D_2 & D_5 & -2D_1 \\ D_2 & 0 & D_6 & D_3 & 2D_1 & D_7 \\ D_5 & D_7 & 3D_8 & -2D_1 & 4D_4 & -4D_6 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha_1 & b_1(1-\alpha_1) \\ 1 & \alpha_2 & b_2(1-\alpha_2) \\ 1 & \alpha_3 & b_3(1-\alpha_3) \end{vmatrix} \cdot \begin{vmatrix} -D_2 & D_5 & -2D_1 \\ D_3 & 2D_1 & D_7 \\ -2D_1 & 4D_4 & -4D_6 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & \alpha_1 & a_1 \\ 1 & \alpha_2 & a_2 \\ 1 & \alpha_3 & a_3 \end{vmatrix} \cdot \begin{vmatrix} D_4 & -D_2 & -2D_1 \\ D_6 & D_3 & D_7 \\ 3D_8 & -2D_1 & -4D_6 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & \alpha_1 & a_1\alpha_1 \\ 1 & \alpha_2 & a_2\alpha_2 \\ 1 & \alpha_3 & a_3\alpha_3 \end{vmatrix} \cdot \begin{vmatrix} D_4 & -D_2 & D_5 \\ D_6 & D_3 & 2D_1 \\ 3D_8 & -2D_1 & 4D_4 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & b_1(1-\alpha_1) & a_1 \\ 1 & b_2(1-\alpha_2) & a_2 \\ 1 & b_3(1-\alpha_3) & a_3 \end{vmatrix} \cdot \begin{vmatrix} D_3 & -D_2 & -2D_1 \\ 0 & D_3 & D_7 \\ D_7 & -2D_1 & -4D_6 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & b_1(1-\alpha_1) & a_1\alpha_1 \\ 1 & b_2(1-\alpha_2) & a_2\alpha_2 \\ 1 & b_3(1-\alpha_3) & a_3\alpha_3 \end{vmatrix} \cdot \begin{vmatrix} D_3 & -D_2 & D_5 \\ 0 & D_3 & 2D_1 \\ D_7 & -2D_1 & 4D_4 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & a_1 & a_1\alpha_1 \\ 1 & a_2 & a_2\alpha_2 \\ 1 & a_3 & a_3\alpha_3 \end{vmatrix} \cdot \begin{vmatrix} D_3 & D_4 & -D_2 \\ 0 & D_6 & D_3 \\ D_7 & 3D_8 & -2D_1 \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_1 & b_1(1-\alpha_1) & a_1 \\ \alpha_2 & b_2(1-\alpha_2) & a_2 \\ \alpha_3 & b_3(1-\alpha_3) & a_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & -D_2 & -2D_1 \\ D_2 & D_3 & D_7 \\ D_5 & -2D_1 & -4D_6 \end{vmatrix}$$

$$- \begin{vmatrix} \alpha_1 & b_1(1-\alpha_1) & a_1\alpha_1 \\ \alpha_2 & b_2(1-\alpha_2) & a_2\alpha_2 \\ \alpha_3 & b_3(1-\alpha_3) & a_3\alpha_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & -D_2 & D_5 \\ D_2 & D_3 & 2D_1 \\ D_5 & -2D_1 & 4D_4 \end{vmatrix}$$

$$- \begin{vmatrix} \alpha_1 & a_1 & a_1\alpha_1 \\ \alpha_2 & a_2 & a_2\alpha_2 \\ \alpha_3 & a_3 & a_3\alpha_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & D_4 & -D_2 \\ D_2 & D_6 & D_3 \\ D_5 & 3D_8 & -2D_1 \end{vmatrix}$$

$$+ \begin{vmatrix} b_1(1-\alpha_1) & a_1 & a_1\alpha_1 \\ b_2(1-\alpha_2) & a_2 & a_2\alpha_2 \\ b_3(1-\alpha_3) & a_3 & a_3\alpha_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & D_3 & -D_2 \\ D_2 & 0 & D_3 \\ D_5 & D_7 & -2D_1 \end{vmatrix}$$

$$= -8D_1^4 + 16D_1^2D_2D_6 - 16D_1^2D_3D_4 - 2D_1^2D_5D_7 + 10D_1D_3D_5D_6$$
$$+ 10D_1D_2D_4D_7 - 8D_2D_3D_4D_6 - 7D_2^2D_7D_8 + 14D_1D_2D_3D_8 - 8D_2^2D_6^2$$
$$+ 7D_3^2D_5D_8 - 8D_3^2D_4^2 + 4D_3^2D_6D_9 + D_2D_7^2D_9 - 4D_1D_3D_7D_9 + D_3D_4D_5D_7$$
$$- D_2D_5D_6D_7 + 4D_1D_2D_5D_{10} + D_3D_5^2D_{10} - 4D_2^2D_4D_{10}.$$

Since $(X, Y, Z) = (f(z), g(z), 1)$ is a common zero of P_1, P_1, P_3 for each $z \in \boldsymbol{C}$ except poles of f or g, we have $R^* \equiv 0$ by Corollary 3.2.

Now we apply the results in §2 to the torsion-free abelian multiplicative group $G = \mathscr{E}/\mathscr{C}$. Let μ_1, μ_2, μ_3 be representations of rank 8 of $[\alpha_1]$, $[\alpha_2]$, $[\alpha_3]$. Then $\mu_j \neq 0$ since α_j are not constant by Proposition 4.1, and $\mu_j \neq \mu_k$ $(j \neq k)$ since $\alpha_j \not\sim \alpha_k$ $(j \neq k)$ by Proposition 4.2.

