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THE NASH PROBLEM OF ARCS AND THE RATIONAL DOUBLE POINT $\mathbf{E_6}$

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Abstract

This paper deals with the Nash problem, which consists in proving that the number of families of arcs on a germ of a normal isolated singularity coincides with the number of essential components of the exceptional set in any resolution of this singularity. We propose a program for an affirmative solution of the Nash problem for special types of normal isolated hypersurface singularities. We illustrate this program by giving an affirmative solution of the Nash problem for the rational double point $\mathbf{E_6}$. We also prove some results on the algebraic structure of the space of k-jets of an arbitrary hypersurface singularity and apply them to the specific case of $\mathbf{E_6}$.

1. Introduction

In this paper, $\mathbf{k}$ is an algebraically closed field of characteristic 0.

Let $(S,0)$ be a germ of a normal isolated singularity over $\mathbf{k}$ and $\pi:(X,E)\to(S,0)$ a *divisorial* resolution of singularities of $(S,0)$ (this means that X is a smooth variety and $E=\pi^{-1}(0)$ is of pure codimension one). Let

$$E=\bigcup_{i\in\Delta}E_i \tag{1}$$

be the decomposition of E into its irreducible components. The set E has two kinds of irreducible components: essential and inessential. For each i let μ_i denote the divisorial valuation determined by E_i.

Definition 1.1. We say that E_i is an ***essential divisor*** if for any other divisorial resolution $\pi':(X',E')\to(S,0)$ the center of μ_i on X' is an irreducible component of E'. The divisor E_i is ***inessential*** if it is not essential.

Remark 1.2. In general (that is, when $\dim S\geqslant 3$) it is quite difficult to show that a given component is essential (see [29] for a discussion of this question as well as some sufficient conditions for essentiality). In dimension two each exceptional divisor of the minimal resolution is essential.

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In order to study the resolution X of S, J. Nash (around 1968, published in 1995 [23]) introduced the space H of *arcs* passing through the singular point 0.

Definition 1.3. An **arc** is a **k**-morphism from the local ring $\mathcal{O}_{S,0}$ to the formal power series ring $\mathbf{k}[[t]]$.

Intuitively, an arc should be thought of as a parametrized formal curve, contained in S and passing through the singular point 0.

Nash had shown that H has finitely many irreducible components, called *families of arcs*, and that there exists a natural injective map, now called the **Nash map**, from the set of families of arcs to the set of essential divisors. The celebrated **Nash problem**, posed in [23], is the question of whether the Nash map is surjective.

Let us fix a divisorial resolution of singularities $(X, E) \to (S, 0)$. Consider the decomposition (1) of E into irreducible components, as above. Let $\Delta' \subset \Delta$ denote the set which indexes the essential divisors.

M. Lejeune-Jalabert [17], inspired by Nash's original paper [23], proposed the following decomposition of the space H: for $i \in \Delta'$ let N_i be the set of arcs whose strict transform in X intersects the essential divisor E_i transversally but does not intersect any other exceptional divisor E_j. M. Lejeune-Jalabert showed that $H = \bigcup_{i \in \Delta'} \overline{N_i}$ and the set $\overline{N_i}$ is an irreducible algebraic subvariety of the space of arcs; therefore the families of arcs are among the $\overline{N_i}$'s. Moreover there are as many $\overline{N_i}$ as essential divisors E_i. Then the Nash problem reduces to showing that the $\overline{N_i}$, $i \in \Delta'$, are precisely the irreducible components of H, that is, to proving $card(\Delta')(card(\Delta') - 1)$ non-inclusions:

Problem 1.4. Is it true that $\overline{N_i} \not\subset \overline{N_j}$ for all $i \neq j$?

This question has been answered affirmatively in the following special cases: for A_n singularities by Nash, for minimal surface singularities by A. Reguera [30] (with other proofs by J. Fernandez-Sanchez [7] and C. Plénat [26]), for sandwiched singularities by M. Lejeune-Jalabert and A. Reguera (cf. [18] and [31]), for toric vareties by S. Ishii and J. Kollar ([14] using earlier work of C. Bouvier and G. Gonzalez-Sprinberg [2] and [3]), for rational double points $\mathbf{D_n}$ by Plénat [26], for a family of non-rational surface singularities, as well as for a family of singularities in dimension higher than 2 by P. Popescu-Pampu and C. Plénat ([28], [29]).

In [14], S. Ishii and J. Kollar gave a counter-example to the Nash problem in dimension greater than or equal to 4.

In 2008, M. Lejeune and A. Reguera [19] give a characterization of essential components which belong to the image of the Nash map and deduce that an irreducible exceptional divisor which is not uniruled is in the image of the Nash map (for uncountable fields). They also deduce that for general surface singularities over **C** Nash problem would follow from the special case of quasi-rational surface singularities.

In this paper we prove the following theorem:

THEOREM 1.5. *The Nash problem has an affirmative answer for the rational double points* $\mathbf{E_6}$.

Once this theorem is proved, we have the following corollary (cf. [26] for a proof):

COROLLARY 1.6. *Let* $(S,0)$ *be a normal surface singularity whose dual graph is obtained from* $\mathbf{E_6}$ *by increasing the weights (that is, allowing the exceptional curves to have self-intersection numbers of the form* $-n$ *for* $n \geqslant 2$*). Then the problem also has an affirmative answer for* $(S,0)$.

But the principal aim of this paper is to present a general strategy for attacking normal isolated hypersurface singularities which has so far been successful in the case of $\mathbf{D_n}$ ([27]) and $\mathbf{E_6}$ (the present paper).

From now on, we shall restrict ourselves to the case of dimension 2. However, we note that our main technique, that of explicitly computing truncated wedges on $(S,0)$, generalizes in an obvious way to isolated normal hypersurface singularities of any dimension. For this reason we hope that this paper will be useful for studying the Nash problem in higher dimension.

Our study of the Nash problem for a normal 2-dimensional hypersurface singularity with equation $F = \sum c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma = 0$ is divided into two main steps. For the first step we use the following valuative criterion:

PROPOSITION 1.7. *Let* $(S,0)$ *be an isolated singularity and* E_i, E_j *two essential divisors. If there exists an element* f *in* $\mathcal{O}_{S,0}$ *such that* $ord_{E_i} f < ord_{E_j} f$ *then* $\overline{N_i} \not\subset \overline{N_j}$.

This result is stated and proved in ([26], Proposition 1.1) for arbitrary singularities in any dimension. It was first proved by A. Reguera [30] in a different, but equivalent formulation for rational surface singularities.

Remark 1.8. Proposition 1.7 allows us to prove at least half of the non-inclusions appearing in Problem 1.4 in the case of rational surface singularities. Indeed, let $(S,0)$ be a rational surface singularity and E_i, E_j two distinct irreducible exceptional curves on the minimal resolution X of S. Let $n = \#\Delta$. Since the intersection matrix $(E_q.E_s)$ is negative definite, there exists a cycle on X of the form $C = \sum_{q\in\Delta} m_q E_q$ such that

$$m_q > 0, \quad C.E_q \leqslant 0 \quad \textit{for all } q \in \Delta \tag{2}$$

In fact, n-tuples $(m_1, \dots, m_n)$ of rational numbers satisfying (2) form an n-dimensional cone in $\mathbf{Q}^n$, called the Lipman cone. There exists a vector in the Lipman cone with integer coefficients such that $m_i \neq m_j$, otherwise the Lipman cone would be contained in the $(n-1)$-dimensional hyperplane $n_i = n_j$. Say, $m_i < m_j$. Since $(S,0)$ is rational, Artin's theorem [1] tells us that there exists

$f \in \mathcal{O}_{S,0}$ with $ord_{E_i} f = m_i$ and $ord_{E_j} f = m_j$, so the non-inclusion $\overline{N}_i \not\subset \overline{N}_j$ is given by the valuative criterion. This proves that for any pair $i, j \in \Delta$, $i \neq j$, at least one of the two non-inclusions $\overline{N}_i \not\subset \overline{N}_j$, $\overline{N}_j \not\subset \overline{N}_i$ is given by the valuative criterion.

The second step consists in proving the remaining non-inclusions. For this, we use the algebraic machinery developed in §3 of this paper. The idea is the following: Let E_i and E_j be two exceptional divisors such that

$$ord_{E_i} f \leqslant ord_{E_j} f \quad \text{for all } f \in \mathcal{O}_{S,0}. \tag{3}$$

For rational surface singularities, the negative definiteness of the intersection matrix $(E_i.E_j)$ implies that *strict* inequality holds for at least one $f \in m_{S,0}$, so $\overline{N_i} \not\subset \overline{N_j}$ by the valuative criterion (Proposition 1.7).

The opposite non-inclusion

$$\overline{N_j} \not\subset \overline{N_i} \tag{4}$$

cannot be obtained from the valuative criterion and must be proved separately.

