

## A FINITE GENERATING SET FOR THE LEVEL 2 MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

BLAŻEJ SZEPIETOWSKI

### Abstract

We obtain a finite set of generators for the level 2 mapping class group of a closed nonorientable surface of genus  $g \geq 3$ . This set consists of isotopy classes of Lickorish's Y-homeomorphisms also called crosscap slides.

### 1. Introduction

For a closed surface  $F$  the *mapping class group*  $\mathcal{M}(F)$  is the group of isotopy classes of all, orientation preserving if  $F$  is orientable, homeomorphisms  $h : F \rightarrow F$ . This is the orbifold fundamental group of the moduli space of Riemann surfaces homeomorphic to  $F$  if  $F$  is orientable, or Klein surfaces if  $F$  is nonorientable (Klein surface is a compact topological surface with a dianalytic structure, see [1]). Every finite index subgroup of  $\mathcal{M}(F)$  is the orbifold fundamental group of some finite orbifold cover of the moduli space. An important family of such subgroups is obtained as follows. For an integer  $m > 1$  define  $\Gamma_m(F)$  to be the subgroup of  $\mathcal{M}(F)$  consisting of the isotopy classes of homeomorphisms inducing the identity on  $H_1(F, \mathbf{Z}_m)$ , where  $\mathbf{Z}_m = \mathbf{Z}/m\mathbf{Z}$ . The group  $\Gamma_m(F)$  is called *level  $m$  mapping class group* and the corresponding finite cover of the moduli space is known as the moduli space of curves with level  $m$  structures. For orientable  $F$  these groups have been studied extensively, see [6, 7]. More recently, Putman [15] and Sato [16] computed independently the abelianization of  $\Gamma_m(F)$  for  $m$  odd and genus  $g \geq 3$ . Sato also computed the abelianization of  $\Gamma_2(F)$ .

In this paper we are interested in the case of a closed nonorientable surface, which will be denoted as  $N$  or  $N_g$ , where  $g$  is the genus (thus  $N_g$  is homeomorphic to the connected sum of  $g$  projective planes). Lickorish defined in [12] a homeomorphism of  $N$  that he called Y-homeomorphism and proved in [12, 13] that  $\mathcal{M}(N_g)$  is generated by Dehn twists and one isotopy class of Y-homeomorphisms for  $g \geq 2$ . Y-homeomorphisms were called crosscap slides in [10, 14] and also in this paper we use this name. Chillingworth found in [3] a finite set of generators for  $\mathcal{M}(N_g)$ .

---

2010 *Mathematics Subject Classification.* Primary 57N05, Secondary 20F38.

*Key words and phrases.* Mapping class group, nonorientable surface, crosscap slide, Y-homeomorphism.

Received February 16, 2012; revised May 16, 2012.

The action of  $\mathcal{M}(F)$  on  $H_1(F, \mathbf{Z})$  preserves the algebraic intersection pairing. For orientable  $F$  this is a symplectic form and we have a representation of  $\mathcal{M}(F)$  into the symplectic group, which is well known to be surjective and whose kernel is known as the *Torelli group*. On a nonorientable surface  $N$  however, the algebraic intersection pairing is only defined modulo 2 and therefore it is very natural to study the action of  $\mathcal{M}(N)$  on  $H_1(N, \mathbf{Z}_2)$  and its kernel  $\Gamma_2(N)$ . It was proved by McCarthy and Pinkall [14] and by Gadgil and Pancholi [4] that all automorphisms of  $H_1(N_g, \mathbf{Z})$  or  $H_1(N_g, \mathbf{Z}_2)$  preserving the  $\mathbf{Z}_2$ -valued intersection form are induced by homeomorphisms. In [18] we proved that for  $g \geq 2$  the group  $\Gamma_2(N_g)$  is equal to the normal closure in  $\mathcal{M}(N_g)$  of one crosscap slide. This result found its application in the work of S. Hirose [8] who proved that a homeomorphism of a closed nonorientable surface  $N$  standardly embedded in the 4-sphere is extendable to a homeomorphism of the 4-sphere if and only if it preserves the Guillou-Marin quadratic form on  $H_1(N, \mathbf{Z}_2)$ .

Since  $\mathcal{M}(N_g)$  is finitely generated, therefore so is  $\Gamma_2(N_g)$  and it is a very natural problem to find an explicit finite generating set for  $\Gamma_2(N_g)$ . This is our motivation for this paper, in which we obtain such set consisting of  $(g-1)^2 + \binom{g}{4}$  crosscap slides for  $g \geq 3$ . Since there is an epimorphism from  $\Gamma_2(N_g)$  onto  $\mathbf{Z}_2^{(g-1)^2}$ , exhibited in Section 4, thus  $(g-1)^2$  is a lower bound for the number of generators of  $\Gamma_2(N_g)$ . We prove that our generating set is minimal with respect to the number of elements for  $g=3$  and  $g=4$  by showing that  $\Gamma_2(N_3)$  is isomorphic to the level 2 congruence subgroup of  $\mathrm{GL}(2, \mathbf{Z})$  and that the abelianization of  $\Gamma_2(N_4)$  is isomorphic to  $\mathbf{Z}_2^{10}$ .

Having a finite set of generators of  $\Gamma_2(N)$ , the next natural problem is to compute its first homology group  $H_1(\Gamma_2(N), \mathbf{Z})$ . It follows from our work that this group is isomorphic to  $\mathbf{Z}_2^d$  for some  $d$  satisfying  $(g-1)^2 \leq d \leq (g-1)^2 + \binom{g}{4}$  (see the proof of Theorem 4.3) but it appears to be a difficult problem to find the exact value of  $d$ . The computations of  $H_1(\Gamma_m(F), \mathbf{Z})$  for orientable  $F$  in [15, 16] use Johnson's work on the Torelli group. Unfortunately no similar results are known for a nonorientable surface. For example it is a completely open problem to find generators for the Torelli subgroup of  $\mathcal{M}(N)$  consisting of the isotopy classes of homeomorphisms inducing the identity on  $H_1(N, \mathbf{Z})$ .

*Acknowledgments.* This paper was partially written during my visit to Institut de Mathématiques de Bourgogne in Dijon in the period 01.10.2011–30.09.2012 supported by the MNiSW “Mobility Plus” Program 639/MOB/2011/0. I wish to thank the Institut for their hospitality. This work was also supported by the MNiSW grant N N201 366436. I am grateful to Susumu Hirose for sending me a preprint of [8] and to the referee for helpful suggestions.

## 2. Preliminaries

Let  $N = N_g$  be a closed nonorientable surface of genus  $g$  and  $\mathcal{M}(N)$  its mapping class group. For  $\phi, \psi \in \mathcal{M}(N)$  the composition  $\phi\psi$  means that  $\psi$  is

applied first. By abuse of notation we will use the same symbol to denote a homeomorphism and its isotopy class.

**2.1. Curves and Dehn twists.** By a *simple closed curve* in  $N$  we mean an embedding  $\gamma : S^1 \rightarrow N$ . Note that  $\gamma$  has an orientation; the curve with the opposite orientation but same image will be denoted by  $\gamma^{-1}$ . By abuse of notation, we will usually identify a simple closed curve with its oriented image and also with its isotopy class. According to whether a regular neighborhood of  $\gamma$  is an annulus or a Möbius strip, we call  $\gamma$  respectively *two-* or *one-sided*.

