Y. JIANG AND Y. QI
KODAI MATH. J.
35 (2012), 415–424

A GLUING THEOREM FOR QUASICONFORMAL MAPPINGS

Yunping Jiang and Yi Qi

Abstract

We prove, by using the main inequality of Reich and Strebel, that any n K-quasiconformal germs defined on n disjoint domains in the Riemann sphere can be glued by one $(K+\varepsilon)$-quasiconformal homeomorphism, where ε is a positive number which can go to zero as the domains of germs shrinking to n points. This generalizes a result in [8] where only the case $K = 1$ has been considered.

1. Introduction

An analytic mapping defined on a domain in the complex plane is very rigid because of the famous uniqueness theorem of analytic functions. This rigid phenomenon causes a big difficulty in the study of dynamics of an analytic mapping, especially when we would like to use the surgery method to discover a new analytic mapping which may demonstrate some new dynamical phenomena. To have certain flexibility, a quasiconformal surgery method is recently introduced into the study of holomorphic dynamical systems and becomes a very successful theory. However, controlling the quasiconformal constant in a quasiconformal surgery is not easy and may still cause a problem in the study of dynamics of an analytic mapping, especially when we would like to construct a new dynamical system close to the old one in an appropriate metric. It is very important to have a general method in the quasiconformal surgery so that we can control the quasiconformal constant. Then we can estimate the Teichmüller distance between the old and new ones.

We have got the following theorem in [8] by using the holomorphic motion method.

Theorem A. *Let $\{z_k\}_{k=1}^n$ be a set of distinct points in the complex plane* $\mathbf{C}$ *and let U_k be a neighborhoods of z_k for every $k = 1, 2, \ldots, n$. Suppose $\{U_k\}_{k=1}^n$ are pairwise disjoint and $f_k(z)$ is a conformal mapping defined on U_k which fixes z_k*

2000 *Mathematics Subject Classification*. Primary 37F30, Secondary 30C70.

Key words and phrases. main inequality, conformal germ, gluing.

The research of YJ is partially supported by PSC-CUNY awards.

The research of YQ is partially supported by the National Natural Science Foundation of China (Grant No. 10971008 and 10571009).

Received September 7, 2011; revised January 20, 2012.

for every $k = 1, 2, \dots, n$. *Then for every* $\varepsilon > 0$ *there exist a number* $r > 0$ *and a* $(1+\varepsilon)$*-quasiconformal self-mapping* f *of* $\mathbf{C}$ *such that*

$$f|_{U(z_k, r)} = f_k|_{U(z_k, r)},$$

where $U(z_k, r) \subset U_k$ *is the open disk of radius* r *centered at* z_k *for* $k = 1, \dots, n$.

The proof of Theorem A in [8] is carried out by using a big theorem called the holomorphic motion theorem (refer to [5]) which has been developed for many years by Mañe-Sad-Sullivan [10], Thurston-Sullivan [14], Bers-Royden [2], and Slodkowski [12] (see also [1], [3], [4], and [5]). Although the proof in [8] is simple, the deep inside mechanism is hidden due to the use of the holomorphic motion theorem. Therefore, we would like to have a straightforward understanding directly from the main inequality of Reich and Strebel in quasiconformal mapping theory. More importantly, we are not only giving a new proof of Theorem A by viewing some inside mechanism but also we prove a more general new result as follows. This is the main purpose of this paper.

Theorem 1. *Let* $\{z_k\}_{k=1}^n$ *be a set of distinct points in the complex plane* $\mathbf{C}$ *and let* U_k *be a neighborhoods of* z_k *for every* $k = 1, 2, \dots, n$. *Suppose* $\{U_k\}_{k=1}^n$ *are pairwise disjoint and* $f_k(z)$ *is a* K*-quasiconformal mapping defined on* U_k *which fixes* z_k *for every* $k = 1, 2, \dots, n$. *Then for every* $\varepsilon > 0$ *there exist a number* $r > 0$ *and a* $(K+\varepsilon)$*-quasiconformal self-homeomorphism* f *of* $\mathbf{C}$ *such that*

$$f|_{U(z_k, r)} = f_k|_{U(z_k, r)},$$

where $U(z_k, r) \subset U_k$ *is the open disk of radius* r *centered at* z_k *for* $k = 1, \dots, n$.

