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LIMITING DISTRIBUTION OF THE MAXIMUM OF A NULL RECURRENT DIFFUSION PROCESS

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Abstract

A limit theorem for the maximum processes of a class of null recurrent linear diffusions is proved. The limiting distribution is a mixture of the Mittag-Leffler distribution.

1. Introduction

Let $X = (X(t))_{t \geq 0}$ be a regular, recurrent diffusion process on an interval $I = (r_1, r_2) \subset \mathbf{R}$ $(-\infty \leq r_1 < 0 < r_2 \leq \infty)$ with the local generator

$$\mathscr{L} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} \quad (a(x) > 0) \tag{1.1}$$

and let $X^*(t) = \max\{X(s); 0 \leq s \leq t\}$. In the present paper we are interested in the limiting laws of

$$\frac{1}{\psi(t)}(X^*(t) - q(t)) \quad (t \to \infty) \tag{1.2}$$

for suitable normalizing functions $\psi(t) > 0$ and $q(t)$.

On this subject we should mention the classical result of Berman [1]. He proved that, if the diffusion is positive recurrent, then the problem is reduced to that for the maximum of i.i.d. random variables and therefore, by the well-known Fisher-Tippet theorem, all possible limit distributions are the Gumbel, the Fréchet, and the Weibull distribution.

On the other hand, in the case of null recurrent diffusions, [1] says that, in some cases, the *Mittag-Leffler distribution* is possible. By Mittag-Leffler distribution we mean the distribution $\mu_{\alpha,t}$ $(0 \leq \alpha \leq 1, t \geq 0)$ on $[0, \infty)$ characterized by

$$\int_{[0,\infty)} e^{-sx}\mu_{\alpha,t}(dx) = \sum_{k=0}^{\infty} \frac{(-s)^k}{\Gamma(k\alpha + 1)} t^{k\alpha}, \quad s > 0$$

(see [4, p. 453] or [11]). Especially, if $\alpha = 0, 1/2$ or 1, then $\mu_{\alpha,t}$ is an exponential

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distribution, a truncated normal distribution, or the unit mass at $x = t$, respectively. The distribution function of $\mu_{\alpha,1}$ is

$$g_\alpha(x) = \frac{1}{\pi\alpha}\int_0^x \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j!} \sin \pi\alpha \cdot \Gamma(\alpha j + 1) u^{j-1}\, du \quad (x > 0)$$

provided that $0 < \alpha < 1$. Another characterization of $\mu_{\alpha,t}$ $(0 < \alpha < 1)$ is the following: Let $Z_\alpha = (Z_\alpha(t))_{t \ge 0}$ be α-stable subordinator (increasing Lévy process) such that

$$E[e^{-sZ_\alpha(t)}] = e^{-ts^\alpha}, \quad s > 0,\ t > 0. \tag{1.3}$$

Then the one-dimensional marginal distribution of the inverse process $Z_\alpha^{-1}(t)$ obeys $\mu_{\alpha,t}$ (cf. [4, p. 453]). Note that $Z_\alpha^{-1}(\cdot)$ is α-self-similar:

$$(Z_\alpha^{-1}(ct))_t \overset{d}{=} (c^\alpha Z_\alpha^{-1}(t))_t, \quad \forall c > 0, \tag{1.4}$$

which follows immediately from $(Z_\alpha(ct))_t \overset{d}{=} (c^{1/\alpha} Z_\alpha(t))_t$ (here, '$\overset{d}{=}$' denotes the equivalence in law). This characterization of $\mu_{\alpha,t}$ in terms of Z_α helps us to understand why [1] says that $\mu_{\alpha,t}$ is possible for the limiting distribution of (1.2) if we recall that the inverse process $(X^*)^{-1}(t)$ has (time-inhomogenous) independent increments due to the strong Markov property of the diffusion. However, as far as the authors know, no concrete examples satisfying the conditions given in [1] are known except for the case $\alpha = 1/2$.

The aim of the present article is to give a limit theorem for (1.2) where the limit distribution is not the Mittag-Leffler distribution itself but is its 'mixture'. Our main result will be given in Section 2, and here we only give a typical example. Let $1 < \rho < 2$ and consider the diffusion corresponding to

$$\mathscr{L} = \frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{\rho - 1}{x} 1_{(-\infty,-1)}(x) \frac{d}{dx}\right), \quad -\infty < x < \infty.$$

Then, X_t^*/t^α $(\alpha = (2-\rho)/2)$, converges in law to the product of two independent random variables; one is $\mu_{\alpha,t}$-distributed and the other Fréche-distributed (see Example 2.4).