Given a two-sided simple closed curve  $\gamma$ ,  $T_\gamma$  denotes a Dehn twist about  $\gamma$ . On a nonorientable surface it is impossible to distinguish between right and left twists, so the direction of a twist  $T_\gamma$  has to be specified for each curve  $\gamma$ . In this paper it is usually indicated by arrows in a figure. Equivalently we may choose an orientation of a regular neighborhood of  $\gamma$ . Then  $T_\gamma$  denotes the right Dehn twist with respect to the chosen orientation. Unless we specify which of the two twists we mean,  $T_\gamma$  denotes any of the two possible twists.

**2.2. Crosscap slide.** Suppose that  $\alpha$  and  $\beta$  are two simple closed curves in  $N$ , such that  $\alpha$  is one-sided,  $\beta$  is two-sided and they intersect in one point. Let  $K \subset N$  be a regular neighborhood of  $\alpha \cup \beta$ , which is homeomorphic to the Klein bottle with a hole. This is shown in Figure 1, where the shaded discs represent crosscaps; this means that their interiors should be removed, and then antipodal points in each resulting boundary component should be identified. Let  $M \subset K$  be a regular neighborhood of  $\alpha$ , which is a Möbius strip. We denote by  $Y_{\alpha,\beta}$  the *crosscap slide*, or *Y-homeomorphism* equal to the identity on  $N \setminus K$  and which may be described as the result of pushing  $M$  once along  $\beta$  keeping the boundary of  $K$  fixed. Figure 1 illustrates the effect of  $Y_{\alpha,\beta}$  on an arc connecting two points in the boundary of  $K$ . For a more rigorous definition see [12]. Up to isotopy,  $Y_{\alpha,\beta}$  does not depend on the choice of the regular neighbourhood  $K$ . The following properties of crosscap slides follow directly from its definition.

$$(2.1) \quad Y_{\alpha^{-1},\beta} = Y_{\alpha,\beta}$$

$$(2.2) \quad Y_{\alpha,\beta^{-1}} = Y_{\alpha,\beta}^{-1}$$

$$(2.3) \quad h Y_{\alpha,\beta} h^{-1} = Y_{h(\alpha),h(\beta)}$$

for all  $h \in \mathcal{M}(N)$ .

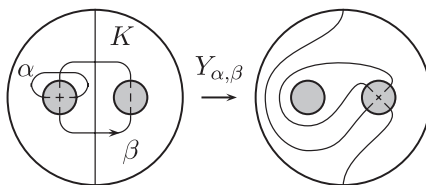


FIGURE 1. Crosscap slide.

Every crosscap slide induces an identity on  $H_1(N, \mathbf{Z}_2)$ , hence it belongs to  $\Gamma_2(N)$ .

**THEOREM 2.1** ([18]). *For  $g \geq 2$  the level 2 mapping class group  $\Gamma_2(N_g)$  is generated by crosscap slides.*

**2.3. Crosscap pushing map.** In this subsection we recall from [18] the definition of the crosscap pushing map which will be an important tool in what follows.

Fix  $x_0 \in N_g$  and define  $\mathcal{M}(N_g, x_0)$  to be the group of isotopy classes of all homeomorphisms of  $h : N_g \rightarrow N_g$  such that  $h(x_0) = x_0$ . Let  $U = \{z \in \mathbf{C} \mid |z| \leq 1\}$  and fix an embedding  $e : U \rightarrow N_g$  such that  $e(0) = x_0$ . The surface  $N_{g+1}$  may be obtained by removing from  $N_g$  the interior of  $e(U)$  and then identifying  $e(z)$  with  $e(-z)$  for  $z \in S^1 = \partial U$ . We define a *blowup homomorphism*

$$\varphi : \mathcal{M}(N_g, x_0) \rightarrow \mathcal{M}(N_{g+1})$$

as follows. Represent  $h \in \mathcal{M}(N_g, x_0)$  by a homeomorphism  $h : N_g \rightarrow N_g$  such that  $h$  is equal to the identity on  $e(U)$  or  $h(x) = e(\overline{e^{-1}(x)})$  for  $x \in e(U)$ . Such  $h$  commutes with the identification leading to  $N_{g+1}$  and thus induces an element  $\varphi(h) \in \mathcal{M}(N_{g+1})$ . We refer the reader to [18] for a proof that  $\varphi$  is well defined.

Forgetting the distinguished point  $x_0$  induces a homomorphism

$$\mathcal{M}(N_g, x_0) \rightarrow \mathcal{M}(N_g),$$

which fits into the Birman exact sequence (see [10])

$$\pi_1(N_g, x_0) \xrightarrow{j} \mathcal{M}(N_g, x_0) \rightarrow \mathcal{M}(N_g) \rightarrow 1.$$

The homomorphism  $j$  is called *point pushing map*. If  $\gamma$  is a loop in  $N_g$  based at  $x_0$  and  $[\gamma] \in \pi_1(N_g, x_0)$  is its homotopy class, then  $j([\gamma])$  may be described as the effect of pushing  $x_0$  once along  $\gamma$ . In order for  $j$  to be a homomorphism, a product  $[\gamma] \cdot [\delta]$  in  $\pi_1(N_g, x_0)$  means go along  $\delta$  first and then along  $\gamma$ .

We define a *crosscap pushing map* to be the composition

$$\psi = \varphi \circ j : \pi_1(N_g, x_0) \rightarrow \mathcal{M}(N_{g+1}).$$

Let  $\alpha$  be the image in  $N_{g+1}$  of  $e(\partial U)$ . Then  $\alpha$  is a one-sided simple closed curve. Every simple loop  $\gamma$  based at  $x_0$  is homotopic to a loop  $\gamma'$  which intersects  $e(U)$  in two antipodal points. If  $\beta$  is the image in  $N_{g+1}$  of  $\gamma' \setminus \text{int}(e(U))$  then  $\beta$  is a simple closed curve, which intersects  $\alpha$  in one-point, and which is two-sided if and only if  $\gamma$  is one-sided. The following lemma follows from the description of the point pushing map for nonorientable surfaces [10, Lemma 2.2 and Lemma 2.3] and the definition of a crosscap slide.

**LEMMA 2.2.** *Suppose that  $\gamma$  is a simple loop in  $N_g$  based at  $x_0$  intersecting  $e(\partial U)$  in two antipodal points. Let  $\alpha$  and  $\beta$  be the images in  $N_{g+1}$  of  $e(\partial U)$  and  $\gamma \setminus \text{int}(e(U))$  respectively. If  $\gamma$  is one-sided, then*

$$\psi([\gamma]) = Y_{\alpha, \beta}.$$

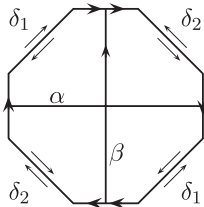


FIGURE 2.  $Y_{\alpha,\beta}$  for two one-sided curves.