Remark. Theorem 1 is also true in the case of that $\{z_k\}_{k=1}^n$ is a set of distinct points in the Riemann sphere $\hat{\mathbf{C}}$. Here, the neighborhood $U(\infty, r)$ of ∞ should be understood as $U(\infty, r) = \{z : |z| > r\}$.

This paper is organized as follows. In §2, we give a brief review of the main inequality of Reich and Strebel and several interesting results in the quasiconformal mapping theory, which will be used in the proof of Theorem 1. In §3, we first prove two theorems, Theorem 2 and Theorem 3, which are of independent interest. Then we prove Theorem 1 by using Theorem 3.

Acknowledgements. The authors would like to thank the referee for their helpful suggestions and comments.

2. Some preparation

To prove Theorem 1, we need some preparation from the quasiconformal mapping theory. The key tool in our proof of Theorem 1 is the famous main inequality of Reich and Strebel [11]. The reader may refer to [6] for some standard notations.

Theorem B [Main Inequality]. *Suppose $f, g : R \to R'$ are two quasiconformal homeomorphisms from a Riemann surface R to another Riemann surface R' which are homotopic modulo the boundary. Then for every integrable holomorphic quadratic differential $\varphi = \varphi(z)\,dz^2$, we have*

$$\|\varphi\| \le \iint_R |\varphi(z)| \frac{\left|1 - \mu_f(z)\dfrac{\varphi(z)}{|\varphi(z)|}\right|^2}{1 - |\mu_f(z)|^2} \cdot \frac{\left|1 - \mu_{g^{-1}}(f(z))\theta \dfrac{\varphi(z)}{|\varphi(z)|}\right|^2}{1 - |\mu_{g^{-1}}(f(z))|^2}\, dxdy, \tag{1}$$

where $\|\varphi\| = \iint_R |\varphi(z)|\, dxdy$ and

$$\theta = \frac{\overline{f_z}}{f_z} \frac{1 - \overline{\mu_f \varphi}/|\varphi|}{1 - \mu_f \varphi/|\varphi|},$$

μ_f and $\mu_{g^{-1}}$ are Beltrami coefficients of the quasiconformal homeomorphisms f and g^{-1} respectively.

We refer to [6] for a proof of Theorem B for arbitrary Riemann surfaces.

In the same condition as Theorem B, the following inequality can be deduced from (1) easily.

$$\|\varphi\| \le \iint_R |\varphi(z)| \frac{\left|1 - \mu_f(z)\dfrac{\varphi}{|\varphi|}\right|^2}{1 - |\mu_f(z)|^2} D_{g^{-1}}(f(z))\, dxdy, \tag{2}$$

where $D_{g^{-1}}(w)$ is the dilatation of g^{-1} at w.

Suppose $f : R \to R'$ is a quasiconformal homeomorphism from a Riemann surface R onto another Riemann surface R'. The boundary dilatation of f is defined as

$$H^*(f) = \inf\{K(f|_{R\setminus E}) \mid E \text{ is a compact subset of } R\},$$

where $K(f|_{R\setminus E})$ is the maximal dilatation of $f|_{R\setminus E}$.

Denote by $[f]$ the set of all quasiconformal homeomorphisms from R to R' which are homotopic to f modulo the boundary. Then the extremal dilatation and the boundary dilatation of $[f]$ are defined as

$$K_0([f]) = \inf\{K(g) \mid g \in [f]\}$$

and

$$H([f]) = \inf\{H^*(g) \mid g \in [f]\}.$$

A quasiconformal homeomorphism $f_0 \in [f]$ is called extremal if

$$K(f_0) = K_0([f]).$$

The frame mapping theorem of Strebel [13] can be now stated as follows.

Theorem C. *Suppose $f : R \to R'$ is a quasiconformal homeomorphism from a Riemann surface R to another Riemann surface R'. Suppose $H([f]) < K_0([f])$.*

Then there is a unique extremal quasiconformal homeomorphism $f_0 \in [f]$. *Moreover, the Beltrami coefficient of* f_0 *has the form* $\mu_{f_0} = k\dfrac{\bar{\varphi}}{|\varphi|}$, *where*

$$0 \le k = \frac{K_0([f]) - 1}{K_0([f]) + 1} < 1$$

and φ *is a holomorphic quadratic differential on* R *with* $\|\varphi\| = 1$.