Remark 1.1. As we mentioned above our problem is closely related to the study of $\tau_x := (X^*)^{-1}(x)$, which is the first-hitting time of X to x. Therefore, our problem may be regarded as the study of the limit theorem for τ_x as $x \to \infty$. On this subject we should mention the results of Yamazato (e.g. [12]). However, we are treating quite different type of diffusions and there seems no direct relations.

2. Main results

We first rewrite

$$\mathscr{L} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} \quad (a(x) > 0) \tag{2.1}$$

into the form of Feller's canonical representation. To this end it is convenient to rewrite (2.1) as

$$\mathscr{L} = a(x)\left(\frac{d^2}{dx^2} - V'(x)\frac{d}{dx}\right) \quad (a(x) > 0), \tag{2.2}$$

where

$$V(x) = -\int_0^x \frac{b(u)}{a(u)}, \quad -\infty < x < \infty.$$

Now define

$$s(x) = \int_0^x e^{V(u)}\, du \quad (x \in I) \tag{2.3}$$

and

$$m(x) = \int_0^x \frac{1}{a(u)} e^{-V(u)}\, du \quad (x \in I). \tag{2.4}$$

Here, $\int_0^x = -\int_x^0$ if $x < 0$ as usual. So far we did not mention detailed conditions on $a(x)$ and $b(x)$, but we shall assume that $a(x)$ and $b(x)$ are measurable functions such that $V(x)$, $s(x)$ and $m(x)$ are finite for all $x \in I$. Throughout the paper we shall confine ourselves to the case where $s(x) \to -\infty (x \downarrow r_1)$, $s(x) \to \infty (x \uparrow r_2)$ so that $s^{-1}(x)$ is defined for all $x \in \mathbf{R}$, which condition means that the process is recurrent. The function $s(x)$ is referred to as the *scale function*, and the Lebesgue-Stieltjes measure $dm(x)$ is called the *speed measure* or the *canonical measure* of X (see e.g. [5]). Using above functions we can rewrite $\mathscr{L}$ as follows:

$$\mathscr{L} = a(x)e^{V(x)}\frac{d}{dx}\left(e^{-V(x)}\frac{d}{dx}\right) = \frac{d}{dm(x)}\frac{d}{ds(x)}.$$

Next, in order to describe the limiting distribution of (1.2) we prepare the following stochastic process (c.f. [3]). By a *canonical extremal process* we mean a nonnegative, nondecreasing process $(\xi(t))_{t \geq 0}$ with the following finite-dimensional marginal distributions; for $0 \leq t_1 < \cdots < t_n$ and $0 < x_1 < \cdots < x_n$,

$$P(\xi(t_1) \leq x_1, \ldots, \xi(t_n) \leq x_n) = G(x_1)^{t_1} G(x_2)^{t_2 - t_1} \cdots G(x_n)^{t_n - t_{n-1}} \tag{2.5}$$

where $G(x) = e^{-1/x}$ (Fréche distribution). Such a process can be obtained as the maximum process of a Poisson point process with the characteristic measure $\nu(dx) = x^{-2}\, dx$ so that $e^{-\nu([x,\infty))} = G(x)$ (for the definition of Poisson point process see [7]). Note that $\xi(\cdot)$ is 1-self-similar;

$$\left(\frac{1}{c}\xi(ct)\right)_{t \geq 0} \overset{d}{=} (\xi(t))_{t \geq 0}, \quad \forall c > 0. \tag{2.6}$$

Also note that $\xi(\cdot)$ is stochastically continuous (i.e., $P\{\xi(t) = \xi(t-0)\} = 1$ $(\forall t > 0)$), which is clear from $E[1/\xi(t)] = 1/t$.

Our main result is the following:

Throughout the paper '$\xrightarrow{f.d.}$' denotes the convergence of all finite-dimensional marginal distributions.

THEOREM 2.1. *Let* $\gamma > 0$ *and put* $\alpha = 1/(\gamma+1)$. *If*

$$\lim_{x\to-\infty}\frac{-m(s^{-1}(x))}{|x|^{\gamma}} = c > 0, \quad \lim_{x\to\infty}\frac{m(s^{-1}(x))}{x^{\gamma}} = 0, \tag{2.7}$$

then,

$$\left(\frac{c^{\alpha}}{\lambda^{\alpha}}s(X^*(\lambda t))\right)_{t\ge 0} \xrightarrow{f.d.} \left(\frac{1}{C_{\alpha}}\xi(Z_{\alpha}^{-1}(t))\right)_{t\ge 0} \quad (\lambda\to\infty),$$

where $(\xi(t))_{t\ge 0}$ *is a canonical extremal process which is independent of* $(Z_{\alpha}(t))_{t\ge 0}$ *and*

$$C_{\alpha} = \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\{\alpha(1-\alpha)\}^{\alpha}. \tag{2.8}$$