If  $\gamma$  is two-sided, then

$$\psi([\gamma]) = (T_{\delta_1} T_{\delta_2}^{-1})^\varepsilon,$$

where  $\delta_1$  and  $\delta_2$  are boundary curves of a regular neighbourhood  $M$  of  $\alpha \cup \beta$ , the twists are right with respect to some orientation of  $M \setminus \alpha$  and  $\varepsilon$  is 1 or  $-1$  depending on the orientation of  $\beta$ .

**2.4. Generalized crosscap slide.** Lemma 2.2 suggests the following generalization of the definition of a crosscap slide. Let  $\alpha$  and  $\beta$  be one-sided curves intersecting in one point and let  $M$ ,  $\delta_1$  and  $\delta_2$  be as in Lemma 2.2. The neighborhood  $M$  is shown in Figure 2 as an octagon whose two pairs of opposite sides should be identified according to arrows on the sides. The other two pairs of opposite sides are the boundary curves  $\delta_1$ ,  $\delta_2$ . The curve  $\alpha$  divides the octagon into halves. Let  $T_{\delta_1}$ ,  $T_{\delta_2}$  be right with respect to the standard orientation of the bottom half (the arrows in Figure 2 indicate the directions of  $T_{\delta_1}$  and  $T_{\delta_2}^{-1}$ ). Then, for  $\beta$  oriented as in Figure 2 we define

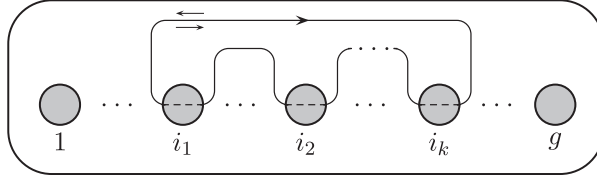
$$Y_{\alpha,\beta} = T_{\delta_1} T_{\delta_2}^{-1}.$$

For the opposite orientation of  $\beta$  we set  $Y_{\alpha,\beta} = T_{\delta_1}^{-1} T_{\delta_2}$ . For such generalized definition the properties (2.1), (2.2) and (2.3) remain valid.

### 3. Generators of the level 2 mapping class group

Let us represent  $N_g$  as a 2-sphere with  $g$  crosscaps. This means that interiors of  $g$  small pairwise disjoint discs should be removed from the sphere, and then antipodal points in each of the resulting boundary components should be identified. Let us arrange the crosscaps as shown on Figure 3 and number them from 1 to  $g$ . For each nonempty subset  $I \subseteq \{1, \dots, g\}$  let  $\alpha_I$  be the simple closed curve shown on Figure 3. For  $I = \{i_1, \dots, i_k\}$  let  $|I| = k$ . Note that  $\alpha_I$  is two-sided if and only if  $|I|$  is even. In such case  $T_{\alpha_I}$  will be Dehn twist about  $\alpha_I$  in the direction indicated by arrows on Figure 3. We will write  $\alpha_i$  instead of  $\alpha_{\{i\}}$  for  $i \in \{1, \dots, g\}$ .

It is well known that  $\mathcal{M}(N_1)$  is trivial and  $\mathcal{M}(N_2) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$  ([12]). It follows easily from the structure of  $\mathcal{M}(N_2)$  that  $\Gamma_2(N_2)$  has order two and it is generated by a crosscap slide.

FIGURE 3. The curve  $\alpha_I$  for  $I = \{i_1, i_2, \dots, i_k\}$ .

**THEOREM 3.1.** *For  $g \geq 4$  the mapping class group  $\mathcal{M}(N_g)$  is generated by the following elements.*

- $Y = Y_{\alpha_{g-1}, \alpha_{\{g-1, g\}}}$ ,
- $A_i = T_{\alpha_{\{i, i+1\}}}$  for  $i = 1, \dots, g-1$ ,
- $B = T_{\alpha_{\{1, 2, 3, 4\}}}$ .

The group  $\mathcal{M}(N_3)$  is generated by  $Y, A_1, A_2$ .

*Proof.* Let  $\mathcal{A}$  denote the set of elements listed in Theorem 3.1 and let  $H$  be the subgroup of  $\mathcal{M}(N_g)$  generated by  $\mathcal{A}$ . Let  $\mathcal{B}$  be the set of Dehn twists about the curves  $\alpha_{\{1, \dots, 2i\}}$  for  $6 \leq 2i \leq g$ . Chillingworth proved in [3] that  $\mathcal{M}(N_g)$  is generated by  $\mathcal{A} \cup \mathcal{B}$ . This generating set is explicitly given on page 427 of [3], where  $\xi_i, \mu_i$  and  $y$  correspond to our  $\alpha_{\{i, i+1\}}, \alpha_{\{i, \dots, 2i\}}$  and  $Y_{\alpha_{g-1}, \alpha_{\{g-1, g\}}}$  respectively, whereas in Theorem on page 426 it is displayed on a different model of  $N_g$ . In order to show  $H = \mathcal{M}(N_g)$  it suffices to prove  $\mathcal{B} \subset H$ . We are assuming  $g \geq 6$ , for otherwise  $\mathcal{B} = \emptyset$ .

Suppose that  $g$  is odd and let  $N'$  be the surface obtained from  $N_g$  by removing an open regular neighborhood of the curve  $\delta = \alpha_{\{1, \dots, g\}}$ . Thus  $N'$  is an orientable surface with one boundary component and by a theorem of Humphries [9] its mapping class group is generated by  $A_i$  for  $i = 1, \dots, g-1$  and  $B$ . It follows that  $T_\gamma \in H$  for every two-sided simple closed curve  $\gamma$  in  $N_g$  such that  $\gamma \cap \delta = \emptyset$ . Since  $\mathcal{B}$  consist entirely of such twists, thus  $\mathcal{B} \subset H$ .

Now suppose that  $g$  is even and let  $N'$  be the surface obtained from  $N_g$  by removing an open regular neighborhood of  $\delta \cup \alpha_g$ , where  $\delta = \alpha_{\{1, \dots, g\}}$ . Again  $N'$  is an orientable surface with one boundary component and we can use the theorem of Humphries to deduce that  $T_\gamma \in H$  for every two-sided simple closed curve  $\gamma$  in  $N_g$  such that  $\gamma \cap (\delta \cup \alpha_g) = \emptyset$ . This proves that all elements of  $\mathcal{B}$  except  $D = T_\delta$  are in  $H$ . It remains to prove  $D \in H$ . We borrow an argument from [8]. Consider the curves  $\varepsilon = \alpha_{\{1, \dots, g-4\}}, \gamma_1 = \alpha_{\{1, \dots, g-2\}}, \gamma_2 = \alpha_{\{g-3, g-2, g-1, g\}}, \gamma_3 = \alpha_{\{1, \dots, g-4, g-1, g\}}$ , and let  $E, C_i$  for  $i = 1, 2, 3$  be the corresponding Dehn twists. Note that the curves  $\delta, \varepsilon, \alpha_{\{g-3, g-2\}}$  and  $\alpha_{\{g-1, g\}}$  bound a four-holed sphere and we have the following lantern relation  $DEA_{g-3}A_{g-1} = C_1C_2C_3$ . Clearly  $A_{g-3}, A_{g-1} \in H$  and since  $\varepsilon \cap (\delta \cup \alpha_g) = \gamma_1 \cap (\delta \cup \alpha_g) = \emptyset$  thus  $E, C_1 \in H$ . It can be easily checked that  $A_{g-1}A_{g-2}A_{g-3}A_{g-4}(\gamma_2) = \alpha_{\{g-4, g-3, g-2, g-1\}}$  and  $A_{g-1}A_{g-2}(\gamma_3) = \alpha_{\{1, \dots, g-4, g-2, g-1\}}$ . Since these curves are disjoint from  $\delta \cup \alpha_g$  the

corresponding twists are in  $H$ . It follows that  $C_2, C_3 \in H$  and the lantern relation gives  $D \in H$ .  $\square$

The following theorem is the main result of this paper.