The proof of the frame mapping theorem for a general Riemann surface can be found, for example, in [6].

A sense preserving self-homeomorphism $h : \mathbf{R} \to \mathbf{R}$ of the real line is called quasisymmetric if there is a constant $M \ge 1$ such that

$$\frac{1}{M} \le \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \le M$$

holds for all $x \in \mathbf{R}$ and $t > 0$. It is known that a sense preserving self-homeomorphism of $\mathbf{R}$ is quasisymmetric if and only if it is the boundary values of a quasiconformal self-homeomorphism of the upper half-plane $\mathbf{H}$. Since every round disk in the Riemann sphere can be mapped onto the upper half-plane by a Möbius transformation, we call a homeomorphism of a circle onto another circle quasisymmetric if it is the boundary values of a quasiconformal homeomorphism between the disks bounded by the circles. The following result can be found in R. Fehlmann's paper [7].

Theorem D. *Suppose* h *is a sense preserving self-homeomorphism of the unit circle* S^1. *If for every* $\zeta \in S^1$, h *can be extended to a neighborhood of* ζ *quasiconformally, then* h *is a quasisymmetric self-homeomorphism of* S^1.

We use the following notations in the rest of this paper:

$$U(z_0, r) = \{z : |z - z_0| < r\} \quad \text{and} \quad A(z_0; r, R) = \{z : r < |z - z_0| < R\}.$$

We denote by $\partial U(z_0, r)$ the boundary circle of $U(z_0, r)$. In case of $z_0 = 0$, we simply denote them by $U(r)$, $A(r, R)$ and $\partial U(r)$.

The following theorem is a known result, which can be found in [9], for example.

Theorem E. *Suppose* $f_1(\theta)$ *and* $f_2(\theta)$ *are two sense preserving homeomorphisms from the real line* $\mathbf{R}$ *to itself and*

$$f_k(0) = 0 \quad \text{and} \quad f_k(\theta + 2\pi) = f_k(\theta) + 2\pi, \quad k = 1, 2.$$

Let $r_0 > 1$ *be a real number. Then there is a quasiconformal self-homeomorphism* F *of the annulus* $A(1, r_0)$ *with*

$$F[\exp(i\theta)] = \exp[if_1(\theta)] \quad \text{and} \quad F[r_0 \exp(i\theta)] = r_0 \exp[if_2(\theta)]$$

if and only if f_1 *and* f_2 *are quasisymmetric.*

As [9] is written in Chinese and published on a Chinese journal, it is not available for people outside China. For the completeness of this paper, we sketch out a proof of Theorem E here.

Sketch of Proof. The necessity is clear. One only need to prove the sufficiency.

Let

$$\pi_1(z) = r_0 e^{iz} : \{z \mid 0 < \Im z < \log r_0\} \to A(1, r_0)$$

be a holomorphic covering of the ring domain $A(1, r_0)$, and let

$$\pi_2(z) = \Re \log z + i \frac{\log r_0}{\pi} \Im \log z,$$

where $\log z$ is the principle value of the logarithm function and $\Re z$ and $\Im z$ are the real and imaginary part of complex number z. Obviously, π_2 is a quasiconformal mapping from $\mathbf{H}$ onto the band domain $\{z \in \mathbf{C} : 0 < \Im z < \log r_0\}$, with the maximal dilatation $K = \max(\pi/\log r_0, \log r_0/\pi)$. Therefore, $\pi = \pi_1 \circ \pi_2$ is a quasiconformal covering of the annulus $A(1, r_0)$ and the boundary mappings

$$z = \exp(i\theta) \mapsto \exp[if_1(\theta)] : \partial U(1) \to \partial U(1)$$

and

$$z = r_0 \exp(i\theta) \mapsto r_0 \exp[if_2(\theta)] : \partial U(r_0) \to \partial U(r_0)$$

can be lift to a sense-preserving self-homeomorphism of $\mathbf{R}$

$$f(x) = \begin{cases} \exp[f_2(\log x)], & 0 < x; \\ 0, & x = 0; \\ -\exp[f_1(\log|x|)], & x < 0. \end{cases}$$