THEOREM 2.2. *If, in addition to the assumptions of Theorem* 2.1,

$$\lim_{\lambda\to\infty}\frac{s^{-1}(\lambda x) - q(\lambda)}{\varphi(\lambda)} = G(x), \quad x > 0, \tag{2.9}$$

for some $\varphi(\lambda)$ (>0), $q(\lambda)$, *and continuous* $G(x)$ $(x>0)$, *then*

$$\frac{1}{\varphi((t/c)^{\alpha})}\{X^*(t) - q((t/c)^{\alpha})\} \xrightarrow{d} G\left(\frac{1}{C_{\alpha}}\xi(1)Z_{\alpha}^{-1}(1)\right) \quad (t\to\infty).$$

The proofs will be given in Section 4.

Remark 2.3. The function $G(x)$ $(x>0)$ in (2.9) is necessarily of the same type as one of the following three functions

$$x^{\beta}, \quad -x^{-\beta}, \quad \log x \quad (\beta > 0),$$

and the law of $G(\xi(1))$ is the Fréche, the Weibull, and the Gumbel distribution, respectively.

Example 2.4. Let $0 < \rho_+ < \rho_- < 2$ and let

$$\mathscr{L} = \frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{\rho(x)-1}{x}\frac{d}{dx}\right), \quad -\infty < x < \infty,$$

where

$$\rho(x) = \begin{cases} \rho_- & (x < -1) \\ 1 & (|x| \le 1) \\ \rho_+ & (x > 1) \end{cases}.$$

Then

$$e^{V(x)} = \begin{cases} |x|^{1-\rho_-} & (x < -1) \\ x & (|x| \le 1) \\ x^{1-\rho_+} & (x > 1) \end{cases},$$

$$s(x) = \begin{cases} \dfrac{-1}{2-\rho_-}(|x|^{2-\rho_-} - 1) - 1 & (x < -1) \\ x & (|x| \le 1) \\ \dfrac{1}{2-\rho_+}(x^{2-\rho_+} - 1) + 1 & (x > 1) \end{cases},$$

$$m(x) = \begin{cases} -\dfrac{2}{\rho_-}(|x|^{\rho_-} - 1) - 2 & (x < -1) \\ 2x & (|x| \le 1) \\ \dfrac{2}{\rho_+}(x^{\rho_+} - 1) + 2 & (x > 1) \end{cases}.$$

Therefore, putting $\gamma = \rho_-/(2-\rho_-)$, $\beta = 1/(2-\rho_+)$, we have

$$\lim_{x\to-\infty} \frac{-m(s^{-1}(x))}{|x|^\gamma} = \frac{2(2-\rho_-)^\gamma}{\rho_-}, \quad \lim_{x\to\infty} \frac{m(s^{-1}(x))}{x^\gamma} = 0$$

and

$$\lim_{x\to\infty} \frac{s(x)}{x^{1/\beta}} = \beta, \quad \text{so that} \quad \lim_{\lambda\to\infty} \frac{s^{-1}(\lambda x)}{\lambda^\beta} = \left(\frac{x}{\beta}\right)^\beta, \quad x > 0.$$

Therefore, we have

$$\left(\frac{c}{t}\right)^{\alpha\beta} X^*(t) \xrightarrow{d} \left(\frac{1}{\beta C_\alpha}\xi(1)Z_\alpha^{-1}(1)\right)^\beta \quad (t\to\infty),$$

where $\alpha = 1/(\gamma+1) = (2-\rho_-)/2$ and

$$c = \frac{2(2-\rho_-)^\gamma}{\rho_-} = \frac{2(2-\rho_-)^{(1/\alpha)-1}}{\rho_-}$$

3. Preliminaries

The basic idea of the proofs is to represent all necessary processes as functionals of a fixed Brownian motion.

Let $B = (B(t))_{t\ge0}$ be a one-dimensional standard Brownian motion starting at 0 and $\{\ell(t,x); t \ge 0, x \in \mathbf{R}\}$ be the local time of B with respect to the measure $2\,dx$:

$$\int_0^t 1_E(B(s))\,ds = 2\int_E \ell(t,x)\,dx, \quad E \in \mathscr{B}(\mathbf{R}).$$

One of the standard ways to construct a diffusion $(X(t))_{t\geq 0}$ with the generator

$$\mathscr{L} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} = \frac{d}{dm(x)}\frac{d}{ds(x)} \tag{3.1}$$

is the following: Let $\tilde{m}(x) = m(s^{-1}(x))$ and let

$$A(t) = \int_{\mathbf{R}} \ell(t,x)\, d\tilde{m}(x), \quad t \geq 0. \tag{3.2}$$