**THEOREM 3.2.** *For  $g \geq 4$  the level 2 mapping class group  $\Gamma_2(N_g)$  is generated by the following elements.*

- (1)  $Y_{\alpha_i, \alpha_{\{i,j\}}}$  for  $i \in \{1, \dots, g-1\}$ ,  $j \in \{1, \dots, g\}$  and  $i \neq j$ ,
- (2)  $Y_{\alpha_{\{i,j,k\}}, \alpha_{\{i,j,k,l\}}}$  for  $i < j < k < l$ .

The group  $\Gamma_2(N_3)$  is generated by the elements (1).

Let  $G$  be the subgroup of  $\mathcal{M}(N_g)$  generated by the elements (1), (2) from Theorem 3.2. Our goal is to prove that  $G = \Gamma_2(N_g)$ . Since by [18, Lemma 3.6]  $\Gamma_2(N_g)$  is generated by crosscap slides conjugate to  $Y_{\alpha_1, \alpha_{\{1,2\}}}$  it suffices to prove that  $G$  is normal in  $\mathcal{M}(N_g)$ . First we need to prove some lemmas.

**LEMMA 3.3.** *Suppose that  $\alpha$  and  $\beta$  are two simple closed curves such that  $\alpha$  is one-sided,  $\beta$  is two-sided and they intersect in one point. Then*

$$T_\beta^2 = Y_{T_\beta(\alpha), \beta} Y_{\alpha, \beta}^{-1}.$$

*Proof.* Since  $Y_{\alpha, \beta}$  preserves the curve  $\beta$  and reverses orientation of its neighbourhood, we have  $Y_{\alpha, \beta} T_\beta Y_{\alpha, \beta}^{-1} = T_\beta^{-1}$ . On the other hand, by (2.3) we have  $T_\beta Y_{\alpha, \beta} T_\beta^{-1} = Y_{T_\beta(\alpha), \beta}$ , therefore,

$$T_\beta^2 = T_\beta Y_{\alpha, \beta} T_\beta^{-1} Y_{\alpha, \beta}^{-1} = Y_{T_\beta(\alpha), \beta} Y_{\alpha, \beta}^{-1}. \quad \square$$

**LEMMA 3.4.** *Suppose that  $\alpha$  and  $\beta$  are one-sided simple closed curves intersecting in one point. Let  $\delta$  be one of the boundary curves of a regular neighbourhood of  $\alpha \cup \beta$ . Then*

$$T_\delta^2 = Y_{\alpha, \beta}^{\varepsilon_1} Y_{\beta, \alpha}^{\varepsilon_2},$$

where  $\varepsilon_i$  is 1 or  $-1$  for  $i = 1, 2$ .

*Proof.* Let  $\delta = \delta_1$  and  $\delta_2$  be the boundary curves of a regular neighbourhood  $M$  of  $\alpha \cup \beta$  and suppose that  $T_{\delta_1}$  and  $T_{\delta_2}$  are right twists with respect to some orientation of  $M \setminus \alpha$ , so that  $Y_{\alpha, \beta} = (T_{\delta_1} T_{\delta_2}^{-1})^{\varepsilon_1}$  (note that this is the generalized crosscap slide defined in Subsection 2.4). Observe that with respect to any orientation of  $M \setminus \beta$  one of the twists  $T_{\delta_i}$  is right, while the other one is left. Hence  $Y_{\beta, \alpha} = (T_{\delta_1} T_{\delta_2})^{\varepsilon_2}$  and the lemma follows.  $\square$

**LEMMA 3.5.** *For  $i \in \{1, \dots, g\}$  and for every simple closed curve  $\beta$  intersecting  $\alpha_i$  in one point we have  $Y_{\alpha_i, \beta} \in G$ . Moreover,  $Y_{\alpha_i, \beta}$  can be written as a product of generators of type (1).*

*Proof.* Let  $G'$  be the subgroup of  $G$  generated by the elements (1). First we assume  $i < g$ . Let  $N'$  be the surface of genus  $g - 1$  obtained from  $N_g$  by replacing the  $i$ -th crosscap by a disc  $U$  with basepoint  $x_0$ . As  $N_g$  may be seen as being obtained from  $N'$  by the blowup construction, we have the corresponding crosscap pushing map

$$\psi : \pi_1(N', x_0) \rightarrow \mathcal{M}(N_g).$$

Note that  $Y_{\alpha_i, \beta}$  is in the image of  $\psi$  (see Lemma 2.2). The group  $\pi_1(N', x_0)$  is generated by homotopy classes of simple loops  $[\gamma_j]$  such that  $\psi([\gamma_j]) = Y_{\alpha_i, \alpha_{\{i, j\}}}$  for  $j \in \{1, \dots, i - 1, i + 1, \dots, g\}$  (in fact  $[\gamma_j]$  can be taken to be standard generators of the fundamental group). It follows that  $\psi(\pi_1(N', x_0)) \subseteq G'$  and  $Y_{\alpha_i, \beta} \in G'$ .

Now suppose  $i = g$ . It suffices to show that  $Y_{\alpha_g, \alpha_{\{j, g\}}} \in G'$  for  $j \in \{1, \dots, g - 1\}$ , the rest of the proof follows as above. Note that  $T_{\alpha_{\{j, g\}}}(\alpha_j) = \alpha_g^{-1}$ . By Lemma 3.3 and (2.1) we have

$$Y_{\alpha_g, \alpha_{\{j, g\}}} = T_{\alpha_{\{j, g\}}}^2 Y_{\alpha_j, \alpha_{\{j, g\}}}$$

and hence it suffices to show that  $T_{\alpha_{\{j, g\}}}^2 \in G'$ . We assume  $j = g - 1$ . Let  $M$  be the surface obtained by cutting  $N_g$  along  $\alpha_{\{g-1, g\}}$ . Thus  $M$  is a nonorientable surface of genus  $g - 2$  with two boundary components. For  $n \in \{1, \dots, g - 1\}$  there exist pairwise disjoint two-sided simple closed curves  $\delta_n$  in  $M$  such that  $\partial M = \delta_1 \cup \delta_{g-1}$  and  $\delta_k \cup \delta_{k+1} = \partial M_k$ , where  $M_k$  is a genus one subsurface containing  $\alpha_k$  for  $k \in \{1, \dots, g - 2\}$ . Choose an orientation of  $M$  ( $\alpha_1 \cup \dots \cup \alpha_{g-2}$ ) and let  $T_{\delta_n}$  be the right Dehn twist with respect to that orientation for  $n \in \{1, \dots, g - 1\}$ . For  $k \in \{1, \dots, g - 2\}$  there exists a one-sided simple curve  $\beta_k$  in  $M_k$  intersecting  $\alpha_k$  in one point and such that  $Y_{\alpha_k, \beta_k} = T_{\delta_k} T_{\delta_{k+1}}^{-1}$ , see Figure 4. We have

$$Y_{\alpha_1, \beta_1} Y_{\alpha_2, \beta_2} \cdots Y_{\alpha_{g-2}, \beta_{g-2}} = T_{\delta_1} T_{\delta_{g-1}}^{-1},$$

and after recovering  $N_g$  by gluing together the boundary curves of  $M$  we obtain

$$Y_{\alpha_1, \beta_1} Y_{\alpha_2, \beta_2} \cdots Y_{\alpha_{g-2}, \beta_{g-2}} = T_{\alpha_{\{g-1, g\}}}^{\pm 2}.$$