If f is quasisymmetric, then it can be extended to a quasiconformal self-homeomorphism $\tilde{F}$ of $\mathbf{H}$ by the Beurling-Ahlfors extension

$$\tilde{F}(x + iy) = \frac{1}{2y} \int_{-y}^{y} f(x + t)\, dt + \frac{i}{2y} \int_0^y [f(x + t) - f(x - t)]\, dt.$$

It is easy to check that

$$f[\exp(2\pi)x] = \exp(2\pi) f(x), \quad \forall x \in \mathbf{R}.$$

So, by the property of integral and the definition of Beurling-Ahlfors extension,

$$\tilde{F}[\exp(2\pi)z] = \exp(2\pi)\tilde{F}(z), \quad z \in \mathbf{C},$$

and consequently, $\tilde{F}(z)$ induces a quasiconformal self-homeomorphism F of $A(1, r_0)$, which is an extension we needed.

Therefore, we only need to check that $f(x)$ is quasisymmetric on the real line. That is, we need to estimate the upper and lower bounds of

$$\Delta(x, t) = \frac{f(x + t) - f(x)}{f(x) - f(x - t)}$$

for all $x \in \mathbf{R}$ and $t > 0$. To do that we divide into 3 cases: (I) $x = 0$, $t > 0$, (II) $x > 0$, $t > 0$, and (III) $x < 0$, $t > 0$. We can easily get the estimates in case (I). For case (II), we can obtain the estimates by divided into 4 subcases further: $0 < t < x/2$, $x/2 \leq t \leq x$, $x < t \leq 2x$ and $t > 2x$. The estimates in case (III) can be done similarly to case (II). $\square$

3. Proof of Theorem 2

In order to prove Theorem 1, we first prove the following theorems, Theorem 2 and Theorem 3, which are of independent interest.

THEOREM 2. *Let f_1 and f_2 be two K-quasiconformal mappings defined on disjoint simply connected subdomains Ω_1 and Ω_2 of $\hat{\mathbf{C}}$ respectively and $f_1(\Omega_1) \cap f_2(\Omega_2) = \emptyset$. Then, for any two Jordan domains D_1 and D_2 with $\overline{D_1} \subset \Omega_1$ and $\overline{D_2} \subset \Omega_2$, there exists a quasiconformal mapping g of $\hat{\mathbf{C}}$, such that*

$$g|_{D_1} = f_1|_{D_1} \quad \text{and} \quad g|_{D_2} = f_2|_{D_2}.$$

To prove Theorem 2, we need the following lemma which can be derived from Theorem E directly.

LEMMA 1. *Suppose $0 < r_1 < r_2$ and $0 < R_1 < R_2$ are real numbers. Suppose $f_k : \partial U(z_0, r_k) \to \partial U(w_0, R_k)$, $k = 1, 2$, are two homeomorphisms. Then there exists a quasiconformal homeomorphism $F : A(z_0; r_1, r_2) \to A(w_0; R_1, R_2)$ satisfying*

$$F|_{\partial U(z_0, r_k)} = f_k, \quad (k = 1, 2)$$

if and only if f_1 and f_2 are both quasisymmetric.

In fact, we may assume, without lose of generality, that $z_0 = w_0 = 0$, $r_1 = R_1 = 1$, $f_1(1) = 1$, and $f_2(r_2) = R_2 e^{i\theta_0}$. Then Lemma 1 is derived by applying Theorem E to

$$\tilde{f}_k = G \circ f_k, \quad k = 1, 2,$$

where

$$G(re^{i\theta}) = \left[1 + \frac{r_2 - 1}{R_2 - 1}(r - 1)\right] e^{i(\theta - (r-1)/(R_2 - 1)\theta_0)} : A(1, R_2) \to A(1, r_2)$$

is a quasiconformal homeomorphism of $A(1, R_2)$ onto $A(1, r_2)$.

Proof of Theorem 2. Since ∂D_1 and ∂D_2 are closed Jordan curves, $f_1(\partial D_1)$ and $f_2(\partial D_2)$ are also closed Jordan curves. Let B and $\tilde{B}$ be ring domains bounded by Jordan curves ∂D_1 and ∂D_2 and by $f_1(\partial D_1)$ and $f_2(\partial D_2)$, respectivly. Then there are conformal mappings $F : B \to A(r_1, r_2)$ and $G : \tilde{B} \to A(R_1, R_2)$ of these ring domains onto annuli, which can be extended to the boundaries of the ring domains continuously.