Then, it is well known that

$$Y(t) = B(A^{-1}(t)), \quad t \geq 0 \tag{3.3}$$

is a diffusion with the generator $\tilde{\mathscr{L}} = \dfrac{d}{d\tilde{m}(x)}\dfrac{d}{dx}$, and therefore,

$$X(t) := s^{-1}(Y(t)), \quad (t \geq 0) \tag{3.4}$$

corresponds to (3.1) with the initial condition $X(0) = 0$ (see Itô-McKean [6]). Therefore, in what follows we shall adopt (3.4) for the 'definition' of $(X(t))_{t\geq 0}$. Note that (3.3) and (3.4) imply

$$X^*(t) = s^{-1}(Y^*(t)) \quad \text{and} \quad Y^*(t) = B^*(A^{-1}(t)), \quad t \geq 0, \tag{3.5}$$

where $X^*(t)$, $Y^*(t)$ and $B^*(t)$ are the maximum processes of $X(t)$, $Y(t)$ and $B(t)$, respectively.

Throughout the paper let us say that a càdlàg stochastic process $(Z(t))_{t\geq 0}$ is *parametrized* by $(x(t), y(t))$ if $x(\cdot)$ is a càdlàg process, $y(\cdot)$ is a non-negative, nondecreasing càdlàg process, and if $Z(t) = x(y^{-1}(t))$ *a.s.*. For example, (3.5) means that $X^*(t)$ and $Y^*(t)$ are parametrized by $(s^{-1}(B^*(t)), A(t))$ and $(B^*(t), A(t))$, respectively. In this way the study of $X^*(t)$ (or $Y^*(t)$) may be reduced to that of $(B^*(t), A(t))$.

LEMMA 3.1. *For every $\lambda > 0$,*

$$\left(\frac{1}{\lambda} Y^*(c\lambda^{1/\alpha} t)\right)_{t\geq 0}$$

is parametrized by

$$\left(\frac{1}{\lambda} B^*(\lambda^2 t), \frac{1}{c\lambda^{1/\alpha}} A(\lambda^2 t)\right)_{t\geq 0}. \tag{3.6}$$

Proof. Simply compute the inverse process of the second component and use (3.5). □

To find the limiting distribution of (3.6) we prepare

Lemma 3.2. *For every* $\lambda > 0$,

$$\left(\frac{1}{\lambda}B^*(\lambda^2 t), \frac{1}{c\lambda^{1/\alpha}}A(\lambda^2 t)\right)_{t\geq 0} \stackrel{d}{=} \left(B^*(t), \frac{1}{c}\int_{\mathbf{R}} \ell(t,x)\, d\tilde{m}_\lambda(x)\right)_{t\geq 0} \tag{3.7}$$

where

$$\tilde{m}_\lambda(x) = \frac{1}{\lambda^{(1/\alpha)-1}}\tilde{m}(\lambda x), \quad x \in \mathbf{R}.$$

Proof. Since

$$\left(\frac{1}{\lambda}B(\lambda^2 t), \ell(\lambda^2 t, x)\right)_{t\geq 0} \stackrel{d}{=} (B(t), \ell(t, x/\lambda))_{t\geq 0},$$

we have $\left(\frac{1}{\lambda}B^*(\lambda^2 t)\right)_t \stackrel{d}{=} (B^*(t))_t$ and, simultaneously,

$$\frac{1}{\lambda^{1/\alpha}}A(\lambda^2 t)\left(= \frac{1}{\lambda^{1/\alpha}}\int_{\mathbf{R}} \ell(\lambda^2 t, x)\, d\tilde{m}(x)\right) \tag{3.8}$$

is equivalent in law to

$$\frac{\lambda}{\lambda^{1/\alpha}}\int_{\mathbf{R}} \ell(t,x)\, d\tilde{m}(\lambda x) = \int_{\mathbf{R}} \ell(t,x)\, d\tilde{m}_\lambda(x). \tag{3.9}$$

□

We next find the limiting process of (3.7):

Lemma 3.3. *Under the assumptions of Theorem* 2.1,

$$\left(\frac{1}{\lambda}B^*(\lambda^2 t), \frac{1}{c\lambda^{1/\alpha}}A(\lambda^2 t)\right)_{t\geq 0} \xrightarrow{d} (B^*(t), A_\alpha(t))_{t\geq 0} \tag{3.10}$$

over the function space $C([0,\infty); \mathbf{R}^2)$, *where*

$$A_\alpha(t) = \int_{\mathbf{R}} \ell(t,x)\, dm^{(\alpha)}(x), \quad m^{(\alpha)}(x) = \begin{cases} -(-x)^{(1/\alpha)-1} & (x < 0) \\ 0 & (x \geq 0) \end{cases}.$$