Assume that $(S, 0)$ is a normal hypersurface singularity, embedded in the three-dimensional affine space $spec\, \mathbf{k}[x, y, z]$. An arc on $(S, 0)$ is described by three formal power series

$$\begin{cases} x(t) = \sum_{k=1}^{\infty} a_k t^k \\ y(t) = \sum_{k=1}^{\infty} b_k t^k \\ z(t) = \sum_{k=1}^{\infty} c_k t^k \end{cases} \tag{5}$$

whose coefficients a_k, b_k, c_k satisfy infinitely many polynomial equations, obtained as follows. Substitute the series (5) in F and write $F(x(t), y(t), z(t)) = \sum_{l=1}^{\infty} f_l(a, b, c)t^l$. Here $a = (a_k)_{k \in \mathbf{N}}$, $b = (b_k)_{k \in \mathbf{N}}$, $c = (c_k)_{k \in \mathbf{N}}$, and the f_l are polynomials in a, b and c. Let $\mathbf{k}^{\{a,b,c\}}$ denote the direct product of infinitely many copies of $\mathbf{k}$, indexed by $a = (a_k)_{k \in \mathbf{N}}$, $b = (b_k)_{k \in \mathbf{N}}$ and $c = (c_k)_{k \in \mathbf{N}}$. We think of $\mathbf{k}^{\{a,b,c\}}$ as an infinite-dimensional space over $\mathbf{k}$ with coordinates a, b, c. Then H is defined inside $\mathbf{k}^{\{a,b,c\}}$ by the equations $f_l = 0$, $l \in \mathbf{N}\backslash\{0\}$. Let I denote the defining ideal of H in $\mathbf{k}^{\{a,b,c\}}$, that is, the ideal generated by $(f) = (f_l)_{l \in \mathbf{N}}$ in $\mathbf{k}[a, b, c]$.

To each arc as above we can associate in a natural way a closed point of the infinite-dimensional scheme $\mathscr{H} = Spec\, \dfrac{\mathbf{k}[a, b, c]}{I}$. This scheme has the following description as a projective limit of schemes of finite type.

Definition 1.9. A k-jet is a $\mathbf{k}$-morphism $\mathcal{O}_{S,0} \to \dfrac{\mathbf{k}[[t]]}{(t^{k+1})}$.

Let us denote the set of all k-jets by $H(k)$. The set $H(k)$ can be naturally identified with the set of closed points of a scheme of finite type, denoted by

$\mathscr{H}(k)$. With the natural maps $\rho_{kk'} : \mathscr{H}(k) \to \mathscr{H}(k')$, $k' < k$, called truncation maps, the $\mathscr{H}(k)$ form a projective system whose inverse limit is $\mathscr{H}$. The natural maps $\rho_k : \mathscr{H} \to \mathscr{H}(k)$ are also called **truncation maps**.

For a natural number k and $i \in \Delta$, let $N_i(k)$ denote the image of N_i in the algebraic variety $H(k)$ of k-jets of S.

We prove the non-inclusion (4) by contradiction: suppose that

$$\overline{N_j} \subset \overline{N_i}. \tag{6}$$

Clearly the inclusion (6) implies that $\overline{N_j(k)} \subset \overline{N_i(k)}$. Therefore we may work with $\mathscr{H}(k)$ for a sufficiently large k instead of $\mathscr{H}$. The precise meaning of "sufficiently large" depends on the specific singularity in question, as well as on the particular non-inclusion (4) we want to show; below we will specify k precisely in each case. Note that ρ_k need not, in general, be surjective onto $H(k)$.

Let $K(N_j(k))$ denote the field of rational functions of $N_j(k)$.

By the Curve Selection Lemma (Lemma 3.6 below) there exists a finite extension L of $K(N_j(k))$ and an L-wedge

$$\phi_{ij} : Spec \frac{L[[t,s]]}{(t^{k+1})} \to S \tag{7}$$

such that the image of the special arc $\{s = 0\}$ is the generic point of $N_j(k)$, while the image of the general arc $\{s \neq 0\}$ is an L-point of $N_i(k) \backslash N_j(k)$. For each pair i, j such that the non-inclusion (4) does not follow from the valuative criterion we study equations satisfied by an L-wedge (7) and prove that such an L-wedge does not exist.

The paper is organized as follows: in §2 we recall the description of the singularity $\mathbf{E_6}$ we will use and carry out the first step of the proof using the valuative criterion. In §3, we partially describe the spaces of k-jets $H(k)$ of a hypersurface singularity for a general k and apply this description to the specific case of the $\mathbf{E_6}$ singularity. We also describe the image of a family of arcs in the truncated space $H(k)$. The last section is devoted to the second step of the proof for $\mathbf{E_6}$. Namely, we go one by one through the various non-inclusions (4) which are not covered by the valuative criterion and prove the non-existence of the L-wedge (7) as above in each case. On four occasions, when the resulting system of equations is too complicated to solve by hand, we use MAPLE to check that it has no non-trivial solutions.

Note that by passing to the k-truncation we avoid using A. Reguera's non-trivial theorem [31], which can be viewed as a version of the Curve Selection Lemma for the pair of infinite dimensional schemes (N_i, N_j). In the present paper, the usual Curve Selection Lemma for finite-dimensional algebraic varieties suffices for our purposes.

1.1. Progress since the first version of this paper. Since the appearance of the first version of this paper, the status of the Nash problem for surfaces has changed completely.

On November 16, 2010 Maria Pe Pereira (based on the work [5] of Javier Fernandez de Bobadilla) solved the problem affirmatively for quotients of $\mathbf{C}^2$ by an action of finite group [24].

On January 30, 2011 Maximiliano Leyton-Alvarez gave an affirmative solution for the following classes of normal hypersurfaces in C^3: hypersurfaces $S(p, h_q)$ given by the equation $z^p + h_q(x, y) = 0$, where h_q is a homogeneous polynomial of degree q without multiple factors, and $p \geqslant 2$, $q \geqslant 2$ are two relatively prime integers. He also applied his methods to give new proofs for the rational double points $\mathbf{D_n}$, $\mathbf{E_6}$ and $\mathbf{E_7}$ [20].

Finally, on February 22, 2011, Javier Fernandez de Bobadilla and Maria Pe Pereira made public their affirmative solution of the Nash problem for all the surface singularities [6].

Somewhat earlier, Ana Reguera announced a positive solution of the Nash problem for the rational surface singularities, though at the moment of the writing of this paper the details of her proof have not yet been made public in written form.

In any case, all the methods are completely different. We hope that our method will one day be useful in a more general context, not covered by the above results, such as normal hypersurface singularities in $\mathbf{C}^n$.

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2. The singularity $\mathbf{E_6}$ and the valuative criterion

The singularity $\mathbf{E_6}$ is, by definition, the hypersurface singularity defined in $\mathbf{k}^3$ by the equation $F = z^2 + y^3 + x^4 = 0$.

The first graph in Figure 1 is the dual graph of $\mathbf{E_6}$; the remaining five graphs show the orders of vanishing of the functions x, y, z, $z - ix^2$ and $z + ix^2$ on the exceptional curves E_1, E_2, E_3, E_4, E_5 and E_6. Although we do not have a conceptual reason for considering these five functions and not others, these functions will appear naturally below in our calculation of the wedges (note that $y^3 = (z - ix^2)(z + ix^2)$). The orders of vanishing of x, y, z, $z - ix^2$ and $z + ix^2$

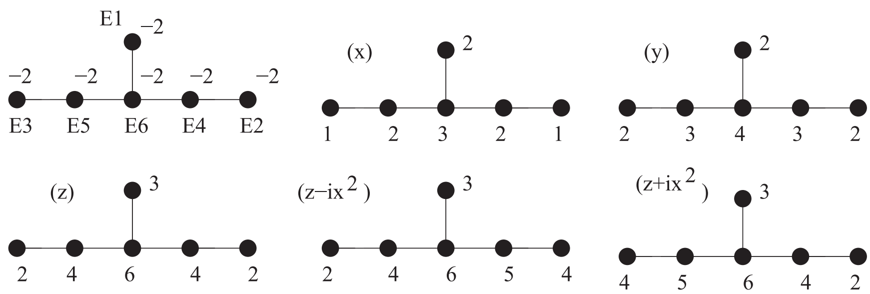

FIGURE 1. Dual graph of $\mathbf{E_6}$ and order of the functions x, y, z, $z - ix^2$ and $z + ix^2$

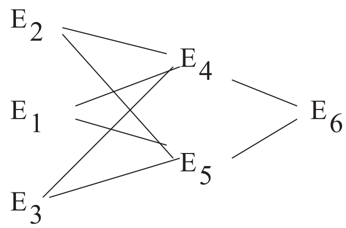

FIGURE 2. The partial order for $\mathbf{E_6}$.

on the exceptional curves E_1, E_2, E_3, E_4, E_5 and E_6 are calculated explicitly by considering the four successive point blowings up needed for the resolution of the $\mathbf{E_6}$ singularity and computing the multiplicities in coordinates at each step.