Assume, without losing generality, that $F(\partial D_k) = \partial U(r_k)$ and $G(f_1(\partial D_k)) = \partial U(R_k)$ for $k = 1, 2$. Then $G \circ f_k \circ F^{-1}$ is a quasiconformal mapping from the ring domain bounded by $\partial U(r_k)$ and $F(\partial \Omega_k)$ onto the ring domain bounded by $\partial U(R_k)$ and $G(\partial f_k(\Omega_k))$ for $k = 1, 2$. Thus, by Theorem D, $G \circ f_k \circ F^{-1}|_{\partial U(r_k)}$ is a quasisymmetric mappings of $\partial U(r_k)$ onto $\partial U(R_k)$. Hence, there is a quasiconformal mapping $\Phi : A(r_1, r_2) \to A(R_1, R_2)$ with

$$\Phi|_{\partial U(r_k)} = G \circ f_k \circ F^{-1}|_{\partial U(r_k)},$$

by Lemma 1. Therefore,

$$g(z) = \begin{cases} f_1(z), & \text{when } z \in D_1; \\ G^{-1} \circ \Phi \circ F(z), & \text{when } z \in \hat{\mathbf{C}} \backslash (D_1 \cup D_2); \\ f_2(z), & \text{when } z \in D_2 \end{cases}$$

is the required mapping and this completes the proof of Theorem 2. □

THEOREM 3. *Let f be a K-quasiconformal mapping defined in $U(z_0, \delta)$ which fixes z_0 and let r_0 be a positive number such that $r_0 > \delta$ and $f(U(z_0, \delta)) \subset U(z_0, r_0)$. Then for every $\varepsilon > 0$, there exist a positive number $r < \delta$ and a $(K + \varepsilon)$-quasiconformal mapping g of the complex plane $\mathbf{C}$, such that*

$$g|_{U(z_0, r)} = f|_{U(z_0, r)} \quad \textit{and} \quad g|_{\mathbf{C} \backslash U(z_0, r_0)} = id_{\mathbf{C} \backslash U(z_0, r_0)}.$$

Proof. By Theorem 2, for every positive number $r < \delta$,

$$h_r(z) = \begin{cases} f(z); & z \in \partial U(z_0, r), \\ z; & z \in \partial U(z_0, r_0) \end{cases}$$

can be extended to the annulus $A(z_0; r, r_0)$ quasiconformally. Let f_r be an extremal quasiconformal extension of h_r to the annulus $A(z_0; r, r_0)$ onto $U(z_0, r_0) \backslash f(U(z_0, r))$ and let

$$F_r(z) = \begin{cases} f(z); & z \in U(z_0, r), \\ f_r(z); & z \in A(z_0; r, r_0), \\ z; & z \in \mathbf{C} \backslash U(z_0, r_0). \end{cases}$$

Then F_r is a quasiconformal mapping of the complex plane $\mathbf{C}$.

We will see that $F_r(z)$ is the required quasiconformal mapping for some sufficiently small number $r > 0$.

To prove this, we only need to show that $K(f_r) \le K + \varepsilon$ for some sufficiently small number $r > 0$, where $K(f_r)$ is the maximal dilatation of f_r.

Assume, by contradiction, that $K(f_r) > K + \varepsilon$ for all positive numbers $r < \delta$. Since $f(z)$ is K-quasiconformal in $U(z_0, \delta)$, the boundary dilatation of $h_r(z)$ is $H(h_r) \le K$ for $0 < r < \delta$. Therefore, $K(f_r) > H(h_r)$, and by the frame mapping theorem (Theorem C), f_r is a Teichmüller mapping with Beltrami coefficient $\mu_r = k_r \bar{\varphi}_r / |\varphi_r|$ $(0 < k_r < 1)$, where $\varphi_r = \varphi_r(z)\, dz^2$ is the associated holomorphic quadratic differential with $\|\varphi_r\| = 1$. We claim that

CLAIM. *The sequence φ_r converges to* 0 *uniformly on any compact subset of* $U(z_0, r_0)\backslash\{z_0\}$ *as* $r \to 0$.

Assume, by contradiction, that there is a sequence $\{r_n\}$ of positive number decreasing to 0 such that $\varphi_{r_n}(z) \to \varphi_0(z) \not\equiv 0$ as $n \to \infty$.