This completes the proof because  $Y_{\alpha_k, \beta_k} \in G'$  for  $k \in \{1, \dots, g - 2\}$  by earlier part of the proof. Similarly, we can show  $T_{\alpha_{\{j, g\}}}^2 \in G'$  for other  $j$ .  $\square$

LEMMA 3.6. *For every  $I \subseteq \{1, \dots, g\}$  such that  $|I| = 2$  or  $|I| = 4$  we have  $T_{\alpha_I}^2 \in G$ .*

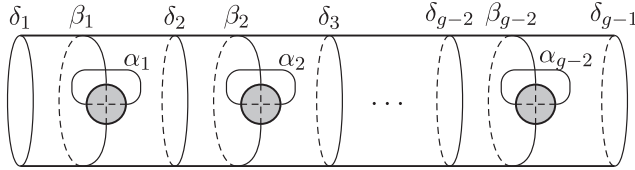


FIGURE 4. The surface  $M$  and curves from the proof of Lemma 3.5.



*Proof.* Suppose that  $I = \{i, j\}$  where  $i < j$ . Observe that  $T_{\alpha_{\{i,j\}}}(\alpha_i) = \alpha_j^{-1}$ . By Lemma 3.3 and (2.1) we have  $T_{\alpha_{\{i,j\}}}^2 = Y_{\alpha_j, \alpha_{\{i,j\}}} Y_{\alpha_i, \alpha_{\{i,j\}}}^{-1} \in G$ .

Suppose that  $I = \{i, j, k, l\}$  where  $i < j < k < l$  and let  $J = \{i, j, k\}$ . Observe that  $T_{\alpha_J}(\alpha_J) = \alpha_l^{-1}$ . By Lemma 3.3 and (2.1) we have  $T_{\alpha_J}^2 = Y_{\alpha_l, \alpha_J} Y_{\alpha_J, \alpha_l}^{-1}$ . Since  $Y_{\alpha_l, \alpha_J} \in G$  by Lemma 3.5, and  $Y_{\alpha_J, \alpha_l} \in G$  by the definition of  $G$ , also  $T_{\alpha_J}^2 \in G$ .  $\square$

LEMMA 3.7. *For every  $I = \{i, j, k\}$ , where  $1 \leq i < j < k \leq g$ , and for every two-sided simple closed curve  $\beta$  intersecting  $\alpha_I$  in one point we have  $Y_{\alpha_I, \beta} \in G$ .*

*Proof.* Note that  $\beta$  actually exists only for  $g \geq 4$ . Indeed, if  $g = 3$  then the surface obtained from  $N_g$  by cutting along  $\alpha_I$  is orientable, and every curve intersecting  $\alpha_I$  in one point must be one-sided. Therefore we are assuming  $g \geq 4$ .

Let  $H$  be the subgroup of  $\mathcal{M}(N)$  generated by  $Y_{\alpha_l, \alpha_i}$ ,  $Y_{\alpha_l, \alpha_k}$  and  $Y_{\alpha_l, \alpha_J}$  for every  $J$  such that  $|J| = 4$  and  $I \subset J$ . We claim that  $H \subseteq G$ . Consider a regular neighborhood of  $\alpha_I \cup \alpha_i$ . One of its boundary curves is  $\alpha_{\{j,k\}}$  and by Lemma 3.4 we have

$$T_{\alpha_{\{j,k\}}}^2 = Y_{\alpha_l, \alpha_i}^{\varepsilon_1} Y_{\alpha_l, \alpha_J}^{\varepsilon_2},$$

where  $\varepsilon_l$  is 1 or  $-1$  for  $l = 1, 2$ . Since  $Y_{\alpha_i, \alpha_l} \in G$  by Lemma 3.5 and  $T_{\alpha_{\{j,k\}}}^2 \in G$  by Lemma 3.6, also  $Y_{\alpha_l, \alpha_i} \in G$ . By a similar argument  $Y_{\alpha_l, \alpha_k} \in G$ . Let  $J = I \cup \{l\}$  for  $l \notin \{i, j, k\}$  and note that  $T_{\alpha_J}^{\pm 1}(\alpha_I) = \alpha_l^{-1}$ . By Lemma 3.3 and (2.1) we have

$$T_{\alpha_J}^{\pm 2} = Y_{\alpha_l, \alpha_J} Y_{\alpha_J, \alpha_l}^{-1},$$

and since  $Y_{\alpha_l, \alpha_J} \in G$  by Lemma 3.5 and  $T_{\alpha_J}^2 \in G$  by Lemma 3.6, also  $Y_{\alpha_J, \alpha_l} \in G$  and the claim is proved.

Now it suffices to show that  $Y_{\alpha_I, \beta} \in H$ . Choose  $n \in \{1, \dots, g\} \setminus \{i, j, k\}$  and observe that for  $J = I \cup \{n\}$  we have  $T_{\alpha_J}^\varepsilon(\alpha_I) = \alpha_n^{-1}$ , where  $\varepsilon = \pm 1$ . Let

$$Y = T_{\alpha_J}^\varepsilon Y_{\alpha_I, \beta} T_{\alpha_J}^{-\varepsilon} = Y_{\alpha_n, T_{\alpha_J}^\varepsilon(\beta)}, \quad H' = T_{\alpha_J}^\varepsilon H T_{\alpha_J}^{-\varepsilon}.$$