Consider the following partial ordering on the set $\{E_1, E_2, E_3, E_4, E_5, E_6\}$. We say that

$$E_i < E_j \tag{8}$$

if for all $f \in m_{S,0}$ the inequality (3) holds (as explained in Remark 1.8, together with the rationality of $\mathbf{E_6}$ this implies that *strict* inequality holds in (3) for some $f \in m_{S,0}$).

Using the functions x, y, z, $z - ix^2$ and $z + ix^2$, we see that our partial ordering contains at most the inequalities shown in Figure 2.

Here an inequality (8) is represented by placing E_i to the left of E_j. We will now show that Figure 2 shows the entire partial ordering: this is all the information we can derive from comparing $ord_{E_i} f$ with $ord_{E_j} f$ for various $f \in m_{S,0}$. Indeed, take an element $f \in m_{S,0}$ and let $p_i = ord_{E_i} f$, $i \in \{1, \dots, 6\}$. Our description of the partial ordering follows from the following Proposition:

PROPOSITION 2.1. *We have*

$$p_6 \leqslant \frac{3}{2} p_4 \tag{9}$$

$$p_6 \leqslant \frac{3}{2} p_5 \tag{10}$$

$$p_3 \leqslant \frac{4}{5} p_5 \tag{11}$$

$$p_2 \leqslant \frac{4}{5} p_4 \tag{12}$$

$$p_4, p_5 \leqslant \frac{5}{6} p_6 \tag{13}$$

$$p_1 \leqslant \frac{2}{3} p_6. \tag{14}$$

Proof. Write the zero cycle $(\pi^* f)_0$ on X as $F = \sum_{i=1}^{6} p_i E_i + C$, where the support of C contains no exceptional curves E_i. The inequalities $F.E_i \leqslant 0$, $i \in \{1, \dots, 6\}$, translate into

$$2p_1 \geqslant p_6 \tag{15}$$

$$2p_3 \geqslant p_5 \tag{16}$$

$$2p_2 \geqslant p_4 \tag{17}$$

$$2p_5 \geqslant p_3 + p_6 \tag{18}$$

$$2p_4 \geqslant p_2 + p_6 \tag{19}$$

$$2p_6 \geqslant p_1 + p_4 + p_5. \tag{20}$$

Now, (9) follows immediately from (17) and (19). The inequality (10) holds by symmetry. The inequality (14) now follows from (20), (9) and (10). The inequality $p_5 \leqslant \frac{5}{6} p_6$ follows from (20), (9) and (15), and $p_4 \leqslant \frac{5}{6} p_6$ is obtained by symmetry. Therefore we have (13). Finally, (11) follows from (18) and (13). The inequality (12) is obtained by symmetry. □

The valuative criterion proves all the non-inclusions (4) such that either $E_j < E_i$ or E_i and E_j are not comparable in the partial ordering. By symmetry, to complete the solution of the Nash problem for $\mathbf{E_6}$, it is sufficient to show the following non-inclusions:

$$\overline{N_4}, \overline{N_6} \not\subset \overline{N_1} \tag{21}$$

$$\overline{N_4}, \overline{N_5}, \overline{N_6} \not\subset \overline{N_2} \tag{22}$$

$$\overline{N_6} \not\subset \overline{N_4}. \tag{23}$$

For these non-inclusions we work in the space of k-jets of the singularity $\mathbf{E_6}$ (with k depending on the non-inclusion). Let $\mathscr{P}_{ik}$ and $\mathscr{P}_{jk}$ be two prime ideals in the coordinate ring of the space $H(k)$ of k-jets such that

$$\overline{N_i(k)} = V(\mathscr{P}_{ik}) \quad \text{and} \tag{24}$$

$$\overline{N_j(k)} = V(\mathscr{P}_{jk}). \tag{25}$$

In order to prove that $N_i \not\subset \overline{N_j}$, we show that $\mathscr{P}_{jk} \not\subset \mathscr{P}_{ik}$. To do this, we partially describe the ideals $\mathscr{P}_{jk}$ and $\mathscr{P}_{ik}$ and the space $H(k)$ of k-jets. This is the aim of §3.

3. The space of k-jets of a hypersurface singularity

In this section we first recall some lemmas about hypersurface singularities, found in [26]. We then study the image of a family of arcs in the truncation spaces $H(k)$.

In what follows we will look at a hypersuface singularity defined by

$$f(x,y,z) = \sum c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma = 0,$$

embedded in $\mathbf{k}^3$ with a singularity at 0.

Assume that f is irreducible.

3.1. Notation. Let N_i be the set of arcs determined by the exceptional divisor E_i, as defined in the introduction.

- For an element $g \in \mathcal{O}_{S,0}$, let $\mu_i(g)$ be the order of vanishing of $g \circ \pi$ on E_i.
- Let $R_k = \mathbf{k}[a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_k]$.
- For $i \in \Delta$, let $o_i = \min\{\alpha\mu_i(x) + \beta\mu_i(y) + \gamma\mu_i(z) \mid c_{\alpha\beta\gamma} \neq 0\}$.
- For $i \in \Delta$, $p \in \mathbf{N}$, let

$$\begin{aligned} o_{ip} = \min\{&[(\alpha - 1)\mu_i(x) + \beta\mu_i(y) + \gamma\mu_i(z)] + p, \\ &[\alpha\mu_i(x) + (\beta - 1)\mu_i(y) + \gamma\mu_i(z)] + p, \\ &[\alpha\mu_i(x) + \beta\mu_i(y) + (\gamma - 1)\mu_i(z)] + p \mid c_{\alpha\beta\gamma} \neq 0\}. \end{aligned}$$

- Let f_l be the coefficient of t^l in $(f \circ \phi) = f(x(t), y(t), z(t)) = 0$.
- Let f_{il} denote the unique element of $\mathbf{k}[a_{\mu_i(x)}, \dots, a_l, b_{\mu_i(y)}, \dots, b_l, c_{\mu_i(z)}, \dots, c_l]$ such that $f_{il} \equiv f_l$ modulo the ideal $(a_1, \dots, a_{\mu_i(x)-1}, b_1, \dots, b_{\mu_i(y)-1}, c_1, \dots, c_{\mu_i(z)-1})$ (here we adopt the obvious convention that the list $a_{\mu_i(x)}, \dots, a_l$ is considered empty whenever $\mu_i(x) > l$, and similarly for the b and c coefficients).

3.2. The k-jets scheme. Fix an integer $k > 0$.

Any k-jet $\phi(t)$ passing through the singularity can be represented by three polynomials of degree k, $\phi(t) = (x(t), y(t), z(t)) = (a_1 t + \dots + a_k t^k, b_1 t + \dots + b_k t^k, c_1 t + \dots + c_k t^k)$ (because the singularity is at 0), satisfying the algebraic constraints given by $f \circ \phi = 0$. Then $\{f_1 = 0, \dots, f_k = 0\}$ are the equations defining the k-jet scheme $H(k)$ in $\mathbf{k}^{3k}$. Below we shall describe the equations $\{f_1 = 0, \dots, f_k = 0\}$ more explicitly.

Let l, m, n be integers such that there exists an exceptional divisor E_i with

$$\mu_i(x) = l \tag{26}$$

$$\mu_i(y) = m \tag{27}$$

$$\mu_i(z) = n. \tag{28}$$

Let K be the subset of the k-jet scheme defined in $H(k)$ by the ideal $(a_1, \dots, a_{l-1}, b_1, \dots, b_{m-1}, c_1, \dots, c_{n-1})$.

Let r be the smallest integer such that

$$f_r \notin (a_1, \dots, a_{l-1}, b_1, \dots, b_{m-1}, c_1, \dots c_{n-1}).$$

The subspace K of $H(k)$ $(k > r)$ is defined by

$$(a_1, \dots, a_{l-1}, b_1, \dots, b_{m-1}, c_1, \dots c_{n-1}, f_r, \dots f_k)$$

in $\mathbf{k}^{3k}$.

Then one can write, for $i \geqslant 0$,

$$f_{r+i} = \left(\frac{\partial f_r}{\partial a_l}\right) a_{l+i} + \left(\frac{\partial f_r}{\partial b_m}\right) b_{m+i} + \left(\frac{\partial f_r}{\partial c_n}\right) c_{n+i} + S_{r+i}(a_l, \dots, a_{l+i-1}, b_m, \dots, b_{m+i-1}, c_n, \dots, c_{n+i-1}),$$

where S_{r+i} is a polynomial (for a proof see [26], §4.2).

Let us recall the main lemma of [26], §1.3, used for the description of the image of a family of arcs in the space of k-jets:

Lemma 3.1. *Consider the polynomial ring* $A = \mathbf{k}[y_1, \dots y_v, x_{21}, \dots, x_{2u}, \dots, x_{q1}, \dots, x_{qu}]$, *where* $y_1, \dots, y_v$, $x_{21}, \dots, x_{2u}, \dots, x_{q1}, \dots, x_{qu}$ *are independent variables. Let* $f_1, \dots, f_q$ *be a sequence of elements of the following form*:

$$f_1 = f_1(y_1, \dots, y_v) = g_1 \cdots g_s$$
$$f_2 = a_1 x_{21} + \cdots + a_u x_{2u} + h_2(y_1, \dots, y_v)$$
$$f_3 = a_1 x_{31} + \cdots + a_u x_{3u} + h_3(y_1, \dots, y_v, x_{21}, \dots, x_{2u})$$
$$\vdots$$
$$f_q = a_1 x_{q1} + \cdots + a_u x_{qu} + h_q(y_1, \dots, y_v, x_{21}, \dots, x_{(q-1)u})$$

with $g_1, \dots, g_s$ *distinct irreducible polynomials and* $a_1, \dots, a_u \in \mathbf{k}[y_1, \dots, y_v]$.