It is clear that the dilatations of f_{r_n} are non-increasing and uniformly bounded. So $k_{r_n} \to k_0$ $(0 \le k_0 < 1)$ and the Beltrami coefficients μ_{r_n} of f_{r_n} converge to $\mu_0 = k_0 \dfrac{\bar{\varphi}_0}{\varphi_0}$ in $\Delta_{r_0}\backslash\{z_0\}$ as $n \to \infty$. Moreover, by the assumption that $K(f_{r_n}) > K + \varepsilon$, we have

$$k_{r_n} > \frac{K + \varepsilon - 1}{K + \varepsilon + 1},$$

and consequently,

$$k_0 \ge \frac{K + \varepsilon - 1}{K + \varepsilon + 1} > 0.$$

Since f_{r_n}'s and their dilatations are uniformly bounded, so $\{F_{r_n}\}$ is a normal family in $\bar{U}(z_0, r_0)\backslash\{z_0\}$. Thus $\{F_{r_n}\}$ has a subsequence, denoted by itself also, which converges to a quasiconformal mapping f_0 uniformly on any compact subset of $\bar{U}(z_0, r_0)\backslash\{z_0\}$.

From the above discussion, we conclude that f_0 is a Teichmüller mapping with Beltrami coefficient $\mu_0 = k_0 \dfrac{\overline{\varphi_0}}{\varphi_0}$ $(0 < k_0 < 1)$. As $\|\varphi_0\| \le \lim_{n\to\infty} \|\varphi_{r_n}\| = 1$, f_0 is uniquely extremal for its boundary values $f_0|_{\partial U(z_0, r_0)\cup\{z_0\}} = id|_{\partial U(z_0, r_0)\cup\{z_0\}}$. However, the identity mapping is obviously extremal for such boundary values. This gives a contradiction and the claim is proved.

Since $H(h_r) \le K$ for all positive number $r < \delta$, so for a given positive number $r_* < \delta$, there are an quasiconformal extension g of h_{r_*} to the annulus $A(z_0; r_*, r_0)$ and a compact subset E of the annulus $A(z_0; r_*, r_0)$, such that

$$K(g|_{A(z_0; r_*, r_0)\backslash E}) < K + \frac{\varepsilon}{2}. \tag{3}$$

Let

$$G(z) = \begin{cases} f(z) & \text{on } U(z_0, r_*), \\ g(z) & \text{on } A(z_0; r_*, r_0). \end{cases}$$

Then f_r $(0 < r < r_*)$ is homotopic to the restriction of $G(z)$ to the annulus $A(z_0; r, r_0)$ modulo the boundary. Using the main inequality (2) to f_r and G on the annulus $A(z_0; r, r_0)$, we have

$$\begin{aligned} 1 = \|\varphi_r\| &\le \iint_{A(z_0; r, r_0)} |\varphi_r(z)| \frac{\left|1 - \mu_r(z)\dfrac{\varphi_r(z)}{|\varphi_r(z)|}\right|^2}{1 - |\mu_r(z)|^2} D_{G^{-1}}(f_r(z))\, dxdy \\ &= \iint_{A(z_0; r, r_0)} \frac{|\varphi_r(z)|}{K(f_r)} D_{G^{-1}}(f_r(z))\, dxdy. \end{aligned}$$

Thus,

$$K(f_r) \leq \iint_{A(z_0;r,r_0)} |\varphi_r(z)| D_{G^{-1}}(f_r(z))\,dxdy \tag{4}$$
$$= \iint_{f_r^{-1}\circ G(E)} |\varphi_r(z)| D_{G^{-1}}(f_r(z))\,dxdy$$
$$+ \iint_{A(z_0;r,r_0)\setminus f_r^{-1}\circ G(E)} |\varphi_r(z)| D_{G^{-1}}(f_r(z))\,dxdy.$$

As $K(f_r) \leq K(G)$, all $f_r^{-1} \circ G(E)$ $(0 < r \leq r_*)$ are contained in a compact subset of $U(z_0, r_0)\setminus\{z_0\}$. By the degenerating property of $\{\varphi_r\}$,

$$\iint_{f_r^{-1}\circ G(E)} |\varphi_r(z)| D_{G^{-1}}(f_r(z))\,dxdy \leq \frac{\epsilon}{2} \tag{5}$$

holds for all sufficiently small numbers r. By the definition of G and (3),

$$\iint_{A(z_0;r,r_0)\setminus f_r^{-1}\circ G(E)} |\varphi_r(z)| D_{G^{-1}}(f_r(z))\,dxdy \leq K + \frac{\epsilon}{2}. \tag{6}$$