We need to show that  $Y \in H'$ . Let  $N'$  be the surface of genus  $g - 1$  obtained from  $N_g$  by replacing the  $n$ -th crosscap by a disc  $U$  with basepoint  $x_0$ . We have the crosscap pushing map

$$\psi : \pi_1(N', x_0) \rightarrow \mathcal{M}(N_g).$$

Since  $Y \in \psi(\pi_1(N', x_0))$  it suffices to show that  $\psi(\pi_1(N', x_0)) \subseteq H'$  and for that it is enough to check that  $\pi_1(N', x_0)$  is generated by loops mapped by  $\psi$  on generators of  $H'$ . Let us assume  $n = 1$ . The proof is similar for other  $n$ . In this case we have  $H' = D^{-1} H D$  for  $D = T_{\alpha_{\{1, i, j, k\}}}$ . For  $s \in \{2, \dots, g\}$  let  $x_s$  be the standard generators of  $\pi_1(N', x_0)$  such that  $\psi(x_s) = Y_{\alpha_1, \alpha_{\{1, s\}}}$ . The group  $H'$  is generated by

$$\begin{aligned}
D^{-1}Y_{\alpha_l, \alpha_i}D &= Y_{\alpha_l, \alpha_{\{1, j, k\}}^{-1}} = \psi((x_j x_k)^{-1}), \\
D^{-1}Y_{\alpha_l, \alpha_k}D &= Y_{\alpha_l, \alpha_{\{1, i, j\}}^{-1}} = \psi((x_i x_j)^{-1}), \\
D^{-1}Y_{\alpha_l, \alpha_{\{1, i, j, k\}}}D &= Y_{\alpha_l, \alpha_{\{1, i, j, k\}}} = \psi(x_i x_j x_k), \\
D^{-1}Y_{\alpha_l, \alpha_{\{l, i, j, k\}}}D &= \psi((x_i x_j x_k)^{-1} x_l x_i x_j x_k) \quad \text{for } 1 < l < i, \\
D^{-1}Y_{\alpha_l, \alpha_{\{i, l, j, k\}}}D &= \psi(x_i x_l x_i^{-1}) \quad \text{for } i < l < j, \\
D^{-1}Y_{\alpha_l, \alpha_{\{i, j, l, k\}}}D &= \psi(x_k^{-1} x_l x_k) \quad \text{for } j < l < k, \\
D^{-1}Y_{\alpha_l, \alpha_{\{i, j, k, l\}}}D &= \psi(x_i x_j x_k x_l (x_i x_j x_k)^{-1}) \quad \text{for } k < l \leq g.
\end{aligned}$$

It is easy to check that each  $x_s$  can be expressed as a product of elements which are mapped by  $\psi$  on the generators of  $H'$ . It follows that  $\psi(\pi_1(N', x_0)) \subseteq H'$  and the lemma is proved.  $\square$

LEMMA 3.8. *Let  $I = \{i_1, \dots, i_5\}$  and  $J = \{i_1, \dots, i_5, i_6\}$ , where  $1 \leq i_1 < \dots < i_5 < i_6 \leq g$ . Then  $Y_{\alpha_I, \alpha_J} \in G$ .*

*Proof.* Note that  $T_{\alpha_I}(\alpha_{\{i_1, i_2, i_3\}}) = \alpha_{\{i_4, i_5, i_6\}}^{-1}$ . By Lemma 3.3 and (2.1) we have

$$T_{\alpha_J}^2 = Y_{\alpha_{\{i_4, i_5, i_6\}}, \alpha_J} Y_{\alpha_{\{i_1, i_2, i_3\}}, \alpha_I}^{-1}.$$

On the other hand, since  $T_{\alpha_J}(\alpha_I) = \alpha_{i_6}^{-1}$ , we also have

$$T_{\alpha_J}^2 = Y_{\alpha_{i_6}, \alpha_J} Y_{\alpha_I, \alpha_J}^{-1}.$$

Hence

$$Y_{\alpha_I, \alpha_J} = Y_{\alpha_{\{i_1, i_2, i_3\}}, \alpha_J} Y_{\alpha_{\{i_4, i_5, i_6\}}, \alpha_J}^{-1} Y_{\alpha_{i_6}, \alpha_J} \in G$$

by Lemmas 3.5 and 3.7.  $\square$

*Proof of Theorem 3.2.* As we explained before Lemma 3.3, we have to prove that  $G$  is normal in  $\mathcal{M}(N_g)$ . By Theorem 3.1 it suffices to check for every generator  $x$  of  $G$  that  $A_i x A_i^{-1} \in G$  or  $A_i^{-1} x A_i \in G$  for  $i \in \{1, \dots, g\}$  and  $B x B^{-1} \in G$  or  $B^{-1} x B \in G$ . Note that  $A_i x A_i^{-1} \in G$  if and only if  $A_i^{-1} x A_i \in G$  since  $A_i^2 \in G$  by Lemma 3.6, and analogously for  $B$ .

For  $i \in \{1, \dots, g\}$  we have  $A_{i-1}^{-1}(\alpha_i) = \alpha_{i-1}^{-1}$ ,  $A_i(\alpha_i) = \alpha_{i+1}^{-1}$  and  $A_k(\alpha_i) = \alpha_i$  for  $k \neq i-1, i$ . It follows that for all  $i \neq j$  and  $k$  the generator  $Y_{\alpha_i, \alpha_{\{i, j\}}}$  is conjugated by  $A_k$  or  $A_k^{-1}$  to  $Y_{\alpha_l, \beta}$  for some  $l$  and some  $\beta$ . Since the last element is in  $G$  by Lemma 3.5, we have proved that  $A_i x A_i^{-1} \in G$  for every  $i \in \{1, \dots, g\}$  and every generator  $x$  of type (1).

If  $g \geq 4$  then  $B(\alpha_i) = \alpha_i$  for  $i > 4$ , while for  $i \in \{1, 2, 3, 4\}$  we have  $B^{\pm 1}(\alpha_i) = \alpha_I^{-1}$ , where  $I = \{1, 2, 3, 4\} \setminus \{i\}$ . It follows by Lemma 3.5 or by Lemma 3.7 that  $B x B^{-1} \in G$  for every generator  $x$  of type (1).

Suppose that  $I = \{i, j, k\}$  and  $J = \{i, j, k, l\}$ , where  $i < j < k < l$ . It can be checked that for  $i > 1$  we have  $A_{i-1}(\alpha_I) = Y_{\alpha_i, \alpha_{\{i-1, i\}}}(\alpha_{\{i-1, j, k\}})$  and  $A_{i-1}(\alpha_J) = Y_{\alpha_i, \alpha_{\{i-1, i\}}}(\alpha_{\{i-1, j, k, l\}})$ . It follows that

$$A_{i-1} Y_{\alpha_i, \alpha_J} A_{i-1}^{-1} = Y_{\alpha_i, \alpha_{\{i-1, i\}}} Y_{\alpha_{\{i-1, j, k\}}, \alpha_{\{i-1, j, k, l\}}} Y_{\alpha_i, \alpha_{\{i-1, i\}}}^{-1} \in G.$$

By similar arguments one can check that  $A_n Y_{\alpha_i, \alpha_J} A_n^{-1} \in G$  for  $n \in \{1, \dots, g-1\}$ .