Proof. We first note that (2.7) implies

$$\begin{aligned}\frac{1}{c}\tilde{m}_\lambda(x) = \frac{1}{c}m(s^{-1}(\lambda x)) &= \frac{1}{c}x^\gamma \frac{m(s^{-1}(\lambda x))}{(\lambda x)^\gamma} \\ &\to m^{(\alpha)}(x) \quad (\lambda \to \infty), \quad \forall x \in \mathbf{R}.\end{aligned}$$

Therefore, $\frac{1}{c}\, d\tilde{m}_\lambda(x)$ converges vaguely to $dm^{(\alpha)}(x)$; i.e.,

$$\frac{1}{c}\int_{\mathbf{R}} f(x)\, d\tilde{m}_\lambda(x) \to \int_{\mathbf{R}} f(x)\, dm^{(\alpha)}(x) \quad (\lambda \to \infty)$$

for all continuous function $f(x)$ vanishing outside a compact set. Thus we have

$$\frac{1}{c}\int_{\mathbf{R}} \ell(t,x)\, d\tilde{m}_\lambda(x) \to \int_{\mathbf{R}} \ell(t,x)\, dm^{(\alpha)}(x) \tag{3.11}$$

for every fixed $t \geq 0$. In fact, this convergence is automatically uniform in t on every finite interval because the both sides are nondecreasing and the right-hand side is continuous by Pólya's extension of Dini's theorem (see e.g. [2, 1.11.22]). Now combining (3.11) with Lemma 3.2 we can deduce (3.10). □

Proposition 3.4. *Under the assumptions of Theorem* 2.1,

$$\left(\frac{c^\alpha}{\lambda^\alpha} Y^*(\lambda t)\right)_{t\geq 0} \xrightarrow{f.d.} (B^*(A_\alpha^{-1}(t)))_{t\geq 0} \quad (\lambda \to \infty).$$

Proof. In (3.10), each side is a parametrization of $(1/\lambda)Y^*(c\lambda^{1/\alpha}t)$ or of $B^*(A_\alpha^{-1}(t))$. Therefore, Lemma 3.1 implies that

$$\left(\frac{1}{\lambda} Y^*(c\lambda^{1/\alpha} t)\right)_{t\geq 0} \xrightarrow{f.d.} (B^*(A_\alpha^{-1}(t))_{t\geq 0} \quad (\lambda \to \infty).$$

For this kind of arguments see Appendix. Now change the variable (replace $c\lambda^{1/\alpha}$ by λ). □

For the proof of Theorem 2.1 our next task is to show that the limit process $B^*(A_\alpha^{-1}(t))$ in Proposition 3.4 is distributed like $\xi(Z_\alpha^{-1}(t))$ in Theorem 2.1 up to a multiplicative constant C_α. To this end let us represent $Z_\alpha(\cdot)$ and $\xi(\cdot)$ as functionals of the Brownian motion $B(\cdot)$:

Let $A_\alpha(t)$ be as before and let

$$T_\alpha(t) = A_\alpha(\ell^{-1}(t,0)) \; \left(= \int_{\mathbf{R}} \ell(\ell^{-1}(t,0),x)\, dm^{(\alpha)}(x)\right), \quad t \geq 0. \tag{3.12}$$

(Here, $\ell^{-1}(t,0) := \inf\{s; \ell(s,0) > t\}$.) Then, it is well-known that $(T_\alpha(t))_{t\geq 0}$ is an α-stable subordinator such that

$$E[e^{-sT_\alpha(t)}] = e^{-C_\alpha t s^\alpha}, \quad t \geq 0,\, s > 0, \tag{3.13}$$

where C_α is the same as in (2.8) (see e.g. [9]). Therefore, comparing (1.3) and (3.13), we see that $(T_\alpha(t/C_\alpha))_{t\geq 0}$ is identical in law to $(Z_\alpha(t))_{t\geq 0}$. Thus in what follows it is harmless to assume that

$$Z_\alpha(t) = T_\alpha(t/C_\alpha). \tag{3.14}$$

We next construct a process $\xi(t)$ given in Theorem 2.1; i.e., a process which is independent of Z_α and has the marginal distribution (2.5). An answer is

$$\xi(t) = B^*(\ell^{-1}(t)), \quad t \geq 0, \quad \ell(t) = \ell(t,0).$$

Indeed, this is a canonical extremal process because the right-hand side is the maximum process of a $(0,\infty)$-valued Poisson point process with characteristic measure $\nu(dx) = x^{-2}\,dx$, which fact is well-known in the excursion theory for the Brownian motion (see [7, Sec. 4.3]). It remains to check that $B^*(\ell^{-1}(\cdot))$ is independent of $Z_\alpha(\cdot)$. However, it is clear because $B^*(\ell^{-1}(\cdot))$ is a functional of positive excursions while $Z_\alpha(\cdot)$ is a functional of negative excursions (positive excursions and negative excursions are independent).