For $\alpha_1^j\alpha_2^k\alpha_k^l$, we call $j+k+l$ its total exponent.

The expansion of R^* is a linear combination of some of $\alpha_1^j\alpha_2^k\alpha_3^l$, $0 \le j,k,l \le 4$, $4 \le j+k+l \le 8$. In the expansion of R^* the maximal total exponent is 8 and the minimal total exponent is 4. The terms of total exponent 8 are produced only from $-8D_1^4 - 16D_1^2D_3D_4 - 8D_2^2D_4^2 = -8(D_1^2 + D_3D_4)^2$, and the part of total exponent 4 in the factor $D_1^2 + D_3D_4$ is

$$\{(b_1-b_2)\alpha_1\alpha_2 + (b_2-b_3)\alpha_2\alpha_3 + (b_3-b_1)\alpha_3\alpha_1\}^2$$
$$+ \{(a_2-a_1)\alpha_1\alpha_2 + (a_3-a_2)\alpha_2\alpha_3 + (a_1-a_3)\alpha_3\alpha_1\}$$
$$\times \{(a_2b_1-a_1b_2)\alpha_1\alpha_2 + (a_3b_2-a_2b_3)\alpha_2\alpha_3 + (a_1b_3-a_3b_1)\alpha_3\alpha_1\}$$
$$= R_{12}\alpha_1^2\alpha_2^2 + R_{23}\alpha_2^2\alpha_3^2 + R_{31}\alpha_3^2\alpha_1^2$$
$$+ \{2(b_1-b_2)(b_2-b_3) + (a_2-a_1)(a_3b_2-a_2b_3)$$
$$+ (a_3-a_2)(a_2b_1-a_1b_2)\}\alpha_1\alpha_2^2\alpha_3$$
$$+ \{2(b_2-b_3)(b_3-b_1) + (a_3-a_2)(a_1b_3-a_3b_1)$$
$$+ (a_1-a_3)(a_3b_2-a_2b_3)\}\alpha_1\alpha_2\alpha_3^2$$
$$+ \{2(b_3-b_1)(b_2-b_1) + (a_1-a_3)(a_2b_1-a_1b_2)$$
$$+ (a_2-a_1)(a_1b_3-a_3b_1)\}\alpha_1^2\alpha_2\alpha_3.$$

The terms of total exponent 4 are produced only from $-8D_1^4 + 16D_1^2D_3D_6 - 8D_2^2D_6^2 = -8(D_1^2 - D_2D_6)^2$, and the part of total exponent 2 in the factor

$D_1^2 - D_2D_6$ is

$$
\begin{aligned}
&\{(b_2 - b_3)\alpha_1 + (b_3 - b_1)\alpha_2 + (b_1 - b_2)\alpha_3\}^2 \\
&\qquad - \{(a_2 - a_3)\alpha_1 + (a_3 - a_1)\alpha_2 + (a_1 - a_2)\alpha_3\} \\
&\qquad \times \{(a_3b_2 - a_2b_3)\alpha_1 + (a_1b_3 - a_3b_1)\alpha_2 + (a_2b_1 - a_1b_2)\alpha_3\} \\
&\quad = R_{23}\alpha_1^2 + R_{31}\alpha_2^2 + R_{12}\alpha_3^2 \\
&\qquad + \{2(b_2 - b_3)(b_3 - b_1) + (a_2 - a_3)(a_1b_3 - a_3b_1) \\
&\qquad\quad + (a_3 - a_1)(a_3b_2 - a_2b_3)\}\alpha_1\alpha_2 \\
&\qquad + \{2(b_3 - b_1)(b_1 - b_2) + (a_3 - a_1)(a_2b_1 - a_1b_2) \\
&\qquad\quad + (a_1 - a_2)(a_1b_3 - a_3b_1)\}\alpha_2\alpha_3 \\
&\qquad + \{2(b_1 - b_2)(b_2 - b_3) + (a_1 - a_2)(a_3b_2 - a_2b_3) \\
&\qquad\quad + (a_2 - a_3)(a_2b_1 - a_1b_2)\}\alpha_3\alpha_1.
\end{aligned}
$$

Without loss of generality we may assume that $\mu_1 < \mu_2 < \mu_3$, and it is enough to consider two cases (I) $0 < \mu_1 < \mu_2 < \mu_3$ and (II) $\mu_1 < 0 < \mu_2 < \mu_3$ by taking $-\mu_j$ for μ_j if necessary.

(I) The case where $0 < \mu_1 < \mu_2 < \mu_3$. In this case $j\mu_1 + k\mu_2 + l\mu_3 \le 4(\mu_2 + \mu_3)$ for integers j, k, l with $0 \le j,k,l \le 4$, $4 \le j+k+l \le 8$, and the equality holds only for $(j,k,l) = (0,4,4)$. Hence only the term $-8R_{23}^2\alpha_2^4\alpha_3^4$ has the maximal index $4(\mu_2 + \mu_3)$, which contradict Lemma 2.3.

(II) The case where $\mu_1 < 0 < \mu_2 < \mu_3$. In this case $j\mu_1 + k\mu_2 + l\mu_3 \ge 4\mu_1$ for integers j, k, l with $0 \le j,k,l \le 4$, $4 \le j+k+l \le 8$, and the equality holds only for $(j,k,l) = (4,0,0)$. Hence $-8R_{23}^2\alpha_1^4$ is the unique term with the minimal index $4\mu_1$, which is a contradiction to Lemma 2.3.

So we have denied all cases, and the assumption (NM) is inconsistent, which completes the proof.

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