If  $i > 4$  then  $B(\alpha_I) = \alpha_I$ , while for  $k \leq 4$  we have  $B^{\pm 1}(\alpha_I) = \alpha_m^{-1}$ , where  $\{m\} = \{1, 2, 3, 4\} \setminus I$ . In both cases we have  $BY_{\alpha_i, \alpha_J} B^{-1} \in G$ . If  $i = 1, j = 2$  and  $k > 4$ , then for  $Y_1 = Y_{\alpha_3, \alpha_{\{2, 3\}}}$ ,  $Y_2 = Y_{\alpha_4, \alpha_{\{2, 4\}}}$  it can be checked that  $Y_1 Y_2(\alpha_I)$  and  $Y_1 Y_2(\alpha_J)$  are disjoint from  $\alpha_{\{1, 2, 3, 4\}}$ . It follows that

$$BY_1 Y_2 Y_{\alpha_i, \alpha_J} Y_2^{-1} Y_1^{-1} B^{-1} = Y_1 Y_2 Y_{\alpha_i, \alpha_J} Y_2^{-1} Y_1^{-1} \in G.$$

From earlier part of the proof we know that  $BY_1 Y_2 B^{-1} \in G$ , hence  $BY_{\alpha_i, \alpha_J} B^{-1} \in G$ . A similar argument, using different  $Y_1, Y_2$ , can be applied to other  $I$  such that  $|I \cap \{1, 2, 3, 4\}| = 2$ . It remains to consider the cases where  $i \leq 4$  and  $j > 4$ . If  $i = 1$  then it can be checked that  $B^{-1}(\alpha_I) = Y_{\alpha_1, \alpha_{\{1, 2\}}}^{-1}(\alpha_{I'})$ , where  $I' = \{2, 3, 4, j, k\}$ , and  $B^{-1}(\alpha_J) = Y_{\alpha_1, \alpha_{\{1, 2\}}}^{-1}(\alpha_{J'})$ , where  $J' = I' \cup \{l\}$ . Since  $Y_{\alpha_{I'}, \alpha_{J'}} \in G$  by Lemma 3.8, we have

$$B^{-1} Y_{\alpha_i, \alpha_J} B = Y_{\alpha_1, \alpha_{\{1, 2\}}}^{-1} Y_{\alpha_{I'}, \alpha_{J'}} Y_{\alpha_1, \alpha_{\{1, 2\}}} \in G.$$

If  $i = 2$  then for  $Y_1 = Y_{\alpha_1, \alpha_{\{1, 2\}}}$  and  $Y_2 = Y_{\alpha_2, \alpha_{\{1, 2\}}}$  we have  $B^{-1} Y_1^{-1}(\alpha_I) = Y_1^{-1} Y_2(\alpha_{I'})$ , where  $I' = \{1, 3, 4, j, k\}$ , and  $B^{-1} Y_1^{-1}(\alpha_J) = Y_1^{-1} Y_2(\alpha_{J'})$ , where  $J' = I' \cup \{l\}$ . Since  $Y_{\alpha_{I'}, \alpha_{J'}} \in G$  by Lemma 3.8 we have

$$B^{-1} Y_1^{-1} Y_{\alpha_i, \alpha_J} Y_1 B = Y_1^{-1} Y_2 Y_{\alpha_{I'}, \alpha_{J'}} Y_2^{-1} Y_1 \in G,$$

and since  $B^{-1} Y_1 B \in G$  by earlier part of the proof, also  $B^{-1} Y_{\alpha_i, \alpha_J} B \in G$ . The proof is similar for  $i = 3$  and  $i = 4$ .  $\square$

*Remark 3.9.* By the proof of Lemma 3.6 for  $i < j < k < l$  we have

$$T_{\alpha_{\{i, j, k, l\}}}^2 = Y_{\alpha_i, \alpha_{\{i, j, k, l\}}} Y_{\alpha_{\{i, j, k\}}, \alpha_{\{i, j, k, l\}}}^{-1},$$

and by Lemma 3.5  $Y_{\alpha_i, \alpha_{\{i, j, k, l\}}}$  can be written as a product of the generators of type (1). It follows that each generator of type (2)  $Y_{\alpha_{\{i, j, k\}}, \alpha_{\{i, j, k, l\}}}$  can be replaced by  $T_{\alpha_{\{i, j, k, l\}}}^2$  in Theorem 3.2.

*Remark 3.10.* There are  $(g-1)^2$  generators of type (1) and  $\binom{g}{4}$  generators of type (2). In particular we have 4 generators for  $\Gamma_2(N_3)$  and 10 generators for  $\Gamma_2(N_4)$ . We will show in the next section that these are minimal numbers of generators for these groups. We do not expect that Theorem 3.2 provides minimal number of generators for  $\Gamma_2(N_g)$  for  $g > 4$ .

#### 4. Low genus cases

For  $i \in \{1, \dots, g\}$  let  $c_i$  denote the homology class of the curve  $\alpha_i$  in  $H_1(N_g, \mathbf{Z})$ . Then  $H_1(N_g, \mathbf{Z})$  has the following presentation as a  $\mathbf{Z}$ -module:

$$H_1(N_g, \mathbf{Z}) = \langle c_1, \dots, c_g \mid 2(c_1 + \dots + c_g) = 0 \rangle.$$

Consider the quotient  $R_g = H_1(N_g, \mathbf{Z}) / \langle c \rangle$ , where  $c = c_1 + \dots + c_g$  is the unique homology class of order 2. It is immediate from the above presentation that  $R_g$  is the free  $\mathbf{Z}$ -module with basis given by the images of  $c_1, \dots, c_{g-1}$  in  $R_g$ . Let us fix this basis and identify  $\text{Aut}(R_g)$  with  $\text{GL}(g-1, \mathbf{Z})$ . Every automorphism of  $H_1(N_g, \mathbf{Z})$  preserves  $c$ , and thus induces an automorphism of  $R_g$ . Thus the action of  $\mathcal{M}(N_g)$  on  $H_1(N_g, \mathbf{Z})$  induces a homomorphism

$$\rho : \mathcal{M}(N_g) \rightarrow \text{GL}(g-1, \mathbf{Z}).$$

In general  $\rho$  is neither surjective nor injective. However, it was shown in [14, Section 2], that the group of automorphisms of  $H_1(N_g, \mathbf{Z})$  which act trivially on  $H_1(N_g, \mathbf{Z}_2)$  is isomorphic to the full group of automorphisms of  $R_g$  which act trivially on  $R_g \otimes \mathbf{Z}_2$ . Consequently, the restriction of  $\rho$  to  $\Gamma_2(N_g)$  yields a surjection

$$\eta : \Gamma_2(N_g) \rightarrow \text{GL}_2(g-1, \mathbf{Z}),$$

where  $\text{GL}_2(n, \mathbf{Z})$  is the level 2 congruence subgroup of  $\text{GL}(n, \mathbf{Z})$ .

Birman and Chillingworth obtained in [2, Theorem 3] a finite presentation for  $\mathcal{M}(N_3)$  from which it is immediate that this group is isomorphic to  $\text{GL}(2, \mathbf{Z})$ . It turns out that such isomorphism can also be deduced from the action on  $H_1(N_3, \mathbf{Z})$ , as shows the following Theorem proved in [5].

**THEOREM 4.1.** *The map  $\rho : \mathcal{M}(N_3) \rightarrow \text{GL}(2, \mathbf{Z})$  is an isomorphism.*

The following corollary is an immediate consequence of Theorem 4.1.