Lemma 3.5. *Let $\xi(t)$ and $Z_\alpha(t)$ be as above. Then, for every $t \geq 0$,*

$$B^*(A_\alpha^{-1}(t)) = \xi(T_\alpha^{-1}(t)) = \xi\left(\frac{1}{C_\alpha} Z_\alpha^{-1}(t)\right) \quad a.s.$$

Proof. Since the latter equality follows from (3.14) we shall prove the first only. By the definition of $T_\alpha(t)$ (see (3.12)), we have

$$T_\alpha^{-1}(t) = \ell(A_\alpha^{-1}(t)),$$

where $\ell(t) = \ell(t, 0)$. Combining this with $\xi(t) = B^*(\ell^{-1}(t))$ we roughly have

$$\xi(T_\alpha^{-1}(t)) = B^*(\ell^{-1} \circ \ell \circ A_\alpha^{-1}(t)) = B^*(A_\alpha^{-1}(t)). \tag{3.15}$$

This heuristic argument involves a problem because, precisely speaking, $\ell^{-1} \circ \ell(t) = t$ fails. To be strict (3.15) should be replaced by

$$\xi(T_\alpha^{-1}(t-0)-0) \leq B^*(A_\alpha^{-1}(t)) \leq \xi(T_\alpha^{-1}(t))$$

(see Theorem 5.1 in Appendix). Therefore, it remains to show that $\xi(T_\alpha^{-1}(t-0)-0) = \xi(T_\alpha^{-1}(t))$ with probability one for every fixed $t \geq 0$. Since $T_\alpha^{-1}(t-0) = T_\alpha^{-1}(t)$ *a.s.* (when t is fixed), it is sufficient to prove

$$P(\xi(T_\alpha^{-1}(t)-0) = \xi(T_\alpha^{-1}(t))) = 1, \quad \forall t > 0.$$

However, by the independence (see (i)), the left-hand side equals

$$\int_{(0,\infty)} P(\xi(s-0) = \xi(s))\mu_{T_\alpha^{-1}(t)}(ds) = 1$$

because $\xi(\cdot)$ is stochastically continuous as we mentioned before. □

Now we have that the limit process in Theorem 2.1 and that in Proposition 3.4 are equivalent in law;

Proposition 3.6.

$$(B^*(A_\alpha^{-1}(t)))_{t\geq 0} \stackrel{d}{=} \left(\frac{1}{C_\alpha} \cdot \xi(Z_\alpha^{-1}(t))\right)_{t\geq 0} \tag{3.16}$$

Proof. By Lemma 3.5 the left-hand side is identical in law to

$$\left(\xi\left(\frac{1}{C_\alpha} \cdot Z_\alpha^{-1}(t)\right)\right)_{t\geq 0}$$

and, by the 1-self-similarity of $\xi(\cdot)$ (see (2.6)), the right-hand side is equivalent in law to the right-hand side of (3.16). □

COROLLARY 3.7.

$$B^*(A_\alpha^{-1}(1)) \stackrel{d}{=} \frac{1}{C_\alpha} \cdot \xi(1) \cdot Z_\alpha^{-1}(1)$$

Proof. The left-hand side is identical in law to $\frac{1}{C_\alpha}\xi(Z_\alpha^{-1}(1))$ by Proposition 3.6. Since $\xi(\cdot)$ and $Z_\alpha^{-1}(1)$ are independent and $\xi(\cdot)$ is 1-self-similar, we see that $\xi(Z_\alpha^{-1}(1))$ is equivalent in law to $Z_\alpha^{-1}(1) \cdot \xi(1)$. □

4. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Combining Propositions 3.4 and 3.6 we have

$$\left(\frac{c^\alpha}{\lambda^\alpha} Y^*(\lambda t)\right)_{t\geq 0} \xrightarrow{f.d.} \left(\frac{1}{C_\alpha} \cdot \xi(Z_\alpha^{-1}(t))\right)_{t\geq 0} \quad (\lambda \to \infty).$$

Then recall that $Y^*(t) = s(X^*(t))$ (see (3.5)). □

Proof of Theorem 2.2. Let

$$G_\lambda(x) = \frac{s^{-1}((\lambda/c)^\alpha x) - q((\lambda/c)^\alpha)}{\varphi((\lambda/c)^\alpha)}, \quad x > 0.$$