For a fixed j, $1 \leqslant j \leqslant s$, *let* $S_j \subset \{a_1, \dots, a_u\}$ *be the set of* a_l *such that* $a_l \notin (g_j)$.

Let us denote $J = (f_1, \dots, f_q)$. *We have*:

(1) *If* $S_j \neq \emptyset$, *there exists a unique minimal prime ideal* $\mathscr{P}_j$ *of* J *such that* $g_j \in \mathscr{P}_j$ *and* $a_w \notin \mathscr{P}_j$ *for all* $a_w \in S_j$.

(2) *Assume* $S_j \neq \emptyset$ *for all* $j \in \{1, \dots, s\}$. *Let* $\mathscr{Q}$ *be a minimal prime ideal of* J *different from* $\mathscr{P}_1, \dots, \mathscr{P}_s$; *then* $(a_1, \dots, a_u) \subset \mathscr{Q}$.

(3) *Let* g_i *and* g_j *be two irreducible factors of* f_1. *Then* $\mathscr{P}_i \neq \mathscr{P}_j$.

Definition 3.2. We call the prime ideal $\mathscr{P}_j$ of the lemma the distinguished ideal of J, associated to g_j.

Lemma 3.1 says that there are exactly s distinguished ideals of J, one associated to each irreducible factor g_j, provided $S_j \neq \emptyset$ for all $j \in \{1, \dots, s\}$.

3.3. Image of a family of arcs in $H(k)$.

In this subsection we describe the defining ideal of $\overline{N_i(k)}$ in $H(k)$. Let I denote the defining ideal of H in $\mathbf{k}^{\{a,b,c\}}$, that is, the ideal generated by $(f) = (f_l)_{l \in \mathbf{N}}$ in $\mathbf{k}[a, b, c]$, as defined in the Introduction.

PROPOSITION 3.3. *Take an integer $k > o_i$.*
Let

$$I_{ik} = (a_1, \dots, a_{\mu_i(x)-1}, b_1, \dots, b_{\mu_i(y)-1}, c_1, \dots, c_{\mu_i(z)-1}, f_{io_i}, \dots, f_{io_{ik}})R_k.$$

and

$$\tilde{I}_{ik} = (I + (a_1, \dots, a_{\mu_i(x)-1}, b_1, \dots, b_{\mu_i(y)-1}, c_1, \dots, c_{\mu_i(z)-1})) \cap R_k.$$

Then

$$I_{ik} \subset \tilde{I}_{ik}. \tag{29}$$

For $d \in \left\{ \frac{\partial f_{io_i}}{\partial a_{\mu_i(x)}}, \frac{\partial f_{io_i}}{\partial b_{\mu_i(y)}}, \frac{\partial f_{io_i}}{\partial c_{\mu_i(z)}} \right\}$, *we have*

$$I_{ik}(R_k)_d = \tilde{I}_{ik}(R_k)_d. \tag{30}$$

Proof. The inclusion (29) is obvious. To prove (30), first note that the left hand side is contained in the right hand side by (29). Conversely, let $d = \frac{\partial f_{io_i}}{\partial a_{\mu_i(x)}}$; the proof for the other two possible choices of d is exactly the same. Take an element $g \in (\tilde{I}_{ik})_d$. By definition of $\tilde{I}_{ik}$, g can be written in the form

$$g = \sum_{l=o_i}^{s} h_l f_{il} + \tilde{g}, \tag{31}$$

where $h_j \in R_d$ and $\tilde{g} \in (a_1, \dots, a_{\mu_i(x)-1}, b_1, \dots, b_{\mu_i(y)-1}, c_1, \dots, c_{\mu_i(z)-1})R_d$. Up to multiplication by a unit of R_d (namely, by $\frac{1}{\partial f_{io_i}/\partial a_{\mu_i(x)}}$), f_{il} has the form $a_l + \lambda_{il}$, where

$$\lambda_{il} \in \mathbf{k}[a_1, \dots, a_{l-1}, b_1, \dots, b_l, c_1, \dots, c_l]_d$$

Thus by adding a suitable multiple of f_{il} to each $h_{l'}$ with $l' < l$, we may assume that $h_{l'}$ does not involve the variable a_l whenever $l' < l$. Also, we may assume that $\tilde{g} = 0$ and that none of the h_l involve the variables $a_1, \dots, a_{\mu_i(x)-1}$, $b_1, \dots, b_{\mu_i(y)-1}, c_1, \dots, c_{\mu_i(z)-1}$. We will now show that under these assumptions $s \leqslant o_{ik}$ in (31). Indeed, the right hand side of (31) contains exactly one term involving a_s. If we had $s > o_{ik}$ then, by definition of o_{ik}, we have $g \notin R_k$, a contradiction. This proves the equality (30). □

Let $\tau = \{\mu(x), \mu(y), \mu(z)\}$ be a triple such that there exists $i \in \Delta$ with

$$\tau = \{\mu_i(x), \mu_i(y), \mu_i(z)\}.$$

Let $E(\tau) = \{E_l : \{\mu_l(x), \mu_l(y), \mu_l(z)\} = \tau\}$. For $E_i \in E(\tau)$ and $j \in \mathbf{N}$, the numbers o_i, o_{ij}, $\mu_i(x)$, $\mu_i(y)$, $\mu_i(z)$, the polynomials f_{ij} and the ideals I_{ik}, $\tilde{I}_{ik}$ depend only on τ and not on the particular choice of $E_i \in E(\tau)$. We will therefore denote these objects by o_τ, $o_{\tau j}$, $\mu_\tau(x)$, $\mu_\tau(y)$, $\mu_\tau(z)$, $f_{\tau j}$, and $I_{\tau k}$, $\tilde{I}_{\tau k}$, respectively.

Proposition 3.4 (Image of a family). *Assume that $f_{\tau o_\tau}$ is reduced but not necessarily irreducible and that it is not divisible by any of $a_{\mu_\tau(x)}$, $b_{\mu_\tau(y)}$, $c_{\mu_\tau(z)}$; let $f_{\tau o_\tau} = g_1 \cdots g_s$ be its factorization into irreducible factors.*

Then:

- *there exists a uniquely determined injective map*
$$\psi : \{1, \ldots, s\} \to E(\tau)$$
such that for $j \in \{1, \ldots, s\}$ and $E_i = \psi(j)$, the variety $\overline{N_i(k)}$ is defined by the distinguished prime ideal of $I_{\tau k}$ associated with g_j.
- *The non-inclusion* (4) *holds for all $E_i, E_j \in Im(\psi)$. In particular, if the map ψ is surjective,* (4) *holds for all $E_i, E_j \in E(\tau)$.*

Remark 3.5. If $s = card(E(\tau))$ then ψ is necessarily bijective. This is the case for rational double points $\mathbf{A_n}$, $\mathbf{D_n}$ (in both cases $s = card(E(\tau)) = 1$ for all values of τ [26]). Below, we will see that for the singularity $\mathbf{E_6}$ we always have $s = card(E(\tau)) \leqslant 2$, so, again, ψ is bijective. Of course, ψ is bijective for any singularity for which the Nash problem has an affirmative answer. Thus, a posteriori, the bijectivity of ψ is now known for an arbitrary isolated 2-dimensional hypersurface singularity thanks to the Fernandez de Bobadilla–Pe theorem [6].

Proof of Proposition 3.4. For the first assertion, note that the ideal $I_{\tau k}$ satisfies the hypotheses of Lemma 3.1, with the partial derivatives $\dfrac{\partial f_{\tau o_\tau}}{\partial a_{\mu_\tau(x)}}$, $\dfrac{\partial f_{\tau o_\tau}}{\partial b_{\mu_\tau(y)}}$, $\dfrac{\partial f_{\tau o_\tau}}{\partial c_{\mu_\tau(z)}}$ playing the roles of a_1, a_2, a_3.