Therefore, by (4), (5) and (6), we have

$$K(f_r) \leq K + \epsilon,$$

for all sufficiently small numbers r. This contradicts to the assumption that $K(f_r) > K + \varepsilon$ for all positive numbers $r < r_0$. The proof of Theorem 3 is then completed. □

Proof of Theorem 1. Let U_k $(k = 1, 2, \dots, n)$ be the disjoint neighborhoods of n distinct points z_k $(k = 1, 2, \dots, n)$ respectively, and let $f_k(z)$ be a mapping defined and K-quasiconformal on U_k, and fixes z_k $(k = 1, 2, \dots, n)$. Then there are two positive numbers r_0 and $\delta < r_0$ such that

$$U(z_i, r_0) \cap U(z_j, r_0) = \emptyset, \quad 1 \leq i \neq j \leq n,$$

and $f(U(z_k, \delta)) \subset U(z_k, r_0)$. Given $\varepsilon > 0$, by Theorem 3, for every f_k there are a positive number $r_k < \delta$ and a $(K + \varepsilon)$-quasiconformal mapping g_k of the complex plane $\mathbf{C}$, such that

$$g_k(z) = \begin{cases} f_k(z) & z \in U(z_k, r_k), \\ z & z \in \mathbf{C}\setminus U(z_k, r_0). \end{cases}$$

Therefore, $f = g_1 \circ g_2 \circ \dots \circ g_n$ is the required mapping and Theorem 1 is proved. □

References

[1] K. Astala and G. Martin, Papers on analysis, a volume dedicated to Olli Martio on the occasion of his 60th birthday, Report, Univ. Jyvaskyla **211** (2001), 27–40.

[2] L. Bers and H. Royden, Holomorphic families of injections, Acta Math. **157** (1986), 259–286.

[3] E. M. Chirka, On the extension of holomorphic motions, Doklady Akademii Nauk **397** (2004), 37–40.
[4] A. Douady, Prolongements de meuvements holomorphes (d'apres Slodkowski et autres), Asterisque **227** (1995), 7–20.
[5] F. Gardiner, Y. Jiang and Z. Wang, Holomorphic motions and related topics, Geometry of Riemann surfaces, London Mathematical Society lecture note series **368**, 156–193.
[6] F. Gardiner and N. Lakic, Quasiconformal Teichmüller theory, American Math. Society, 2000.
[7] R. Fehlmann, Ueber extremale quasikonforme abbidungen, Comment. Math. Helv. **56** (1981), 558–580.
[8] Y. Jiang, Holomorphic motions, Fatou linearization, and quasiconformal rigidity for parabolic germs, The Michigan Mathematical Journal **58** (2009), 517–534.
[9] Y. Liu, The boundary problem of quasiconformal mappings on annuli, Chinese J. Engineering Math. **9** (1992), 120–124.
[10] R. Mañe, P. Sad and D. Sullivan, On the dynamics of rational maps, Ann. Sci. École Norm. Sup. **16** (1983), 193–217.
[11] E. Reich and K. Strebel, Extremal quasiconformal mapping with given boundary values, Contributions to analysis, Academic Press, New York, 1974, 375–391.
[12] Z. Slodkowski, Holomorphic motions and polynomial hulls, Proc. Amer. Math. Soc. **111** (1991), 347–355.
[13] K. Strebel, On the existence of extremal Teichmüller mappings, J. d'analysis Math. **30** (1976), 464–480.
[14] D. Sullivan and W. Thurston, Extending holomorphic motions, Acta Math. **157** (1986), 243–257.

Yunping Jiang
Department of Mathematics
Queens College of the City University of New York
Flushing, NY 11367-1597
USA

Department of Mathematics
Graduate School of the City University of New York
365 Fifth Avenue, New York
NY 10016
USA
E-mail: yunping.jiang@qc.cuny.edu

Yi Qi
School of Mathematics and Systems Science
Beihang University
Beijing, 100191
China
E-mail: yiqi@buaa.edu.cn