Then (2.9) implies

$$\lim_{\lambda\to\infty} G_\lambda(x) = G(x), \quad x > 0. \tag{4.1}$$

Note that the convergence in (4.1) is uniform on every compact set in $(0, \infty)$ because $G_\lambda(x)$ is monotone and $G(x)$ is continuous. Therefore, (4.1) and Theorem 2.1 imply

$$G_\lambda\left(\frac{c^\alpha}{\lambda^\alpha} s(X^*(\lambda t))\right)_{t\geq 0} \xrightarrow{f.d.} G\left(\frac{1}{C_\alpha}\xi(Z_\alpha^{-1}(t))\right)_{t\geq 0} \quad (\lambda \to \infty), \tag{4.2}$$

that is,

$$\frac{1}{\varphi((\lambda/c)^\alpha)}\{X^*(\lambda t) - q((\lambda/c)^\alpha)\} \xrightarrow{f.d.} G\left(\frac{1}{C_\alpha}\xi(Z_\alpha^{-1}(t))\right)_{t\geq 0}.$$

Especially,

$$\frac{1}{\varphi((\lambda/c)^\alpha)}\{X^*(\lambda) - q((\lambda/c)^\alpha)\} \xrightarrow{d} G\left(\frac{1}{C_\alpha}\xi(Z_\alpha^{-1}(1))\right).$$

Since $\xi(Z_\alpha^{-1}(1))$ is equivalent in law to $Z_\alpha^{-1}(1)\xi(1)$ by the self-similarity of $(\xi(t))_{t\geq 0}$, we have the assertion of the theorem. □

5. Appendix

In the present paper we said that a càdlàg process $(Z(t))_{t\geq 0}$ is parametrized by two càdlàg processes $X(\cdot)$ and $Y(\cdot)$ if $Y(\cdot)$ is nondecreasing and if $Z(t) = X(Y^{-1}(t))$ *a.s.* (see Section 3). In this section we prove two theorems on the parametrized processes.

THEOREM 5.1. *Let $f(t)$, $g(t)$, $h(t)$ be nondecreasing, right-continuous and nonnegative functions defined on $[0,\infty)$ and define $f_h(t) = f(h(t))$ and $g_h(t) = g(h(t))$. Then,*

$$f_h(g_h^{-1}(t-0)-0) \leq f(g^{-1}(t)) \leq f_h(g_h^{-1}(t)), \quad t > 0.$$

Proof. Draw the graph $G(g,f) = \{(g(s), f(s)); s \geq 0\}$ and see how $f(g^{-1}(t))$ is determined. Then observe that $G(g_h, f_h) \subset G(g,f)$. □

Let $D = D([0,\infty) : \mathbf{R})$ be the space of all $\mathbf{R}$-valued càdàg functions endowed with the usual Skorohod J_1-topology (see [10] for the definition). We denote by Φ $(\subset D)$ the totality of càdlàg nondecreasing functions $f : [0,\infty) \to [0,\infty)$ and let $\Phi_\infty = \{f \in \Phi : \lim_{x\to\infty} f(x) = \infty\}$. For $f \in \Phi$, we always define $f(-0) = 0$ for convenience' sake.

THEOREM 5.2. *Let $(X_\lambda(t))_{t\geq 0}$, $(Y_\lambda(t))_{t\geq 0}$, $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ be stochastic processes with sample paths in Φ and suppose that $P(Y_\lambda \in \Phi_\infty) = P(Y \in \Phi_\infty) = 1$ so that the inverse processes $(Y_\lambda^{-1}(t))_{t\geq 0}$ and $(Y^{-1}(t))_{t\geq 0}$ make sense.*

(i) *If*

$$(X_\lambda(t), Y_\lambda(t)) \xrightarrow{d} (X(t), Y(t)) \quad \text{in } D \times D \tag{5.1}$$

and if

$$P\{X(Y^{-1}(t-0)-0) = X(Y^{-1}(t))\} = 1, \quad \forall t \geq 0 \tag{5.2}$$

then,

$$X_\lambda(Y_\lambda^{-1}(t)) \xrightarrow{f.d.} X(Y^{-1}(t)).$$

(ii) *Each of the following two conditions is sufficient for* (5.2):

(A1) *$(X(t))_{t\geq 0}$ has continuous paths and*

$$P\{Y^{-1}(t-0) = Y^{-1}(t)\} = 1 \quad (\forall t \geq 0).$$

(A2) *$(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ are independent and*

$$P\{X(t) = X(t-0)\} = P\{Y^{-1}(t) = Y^{-1}(t-0)\} = 1 \quad (\forall t \geq 0).$$

Proof. By Skorohod's theorem (5.1) can be realized by an almost-sure convergence: On a suitable probability space we can construct càdàg processes $\hat{X}_\lambda$, $\hat{X}$, $\hat{Y}_\lambda$, $\hat{Y}$ with the following properties.