By definitions

$$V(\tilde{I}_{\tau k}) = \bigcup_{\substack{\mu_i(x) \geqslant \mu_\tau(x) \\ \mu_i(x) \geqslant \mu_\tau(x) \\ \mu_i(x) \geqslant \mu_\tau(x)}} \overline{N_i(k)}. \tag{32}$$

Let d be one of the partial derivatives of $f_{\tau o_\tau}$, which is not identically zero. The fact that $f_{\tau o_\tau}$ is reduced implies that $I_{\tau k} R_d$ is not the unit ideal. Now Proposition 3.3 (particularly, (30)) implies that the distinguished prime ideals $\mathscr{P}_{jk}$, $j \in \{1, \ldots, s\}$ of $I_{\tau k}$ are also minimal primes of $\tilde{I}_{\tau k}$. Since the varieties $\overline{N_i(k)}$ are irreducible, (32) shows that for each $j \in \{1, \ldots, s\}$ there exists i with

$$\mu_i(x) \geqslant \mu_\tau(x), \tag{33}$$

$$\mu_i(x) \geqslant \mu_\tau(x), \tag{34}$$

$$\mu_i(x) \geqslant \mu_\tau(x), \tag{35}$$

such that $V(\mathscr{P}_{jk}) = \overline{N_i(k)}$. Furthermore, since g_j is not divisible by $a_{\mu_\tau(x)}$, $b_{\mu_\tau(y)}$ or $c_{\mu_\tau(z)}$ and has no common factors with d by assumption, by Nullstellensatz there exist triples $(a', b', c') \in \mathbf{k}^3$ such that $g_j(a', b', c') = 0$, $d(a', b', c') \neq 0$ and a', b', c' are different from 0. Then there exists an arc in $V(\mathscr{P}_{jk})$ of the form $\phi(t) = (a' t^{\mu_\tau(x)} + \cdots, b' t^{\mu_\tau(y)} + \cdots, c' t^{\mu_\tau(z)} + \cdots)$. Namely, we construct such an arc by describing the values of $a_{\mu_\tau(x)+r}$, $b_{\mu_\tau(y)+r}$ and $c_{\mu_\tau(z)+r}$. We put $(a_{\mu_\tau(x)}, b_{\mu_\tau(y)}, c_{\mu_\tau(z)}) = (a', b', c')$. Then, for each positive integer r, we let $b_{\mu_\tau(y)+r}$ and $c_{\mu_\tau(z)+r}$ be arbitrary elements of $\mathbf{k}$ and set

$$a_{\mu_\tau(x)+r} = -\frac{f_{\tau, o_\tau+r} - a_{\mu_\tau(x)+r}d}{d}.$$

This proves that $E_i \in E(\tau)$. We define $E_i = \psi(j)$.

The injectivity of ψ is obvious from the definition. Also by definition, the non-inclusion (4) is satisfied for all $E_i, E_j \in Im(\psi)$. Thus, if ψ is surjective, (4) holds for all $E_i, E_j \in E(\tau)$, as desired. This completes the proof. □

Example. Let us apply the above ideas to the special case of the $\mathbf{E_6}$ singularity. According to Figure 1, there are four possible values of τ: $(2,2,3)$, $(1,2,2), (2,3,4)$ and $(3,4,6)$. We have $E(2,2,3) = \{E_1\}$, $E(1,2,2) = \{E_2, E_3\}$, $E(2,3,4) = \{E_4, E_5\}$, and $E(3,4,6) = \{E_6\}$. Thus, for $\tau = (2,2,3)$ or $\tau = (3,4,6)$ the bijectivity of the map ψ is immediate.

Next, let $\tau = (1,2,2)$. We have $o_\tau = 4$ and $f_{\tau o_\tau} = c_2^2 + a_1^4 = (c_2 + ia_1^2)\cdot(c_2 - ia_1^2)$, so $f_{\tau o_\tau}$ is a product of two distinct irreducible factors.

Similarly, if $\tau = (2,3,4)$, we have $o_\tau = 8$ and $f_{\tau o_\tau} = c_4^2 + a_2^4 = (c_4 + ia_2^2)\cdot(c_4 - ia_2^2)$, so, again $f_{\tau o_\tau}$ is a product of two distinct irreducible factors.

Since in the last two cases f_{τ, o_τ} has two irreducible factors and $\#E(\tau) = 2$, the map ψ is bijective also in these two cases. It follows from Proposition 3.4 that for a sufficiently large k each $\overline{N_i(k)}$ is of the form $V(\mathscr{P}_{ik})$, where $\mathscr{P}_{ik}$ is a distinguished prime ideal, associated to I_{ik}.

We recall that the goal is to prove that

$$\mathscr{P}_{ik} \not\subset \mathscr{P}_{jk} \tag{36}$$

whenever

$$E_i < E_j. \tag{37}$$

3.4. The strategy for proving the non-inclusion (36).

By the valuative criterion we already have the opposite non-inclusion in (36). Inequality (37) means that $ord_{E_i}\, g \leqslant ord_{E_j}\, g$ for all $g \in \mathcal{O}_{S,0}$. We thus have the following inclusions:

$$\begin{cases} I_{ik} \subset \mathscr{P}_{ik} \\ I_{ik} \subset I_{jk} \subset \mathscr{P}_{jk} \end{cases}$$

Assume $N_j(k) \subset \overline{N_i(k)}$ for a certain order k.

We will need the Curve Selection lemma (for usual finite-dimensional algebraic varieties). The original Curve Selection Lemma was proved by Milnor in his book [22] (Lemma 3.1, p. 25) in the context of real algebraic varieties. The elementary lemma which follows is inspired by this. We doubt that this result is new, but we could not find the exact statement we needed in the literature, so we include a proof.

PROPOSITION 3.6 (Curve Selection Lemma). *Let V be a reduced algebraic variety over an algebraically closed field* $\mathbf{k}$ *and W a proper reduced irreducible subvariety of V. Let $K(W)$ denote the field of rational functions of W.*

There exists a finite field extension L of $K(W)$ and an arc $\phi : Spec\, L[[s]] \to V$ whose generic point maps to $V \setminus W$, and the special point to the generic point of W.

Proof. Replacing V by a suitable affine open subset of it, we may assume, without loss of generality, that V is an affine variety. Let A denote the coordinate ring of V and write $W = V(P)$ where P is a prime ideal of A. Let Q denote a prime ideal of A, contained in P, such that $ht\, Q = ht\, P - 1$. Let B denote the normalization of the ring $\frac{A_P}{QA_P}$, $\hat{B}$ the completion of B at some fixed maximal ideal and L the residue field of $\hat{B}$. The field L is a finite extension of $K(W)$. Then $\hat{B}$ is a complete regular 1-dimensional local ring; let s be a regular parameter of $\hat{B}$. We have $\hat{B} \cong L[[s]]$; the composition of the natural maps $A \to A_P \to \frac{A_P}{QA_P} \to B \to \hat{B}$ induces the morphism ϕ required in the Proposition. □

Let $W = N_j(k)$. In our context, the curve is an arc of the form $\phi_{ij} : Spec\, L[[s]] \to N_i(k)$, which corresponds to a "truncated" L-wedge

$$\phi_{ij} : Spec\, \frac{L[[t,s]]}{(t^{k+1})} \to (S, 0) \tag{38}$$

whose special arc $(s = 0)$ maps to the generic arc of $N_j(k)$ and whose general arc maps to an L-point of $N_i(k) \setminus N_j(k)$. A wedge as in (38) is given by three polynomials of the form

$$\begin{cases} x(t,s) = \sum_{n=0}^{k} \mathbf{a}_n(s) t^n \\ y(t,s) = \sum_{n=0}^{k} \mathbf{b}_n(s) t^n \\ z(t,s) = \sum_{n=0}^{k} \mathbf{c}_n(s) t^n \end{cases}$$

Write the coefficients $\mathbf{a}_n(s)$, $\mathbf{b}_n(s)$, $\mathbf{c}_n(s)$ of the wedge in the form

$$\begin{cases} \mathbf{a}_n(s) = \sum_{p=0}^{\infty} a_{np} s^p \\ \mathbf{b}_m(s) = \sum_{p=0}^{\infty} b_{mp} s^p \\ \mathbf{c}_l(s) = \sum_{p=0}^{\infty} c_{lp} s^p, \end{cases}$$

with $a_{np}, b_{mp}, c_{lp} \in L$, where a_{n0}, b_{m0}, c_{l0} satisfy the equations of $N_j(k)$. In particular, $a_{n0} = 0$ when $n < \mu_j(x)$, $b_{m0} = 0$ when $m < \mu_j(y)$ and $c_{l0} = 0$ when $l < \mu_j(z)$. Let us denote by α_n (resp. β_m and γ_l) the smallest order q for which a_{nq} (resp. b_{mq} and c_{lq}) is not 0. We need to compute these exponents in order to construct the wedge ϕ_{ij}. Note that $a_{n0} \neq 0$ if and only if $\alpha_n = 0$, and similarly for the b and c coefficients; we always have $a_{n0} \neq 0$ if $n = \mu_j(x)$.

The morphism (38) is given by a ring homomorphism

$$\mathcal{O}_{S,0} \to \frac{L[[t,s]]}{(t^{k+1})}. \tag{39}$$

Localizing $\frac{L[[t,s]]}{(t^{k+1})}$ by the element s, we obtain an $L((s))$-point of $N_i(k)$ (informally, an $L((s))$-arc lying in $N_i(k)$). Thus the coefficients $\mathbf{a}_n(s)$, $\mathbf{b}_m(s)$, $\mathbf{c}_l(s)$ satisfy the equations f_{iu} of $N_i(k)$ and their constant terms a_{n0}, b_{m0}, c_{l0} satisfy the equations f_{ju} of $N_j(k)$ (here f_{iu} is the coefficient of t^u in $Fo\phi_{ij}$ and similarly for f_{ju}; see §3.1 where this notation was introduced).