**COROLLARY 4.2.** *The map  $\eta : \Gamma_2(N_3) \rightarrow \text{GL}_2(2, \mathbf{Z})$  is an isomorphism.*

Let  $\text{Mat}_n(\mathbf{Z}_2)$  denote the additive group of  $n \times n$  matrices with entries in  $\mathbf{Z}_2$ . This is an abelian group isomorphic to  $\mathbf{Z}_2^{n^2}$ . Let us define an epimorphism  $f : \text{GL}_2(n, \mathbf{Z}) \rightarrow \text{Mat}_n(\mathbf{Z}_2)$ . Let  $X$  be any matrix in  $\text{GL}_2(n, \mathbf{Z})$ . Write  $X = I + 2A$ , where  $I$  is the identity matrix and define  $f(X) = A \pmod{2}$ . To see that this is a homomorphism take  $Y = I + 2B$ . Then

$$f(XY) = f(I + 2(A + B) + 4AB) = A + B \pmod{2}.$$

Let  $E_{i,j}$  be the elementary  $n \times n$  matrix with 1 at position  $(i, j)$  and 0's elsewhere. Since for each pair  $(i, j)$  the matrix  $I - 2E_{i,j}$  is in  $\text{GL}_2(n, \mathbf{Z})$ , thus  $f$  is onto. The map  $f$  was defined in [11] to determine abelianizations of congruence subgroups of  $\text{SL}(n, \mathbf{Z})$ . Now let  $g > 1$  and consider the composition

$$f \circ \eta : \Gamma_2(N_g) \rightarrow \text{Mat}_{g-1}(\mathbf{Z}_2).$$

Since  $f \circ \eta$  is surjective, we see that  $\Gamma_2(N_g)$  cannot be generated by less than  $(g-1)^2$  elements. In particular Theorem 3.2 provides the minimal number of generators for  $\Gamma_2(N_3)$ . It follows from the next theorem that this is also the case for  $g = 4$ .

**THEOREM 4.3.** *The group  $H_1(\Gamma_2(N_4), \mathbf{Z})$  is isomorphic to  $\mathbf{Z}_2^{10}$ .*

*Proof.* By [18, Theorem 3.7]  $\Gamma_2(N_g)$  is generated by involutions. It follows that  $H_1(\Gamma_2(N_4), \mathbf{Z}) \approx \mathbf{Z}_2^d$  for some integer  $d$ . From Theorem 3.2 we have  $d \leq 10$  and the existence of  $f \circ \eta$  gives  $d \geq 9$ . Let

$$h : H_1(\Gamma_2(N_4), \mathbf{Z}) \rightarrow \text{Mat}_3(\mathbf{Z}_2)$$

be the map induced by  $f \circ \eta$ . To prove  $d = 10$  it suffices to show that  $\ker h$  is not trivial. Let  $B = T_{\alpha_{(1,2,3,4)}}$  and observe that  $B^2$  induces the identity on  $H_1(N_4, \mathbf{Z})$ , hence it belongs to  $\ker \eta$ . We claim that  $B^2$  is not in the commutator subgroup  $[\Gamma_2(N_4), \Gamma_2(N_4)]$ , hence it represents a nontrivial element of  $\ker h$ . To prove this claim we need to refer to the presentation of  $\mathcal{M}(N_4)$  given in [17]. It follows from this presentation that there exists an epimorphism  $\theta : \mathcal{M}(N_4) \rightarrow \mathbf{Z}_2$  such that  $\theta(B) = 1$  and  $\theta(x) = 0$  for every generator  $x$  different from  $B$ . In particular  $\theta(Y) = 0$ , where  $Y = Y_{\alpha_3, \alpha_{(3,4)}}$ . Since  $\Gamma_2(N_4)$  is the normal closure of  $Y$  we have  $\Gamma_2(N_4) \subset \ker \theta$ . It is a routine to obtain from the presentation of  $\mathcal{M}(N_4)$  a presentation for the index 2 subgroup  $\ker \theta$  and check that  $B^2$  survives in its abelianization, that is  $B^2 \notin [\ker \theta, \ker \theta]$ . Since  $\Gamma_2(N_4) \subset \ker \theta$  also  $B^2 \notin [\Gamma_2(N_4), \Gamma_2(N_4)]$ .  $\square$

## REFERENCES

- [1] N. L. ALLING AND N. GREENLEAF, Foundations of the theory of Klein surfaces, Lect. notes in math. **219**, Springer-Verlag, 1971.
- [2] J. S. BIRMAN AND D. R. J. CHILLINGWORTH, On the homeotopy group of a non-orientable surface, Proc. Camb. Phil. Soc. **71** (1972), 437–448.
- [3] D. R. J. CHILLINGWORTH, A finite set of generators for the homeotopy group of a non-orientable surface, Proc. Camb. Phil. Soc. **65** (1969), 409–430.
- [4] S. GADGIL AND D. PANCHOLI, Homeomorphisms and the homology of non-orientable surfaces, Proc. Indian Acad. Sci. Math. Sci. **115** (2005), 251–257.
- [5] F. J. GONZÁLEZ-ACUÑA AND J. M. MÁRQUEZ-BOBADILLA, On the homeotopy group of the non orientable surface of genus three, Rev. Colombiana Mat. **40** (2006), 75–79.
- [6] N. V. IVANOV, Subgroups of Teichmüller modular groups, Translations of mathematical monographs **115**. Amer. Math. Soc., Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.
- [7] R. M. HAIN, Torelli groups and geometry of moduli spaces of curves, Current topics in complex algebraic geometry, Math. Sci. Res. Inst. Publ. **28**, Cambridge Univ. Press, Cambridge, 1995, 97–143.
- [8] S. HIROSE, On diffeomorphisms over non-orientable surfaces standardly embedded in the 4-sphere, Algebr. Geom. Topol. **12** (2012), 109–130.

- [9] S. HUMPHRIES, Generators of the mapping class group, *Topology of low dimensional manifolds* (R. Fenn, ed.), Lecture notes in math. **722**, Springer-Verlag, Berlin, 1979, 44–47.
- [10] M. KORKMAZ, Mapping class groups of nonorientable surfaces, *Geom. Dedicata* **89** (2002), 109–133.
- [11] R. LEE AND R. H. SZCZARBA, On the homology and cohomology of congruence subgroups, *Invent. Math.* **33** (1976), 15–53.
- [12] W. B. R. LICKORISH, Homeomorphisms of non-orientable two-manifolds, *Proc. Camb. Phil. Soc.* **59** (1963), 307–317.
- [13] W. B. R. LICKORISH, On the homeomorphisms of a non-orientable surface, *Proc. Camb. Phil. Soc.* **61** (1965), 61–64.
- [14] J. D. MCCARTHY AND U. PINKALL, Representing homology automorphisms of nonorientable surfaces, Max Planck Inst., preprint MPI/SFB 85-11, revised version written in 2004, <http://www.math.msu.edu/~mccarthy>.
- [15] A. PUTMAN, The abelianization of the level  $L$  mapping class group, preprint, 2008, [arXiv:0803.0539](https://arxiv.org/abs/0803.0539).
- [16] M. SATO, The abelianization of the level  $d$  mapping class group, *J. Topol.* **3** (2010), 847–882.
- [17] B. SZEPIETOWSKI, A presentation for the mapping class group of the closed non-orientable surface of genus 4, *J. Pure Appl. Algebra* **213** (2009), 2001–2016.
- [18] B. SZEPIETOWSKI, Crosscap slides and the level 2 mapping class group of a nonorientable surface, *Geom. Dedicata* **160** (2012), 169–183.

Błażej Szepietowski  
INSTITUTE OF MATHEMATICS  
GDAŃSK UNIVERSITY  
WITA STWOSZA 57  
80-952 GDAŃSK  
POLAND  
E-mail: blaszep@mat.ug.edu.pl