(1) $(\hat{X}_\lambda, \hat{Y}_\lambda)$ is equivalent in law to (X_λ, Y_λ)

(2) $(\hat{X}, \hat{Y})$ is equivalent in law to (X, Y)

(3) $(\hat{X}_\lambda, \hat{Y}_\lambda) \stackrel{J_1}{\to} (\hat{X}, \hat{Y})$ with probability one.

Since J_1-convergence implies the convergence at all continuity points of the limit function, it follows from (1) that, with probability one,

$$\hat{X}(t-0) \le \liminf_{\lambda\to\infty} \hat{X}_\lambda(t-0) \le \limsup_{\lambda\to\infty} \hat{X}_\lambda(t) \le \hat{X}(t), \quad \forall t \ge 0$$

and

$$\hat{Y}^{-1}(t-0) \le \liminf_{\lambda\to\infty} \hat{Y}_\lambda^{-1}(t-0) \le \limsup_{\lambda\to\infty} \hat{Y}_\lambda^{-1}(t) \le \hat{Y}^{-1}(t), \quad \forall t \ge 0.$$

(Recall that we defined $X(t-0) = Y^{-1}(t-0) = 0$ when $t = 0$.) Therefore,

$$\begin{aligned}\hat{X}(\hat{Y}^{-1}(t-0)-0) &\le \liminf_{\lambda\to\infty} \hat{X}_\lambda(\hat{Y}_\lambda^{-1}(t-0)-0) \\ &\le \limsup_{\lambda\to\infty} \hat{X}_\lambda(\hat{Y}_\lambda^{-1}(t)) \le \hat{X}(\hat{Y}^{-1}(t)), \quad \forall t \ge 0\end{aligned}$$

Thus we can deduce the assertion of (i). Let us prove (ii). Since it is clear that (A1) is sufficient, let us see that (A2) implies (5.2). For every fixed $t \ge 0$, we assume that $P\{Y^{-1}(t) = Y^{-1}(t-0)\} = 1$. Therefore, it is sufficient to show that

$$P\{X(Y^{-1}(t)-0) = X(Y^{-1}(t))\} = 1.$$

But this is easy because X and Y are independent;

$$P\{X(Y^{-1}(t)-0) = X(Y^{-1}(t))\} = \int_{[0,\infty)} P\{X(u) = X(u-0)\}\mu_{Y^{-1}(t)}(du) = 1. \qquad \square$$

References

[1] S. M. Berman, Limiting distribution of the maximum of a diffusion process, Ann. Math. Statist. **35** (1964), 319–329.

[2] N. H. Bingham, C. M. Goldie and J. L. Teugels, Regular variation, Encyclopedia of mathematics and its applications **27**, Cambridge University Press, Cambridge, 1987.

[3] M. Dwass, Extremal processes, Ann. Math. Statist **35** (1964), 1718–1725.

[4] W. Feller, An introduction to probability theory and its applications, II, 2nd ed., John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[5] K. Itô, Essentials of stochastic processes, Translations of mathematical monographs **231**, AMS, 2006.

[6] K. Itô and H. P. McKean, Jr., Diffusion processes and their sample paths, Springer, 1965, Classics in Math., 1974.

[7] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, 2nd ed., North-Holland Mathematical Library **24**, North-Holland, Amsterdam, Kodansha, Tokyo, 1989.

[8] Y. Kasahara and S. Watanabe, Brownian representation of a class of Lévy processes and its application to occupation times of diffusion processes, Illinois J. Math. **50** (2006), 515–539.

[9] S. Kotani and S. Watanabe, Krein's spectral theory of strings and generalized diffusion processes, Functional analysis in Markov processes (Katata/Kyoto, 1981), Lecture notes in math. **923**, Springer, Berlin-New York, 1982, 235–259.

[10] T. Lindvall, Weak convergence of probability measures and random functions in the function space $D[0,\infty)$, J. Appl. Probability **10** (1973), 109–121.

[11] H. Pollard, The representation of $e^{-x^{\lambda}}$ as a Laplace integral, Bull. Amer. Math. Soc. **52** (1946), 908–910.

[12] M. Yamazato, Hitting time distributions of 1-dimensional generalized diffusions, Trends in probability and related analysis, Taipei, 1996, World Sci. Publ., River Edge, NJ, 1997, 325–338.

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