Let A_{np}, B_{mp}, C_{lp}, $p \geqslant 0$, be independent variables and write

$$\begin{cases} A_n(s) = \sum_{p=0}^{\infty} A_{np}s^p \\ B_m(s) = \sum_{p=0}^{\infty} B_{mp}s^p \\ C_l(s) = \sum_{p=0}^{\infty} C_{lp}s^p. \end{cases}$$

We have finitely many equalities of the form

$$0 = f_{iu}(A(s), B(s), C(s)) = \sum_{v=0}^{\infty} f'_{ivu}s^v, \quad u \leqslant o_{ij}, \tag{40}$$

where $A(s)$ stands for $\{A_n(s)\}_{n \in \mathbf{N}}$, and similarly for B and C. Here the coefficients f'_{ivu} are polynomials in A_{np}, B_{mp}, C_{lp} which vanish after substituting $A_{np} = a_{np}$, $B_{mp} = b_{mp}$, $C_{lp} = c_{lp}$.

Let J denote the ideal of $L[A, B, C]$ generated by all the elements of the form A_{np} with $p < \alpha_n$, B_{mp} with $p < \beta_m$ and C_{lp} with $p < \gamma_l$, where A stands for $\{A_{np}\}_{p \in \mathbf{N}}$, and similarly for B and C. Let $\theta_u = \min\{v \mid f'_{ivu}(A, B, C) \notin J\}$. Write $g_{\theta_u} = f'_{i\theta_u u}$. In other words, g_{θ_u} is the first non-zero coefficient of $f_{i,u}(A(s), B(s), C(s))$, viewed as a series in s, not belonging to the ideal J.

Notation. For the rest of this paper, we will write a_n for $a_{n\alpha_n}$, b_m for $b_{m\alpha_m}$ and c_l for $c_{l\gamma_l}$.

Remark 3.7. • *The coefficient g_{θ_u} depends only on $A_{n\alpha_n}$, $B_{m\beta_m}$ and $C_{l\gamma_l}$. Since $a_n \neq 0$, $b_m \neq 0$, $c_l \neq 0$ and*

$$g_{\theta_u}(a_n, b_m, c_l) = 0,$$

the coefficient g_{θ_u} *cannot be a monomial in* a_n, b_m, c_l. *In general,* g_{θ_u} *is a quasi-homogeneous polynomial in which* $A_{n\alpha_n}$ *has weight* α_n, $B_{m\beta_m}$ *weight* β_m *and* $C_{l\gamma_l}$ *weight* γ_l. *Equality of weights of different monomials appearing in* g_{θ_u} *will give us a system of conditions on the exponents* α_n, β_m *and* γ_l. *More precisely, we are not interested in the values of* α_n, β_m *and* γ_l *per se but rather in the ratios of the form* $\frac{\alpha_n}{\delta}$, *where* δ *is some fixed element of the set* $\{\alpha_{\mu_i(x)}, \beta_{\mu_i(y)}, \gamma_{\mu_i(z)}\}$. *In other words, we are interested in the "normalized" weights* α_n, β_m *and* γ_l, *where we set, for example, the first non-trivial weight* $\alpha_{\mu_i(x)}$ *equal to* 1.

- *The hardest part of the proof is to recover the coefficients* g_{θ_u}. *In order to do this, we will use the fact that* g_{θ_u} *are not monomials to give lower bounds on* α_n, β_m *and* γ_l.

The equation $g_{\theta_u} = 0$ plus the equations $f_{jk}(a_{n0}, b_{m0}, c_{l0}) = 0$ form a system satisfied by the coefficients of the wedge. If this system has no solutions then the wedge does not exist.

In some exceptional cases, the above system of equations does not suffice and one is led to use $f'_{i,\theta_u+1,u}$, the next coefficient of $f_{iu}(A_n(s), B_m(s), C_l(s))$ after g_{θ_u}, to arrive at a contradiction. In our work on $\mathbf{E_6}$ such will be the case for the non-inclusion $\overline{N_4} \not\subset \overline{N_2}$.

In the next section we compute the weights α_n, β_m and γ_l for the singularity E_6 and show that the system

$$\begin{cases} g_{\theta_u} = 0 \\ f_{ju}(a_{n0}, b_{mu}, c_{l0}) = 0 \end{cases}$$

for the remaining non-inclusions other than $\overline{N_4} \not\subset \overline{N_2}$, as well as the augmented system

$$\begin{cases} g_{\theta_u} = 0 \\ f'_{i,\theta_u+1,u} = 0 \\ f_{ju}(a_{n0}, b_{m0}, c_{l0}) = 0 \end{cases}$$

in the case of the non-inclusion $\overline{N_4} \not\subset \overline{N_2}$, have no solutions.

4. Computations and proof for the $\mathbf{E_6}$ singularity

Let us consider the $\mathbf{E_6}$ singularity and study the different non-inclusions. For each non-inclusion $\overline{N_j} \not\subset \overline{N_i}$ appearing in (21)–(23), we will denote $R(k) = \frac{R_k}{\mathscr{P}_{ik}}$.

Notation: When talking about the non-inclusion $\overline{N_j} \not\subset \overline{N_i}$, the notation $a|b$ will mean "a divides b in $\overline{R(k)}$", unless otherwise specified (here $\overline{R(k)}$ stands for the integral closure of $R(k)$ in its field of fractions). For some non-inclusions, we will study divisibility in a suitable localization of $\overline{R(k)}$, which will be specified explicitly in each case.

For each of the six non-inclusions involved, it is sufficient to prove that

$$\mathscr{P}_{ik} \not\subset \mathscr{P}_{jk} \tag{41}$$

for some k, in particular for $k = o(j)$. Take $k = o(j)$.

We prove the non-inclusion (41) by contradiction. Assume that $\mathscr{P}_{ik} \subset \mathscr{P}_{jk}$. By the Curve Selection lemma there exists an L-wedge whose special arc is the generic point of $N_j(k)$ and whose generic arc is in $N_i(k)$. The first coefficient g_{θ_u} of f_{iu} cannot be a monomial as generically on $N_i(k)$ each monomial in a_n, b_m, c_l is not zero.

As explained above, we are interested in computing ratios of the form $\frac{\alpha_n}{\delta}$, where δ is some fixed element of the set $\{\alpha_{\mu_i(x)}, \beta_{\mu_i(y)}, \gamma_{\mu_i(z)}\}$, and $\mu_i(x) \leqslant n < \mu_j(x)$, and similarly for $\frac{\beta_m}{\delta}$, $\mu_i(y) \leqslant m < \mu_j(y)$, and $\frac{\gamma_l}{\delta}$, $\mu_i(z) \leqslant l < \mu_j(z)$ (we will pick and fix a specific δ in the proof of each non-inclusion, but the choice of δ will depend on the non-inclusion we want to prove). For example, suppose $\delta = \alpha_{\mu_i(x)}$. Then our problem is closely related to studying, for each n, the totality of pairs $(\alpha, \delta') \in \mathbf{N}^2$ such that

$$\mathbf{a}_n(s)^{\alpha} \mid \mathbf{a}_{\mu_i(x)}(s)^{\delta'}, \tag{42}$$

and similarly for $\mathbf{b}_m(s)^{\beta} \mid \mathbf{a}_{\mu_i(x)}(s)^{\delta'}$ and $\mathbf{c}_l(s)^{\alpha} \mid \mathbf{a}_{\mu_i(x)}(s)^{\delta'}$. Precisely, we have

$$\frac{\alpha_n}{\alpha_{\mu_i(x)}} = \inf\left\{\frac{\alpha}{\delta'}\right\},$$

where (α, δ') runs over all the pairs satisfying (42).

Remark 4.1. In [26] and [27] a different method is used to prove the non-inclusions not covered by the valuative criterion. Namely, we use the fact that the ideal $\mathscr{P}_{ik}$ can be expressed as the saturation $(\mathscr{P}_{ik} R(k) : d^{\infty})$, where $d \in \{a_{\mu_i(x)}, b_{\mu_i(y)}, c_{\mu_i(z)}\}$. For most non-inclusions, we explicitly construct elements of $(\mathscr{P}_{ik} R(k) : d^{\infty})$, not belonging to $\mathscr{P}_{jk}$, which settles the problem. In both the saturation and the wedge methods, the key point is to compute the weight ratios of the form $\frac{\alpha_n}{\delta}$, $\frac{\beta_m}{\delta}$ and $\frac{\gamma_l}{\delta}$ as above. One advantage of the wedge method is that it gives a more geometric vision of the proof.

In what follows we truncate at the order o_j.

(1) • $\overline{N_4} \not\subset \overline{N_1}$. In this case we truncate at the order $o_4 = 8$. We have $o_1 = 6$.

Assume that $\overline{N_4(8)} \subset \overline{N_1(8)}$, aiming for contradiction. Let ϕ_{42} be a wedge with generic arc living in $N_1(8)$ and special arc mapping to $N_4(8)$. Then the wedge is of the form:

$$(43)\qquad \begin{cases} \mathbf{b}_2(s) = b_2 s^{\beta_2} + \sum_{q=\beta_2+1}^{\infty} b_{2q} s^q \\ \mathbf{c}_3(s) = c_3 s^{\gamma_3} + \sum_{q=\gamma_3+1}^{\infty} c_{3q} s^q \\ \mathbf{a}_n(s) = a_n + \sum_{q=1}^{\infty} a_{nq} s^q, \quad n \geqslant 2 \\ \mathbf{b}_m(s) = b_m + \sum_{q=1}^{\infty} b_{mq} s^q, \quad m \geqslant 3 \\ \mathbf{c}_l(s) = c_l + \sum_{q=1}^{\infty} c_{lq} s^q, \quad l \geqslant 4 \end{cases}$$

where a_n, b_m, c_l satisfy the equations of $N_4(6)$, and are non-zero elements of L.

The following equations hold on $N_1(8)$:

$$\mathbf{a}_1 = \mathbf{b}_1 = \mathbf{c}_1 = \mathbf{c}_2 = 0$$
$$f_{1,6} = \mathbf{c}_3^2 + \mathbf{b}_2^3 = 0$$
$$f_{1,7} = 2\mathbf{c}_3\mathbf{c}_4 + 3\mathbf{b}_2^2\mathbf{b}_3 = 0.$$

The following equations hold on $N_4(8)$:

$$a_1 = b_1 = c_1 = c_2 = c_3 = b_2 = 0$$
$$f_{4,8} = c_4^2 + a_2^4 = 0.$$

The generic arc lives in $N_1(8)$, and thus satisfies the equations of $N_1(8)$. This leads to finitely many equations (as we are in $R(8)$):

$$0 = f_{1,6}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s)) = c_3^2 s^{2\gamma_3} + b_2^3 s^{3\beta_2} + \cdots$$
$$0 = f_{1,7}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s)) = 2c_3c_4 s^{\gamma_3} + 3b_2^2 b_3 s^{2\beta_2} + \cdots$$
$$\vdots$$

As $c_3 \neq 0$ and $b_2 \neq 0$, we obtain a relation between γ_3 and β_2:

$$2\gamma_3 = 3\beta_2$$

which implies that

$$\beta_2 \leqslant \gamma_3 < 2\beta_2$$

and that

$$c_3^2 + b_2^3 = 0$$
$$2c_3c_4 = 0.$$

Thus for the equation $f_{4,8}$ we have $\theta_8 = \gamma_3$ and $g_{\theta_8} = 2c_3c_4$, which is impossible.

(2) • $\overline{N_5} \not\subset \overline{N_2}$. In this case we truncate at the order $o_5 = 8$. We have $o_2 = 6$. Assume that $\overline{N_5} \subset \overline{N_2}$, aiming for contradiction. Let ϕ_{52} be a wedge with generic arc living in $N_2(8)$ and special arc mapping to the generic arc in $N_5(8)$.

The following equations hold on $N_2(8)$:

$$\mathbf{b}_1 = \mathbf{c}_1 = 0$$
$$f_{2,4} = \mathbf{c}_2^2 + \mathbf{a}_1^4 = 0$$
$$f_{2,5} = 2\mathbf{c}_2\mathbf{c}_3 + 4\mathbf{a}_1^3\mathbf{a}_2 = 0$$
$$f_{2,6} = \mathbf{c}_3^2 + 2\mathbf{c}_2\mathbf{c}_4 + \mathbf{b}_2^3 + 4\mathbf{a}_1^3\mathbf{a}_3 + 6\mathbf{a}_1^2\mathbf{a}_2^2 = 0$$
$$f_{2,7} = 2\mathbf{c}_3\mathbf{c}_4 + 2\mathbf{c}_2\mathbf{c}_5 + 3\mathbf{b}_2^2\mathbf{b}_3 + 4\mathbf{a}_1^3\mathbf{a}_4 + 12\mathbf{a}_1^2\mathbf{a}_2\mathbf{a}_3 + 4\mathbf{a}_2^3\mathbf{a}_1 = 0$$

We have $f_{2,4} = (\mathbf{c}_2 + i\mathbf{a}_1^2)(\mathbf{c}_2 - i\mathbf{a}_1^2)$. As can be seen from Figure 1, $\mathscr{P}_{2,8}$ is the distinguished ideal corresponding to the irreducible factor $\mathbf{c}_2 - i\mathbf{a}_1^2$. Let us use the notation

$$g_{2,2} := \mathbf{c}_2 - i\mathbf{a}_1^2.$$

Combining $f_{2,5}$ and $g_{2,2}$ we see that $2i\mathbf{c}_3\mathbf{a}_1^2 + 4\mathbf{a}_1^2\mathbf{a}_2 = 0$ on $N_2(8)$. Since $\mathbf{a}_1$ does not vanish identically on $N_2(8)$, we have

$$\bar{f}_{2,3} := \mathbf{c}_3 - 2i\mathbf{a}_1\mathbf{a}_2 = 0$$

on $N_2(8)$.

We claim that

$$\gamma_2 = 2\alpha_1, \tag{44}$$
$$\gamma_3 = \alpha_1 \tag{45}$$
$$\beta_2 \geqslant \frac{2}{3}\alpha_1 \tag{46}$$

Now,

- (44) holds thanks to the equation $g_{2,2} = 0$.
- (45) holds by the equation $\bar{f}_{2,3} = 0$ and the fact that $\alpha_2 = 0$.
- (46) holds by the equation $\bar{f}_{2,6} = 0$, (44) and (45).

After a suitable automorphism of $L[[s]]$, we may assume that $a_1 = 1$.

The vanishing of the first non-trivial coefficients of the power series $\bar{f}_{2,3}(\mathbf{a}_1(s), \mathbf{a}_2(s), \mathbf{c}_3(s))$ and $f_{2,7}$ gives the equations

$$c_3 - 2ia_2 = 0 \tag{47}$$

$$2c_3c_4 + 4a_2^3 = 0 \tag{48}$$

and we have (first equation of N_5):

$$c_4 + ia_2^2 = 0. \tag{49}$$

Substituting (47) into (48) and dividing through by $4ia_2^2$, we obtain the equation

$$c_4 - ia_2^2 = 0,$$

which contradicts (49) and the fact that c_4 and a_2 are non-zero elements of L.

(3) • $\overline{N_4} \not\subset \overline{N_2}$. In this case we truncate at the order $o_4 = 8$.

Assume that $N_4(8) \subset \overline{N_2(8)}$, aiming for contradiction. We can construct an L-wedge $Spec\,L[[t,s]] \to E_6$, with the special arc mapping to the generic arc of N_4 and with the general arc lifting to E_2.

The following equations hold on $N_2(8)$:

$$\mathbf{b}_1 = \mathbf{c}_1 = 0$$

$$g_{2,2} = \mathbf{c}_2 - i\mathbf{a}_1^2 = 0$$

$$\bar{f}_{2,3} = \mathbf{c}_3 - 2i\mathbf{a}_1\mathbf{a}_2 = 0$$

$$f_{2,6} = \mathbf{c}_3^2 + 2\mathbf{c}_2\mathbf{c}_4 + \mathbf{b}_2^3 + 4\mathbf{a}_1^3\mathbf{a}_3 + 6\mathbf{a}_1^2\mathbf{a}_2^2 = 0$$

$$f_{2,7} = 2\mathbf{c}_3\mathbf{c}_4 + 2\mathbf{c}_2\mathbf{c}_5 + 3\mathbf{b}_2^2\mathbf{b}_3 + 4\mathbf{a}_1^3\mathbf{a}_4 + 12\mathbf{a}_1^2\mathbf{a}_2\mathbf{a}_3 + 4\mathbf{a}_2^3\mathbf{a}_1 = 0$$

$$f_{2,8} = \mathbf{c}_4^2 + 2\mathbf{c}_3\mathbf{c}_5 + 2\mathbf{c}_2\mathbf{c}_6 + 3\mathbf{b}_2^2\mathbf{b}_4 + \mathbf{a}_2^4 + 4\mathbf{a}_1^3\mathbf{a}_5 + 12\mathbf{a}_1^2\mathbf{a}_2\mathbf{a}_4 + 12\mathbf{a}_1\mathbf{a}_2^2\mathbf{a}_3 + 6\mathbf{a}_1^2\mathbf{a}_3^2 = 0.$$

Modifying $f_{2,6}$ and $f_{2,7}$ by suitable multiples of $g_{2,2}$ and $\bar{f}_{2,3}$, we may replace them by

$$\bar{f}_{2,6} := 2i\mathbf{a}_1^2(\mathbf{c}_4 - i\mathbf{a}_2^2) + \mathbf{b}_2^3 + 4\mathbf{a}_1^3\mathbf{a}_3 = 0$$

$$\bar{f}_{2,7} := 4i\mathbf{a}_1\mathbf{a}_2(\mathbf{c}_4 - i\mathbf{a}_2^2) + 2\mathbf{c}_2\mathbf{c}_5 + 3\mathbf{b}_2^2\mathbf{b}_3 + 4\mathbf{a}_1^3\mathbf{a}_4 + 12\mathbf{a}_1^2\mathbf{a}_2\mathbf{a}_3 = 0$$

$$\bar{f}_{2,8} = \mathbf{c}_4^2 + 4i\mathbf{a}_1\mathbf{a}_2\mathbf{c}_5 + 2i\mathbf{a}_2\mathbf{c}_6 + 3\mathbf{b}_2^2\mathbf{b}_4 + \mathbf{a}_2^4 + 4\mathbf{a}_1^3\mathbf{a}_5 + 12\mathbf{a}_1^2\mathbf{a}_2\mathbf{a}_4 + 12\mathbf{a}_1\mathbf{a}_2^2\mathbf{a}_3 + 6\mathbf{a}_1^2\mathbf{a}_3^2 = 0$$

Note that the equation $f_{4,8} = c_4 - ia_2^2 = 0$ vanishes on $\overline{N_4(8)}$.

Let μ denote the s-adic valuation of $L[[s]]$. We define $\alpha := \mu(\mathbf{c}_4(s) - i\mathbf{a}_2(s)^2)$. We claim that

$$\gamma_2 = 2\alpha_1 \tag{50}$$

$$\gamma_3 = \alpha_1 \tag{51}$$

$$\beta_2 \geqslant \alpha_1 \tag{52}$$

$$\alpha \geqslant \alpha_1. \tag{53}$$

Indeed,

- (50) holds thanks to the equation $g_{22} = 0$.
- (51) is given by $\bar{f}_{2,3}$ and the fact that $a_{20} = a_2 \neq 0$, and hence

$$\alpha_2 = 0. \tag{54}$$

We have $\alpha_1 > 0$. Using (54) once again, we obtain from the equations $\bar{f}_{2,6} = 0$ and $\bar{f}_{2,7} = 0$ that

- $3\beta_2 \geqslant \min\{3, 2 + \alpha\}$
- $\alpha \geqslant \min\{1, 2\beta_2 - 1\}$.

We will now prove (52) and (53) by contradiction. Assume that at least one of (52) and (53) is false. Then both (52) and (53) are false according to the above inequalities. We see that

- $3\beta_2 \geqslant 2 + \alpha$
- $\alpha \geqslant 2\beta_2 - 1$

which implies that $\frac{2}{3} + \frac{1}{3}\alpha \leqslant \frac{1}{2} + \frac{1}{2}\alpha$, hence $\alpha \geqslant 1$, a contradiction. This completes the proof of the relations (50)–(53).

After a suitable automorphism of $L[[s]]$, we may assume that

$$\mathbf{a}_1(s) = s^{\alpha_1}.$$

Generically, each arc lives in N_2, and thus satisfies the equations of $N_2(8)$. Let $\tilde{c}$ denote the coefficient of s^{α_1} in the formal power series $\mathbf{c}_4(s) - i\mathbf{a}_2(s)^2$ (a priori, $\tilde{c}$ may or may not be zero). Expanding the equations $\bar{f}_{2,6}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s))$, $\bar{f}_{2,7}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s))$, $\bar{f}_{2,8}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s))$ as power series in s gives:

$$\begin{aligned} 0 &= \bar{f}_{2,6}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s)) = g_{2,6}s^{3\alpha_1} + h_{2,6}s^{3\alpha_1+1} \\ 0 &= \bar{f}_{2,7}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s)) = g_{2,7}s^{2\alpha_1} + h_{2,7}s^{2\alpha_1+1} \\ 0 &= \bar{f}_{2,8}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s)) = g_{2,8}s^{\alpha_1} + h_{2,8}s^{\alpha_1+1}, \end{aligned}$$

where $g_{2,6}$, $g_{2,7}$, $g_{2,8}$ are polynomials in a_{np}, b_{np}, c_{np} and $h_{2,6}, h_{2,7}, h_{2,8} \in L[[s]]$.

Since $\bar{f}_{2,6}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s))$, $\bar{f}_{2,7}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s))$, $\bar{f}_{2,8}(\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s))$ must vanish identically as power series in s, we must have $g_{2,6} = g_{2,7} = g_{2,8} = 0$. Let us look at the $g_{2,i}$'s. They are:

$$g_{2,6} = \tilde{c} - 2ia_3 + b_{2,\alpha_1}^3 = 0 \tag{55}$$

$$g_{2,7} = 2i(c_5 - 6ia_2a_3) + 4i\tilde{c} + 3b_{2,\alpha_1}^2 b_3 = 0 \tag{56}$$

$$g_{2,8} = 12ia_2^2a_3 + 4ia_2c_5 + 2ia_2^2\tilde{c} + 3b_{2,\alpha_1}b_3^2 = 0. \tag{57}$$

Elements a_2, a_3, b_3, c_5 lie in $K(N_4(8)) \subset L$ and are different from 0. Let us regard (55)–(57) as a system of three equations over L in two unknowns b_{2,α_1}, $\tilde{c}$; if the wedge exists, these equations should have a solution. Let us prove that this is in fact not the case, thus obtaining the desired contradiction.

The subfield of $K(N_4(8))$ generated by a_2, a_3, b_3, c_5 is isomorphic to the field of fractions of the ring $B = \dfrac{\mathbf{k}[a_2, a_3, b_3, c_5]}{(b_3^3 + 2ia_2^2 c_5 + 4a_2^3 a_3)}$. Let Y denote the affine subscheme of $\mathbf{A}_B^2$ defined by the equations (55)–(57) and let $\overline{Y}$ denote its closure in $\mathbf{P}_B^2$. The scheme $\overline{Y}$ is defined in $\mathbf{P}_B^2$ by the system of three equations

$$G_{2,6} = Z^2 \tilde{C} - 2ia_3 Z^3 + B_{2,\alpha_1}^3 = 0 \tag{58}$$

$$G_{2,7} = 2i(c_5 - 6ia_2 a_3) Z^2 + 4i\tilde{C} Z + 3B_{2,\alpha_1}^2 b_3 = 0 \tag{59}$$

$$G_{2,8} = (12ia_2^2 a_3 + 4ia_2 c_5) Z + 2ia_2^2 \tilde{C} + 3B_{2,\alpha_1} b_3^2 = 0, \tag{60}$$

homogeneous in the variables Z, $\tilde{C}$, B_{2,α_1}.

Suppose the system (55)–(57) had a solution in L. This means that the natural map $Y \to \mathit{Spec}\, B$ is dominant, and hence the map $Y \to \mathit{Spec}\, B$ is surjective by the Proper Mapping Theorem. Thus to prove non-existence of solutions of (55)–(57) it is sufficient to find one specific $\mathbf{k}$-rational point of $\mathit{Spec}\, B$ which is not in the image of $\overline{Y}$. In other words, it suffices to find specific elements of $\mathbf{k}$ such that when these elements are substituted for a_2, a_3, b_3, c_5, the resulting system of homogeneous equations in Z, $\tilde{C}$, B_{2,α_1} has no non-zero solutions. We can easily find such elements. For example, put

$$b_3 = 0. \tag{61}$$

Then

$$2ia_2^2 c_5 + 4a_2^3 a_3 = 0. \tag{62}$$

We will take

$$a_2 \neq 0. \tag{63}$$

Then equation (62) implies that

$$c_5 - 2ia_2 a_3 = 0. \tag{64}$$

Substituting (61) and (64) into $G_{2,7}$ and $G_{2,8}$, we obtain

$$\overline{G}_{2,7} = 8a_2 a_3 Z^2 + 4i\tilde{C} Z = 0 \tag{65}$$

$$\overline{G}_{2,8} = (12i - 8) a_2^2 a_3 Z + 2ia_2^2 \tilde{C} = 0. \tag{66}$$

If $Z = 0$ then, in view of (63) and the equation $G_{2,6} = 0$, we have $\tilde{C} = B_{2,\alpha_1} = 0$. Thus there are no non-trivial solutions with $Z = 0$. Assume

$Z \neq 0$ and divide $\bar{G}_{2,7}$ by Z. Now it is easy to see that there exist $a_2, a_3 \in \mathbf{k}$ with $a_2 \neq 0$ such that the system

$$8a_2a_3Z + 4i\tilde{C} = 0 \tag{67}$$

$$\bar{G}_{2,8} = (12i - 8)a_2^2a_3Z + 2ia_2^2\tilde{C} = 0 \tag{68}$$

has

$$Z = \tilde{C} = 0 \tag{69}$$

as the only solution. (69) together with $G_{2,6}$ implies that $B_{2,\alpha_1} = 0$. We have proved that there exists a choice of elements $a_2, a_3, b_3, c_5 \in \mathbf{k}$, satisfying $b_3^3 + 2ia_2^2c_5 + 4a_2^3a_3 = 0$, such that after substituting these values into $G_{2,6} = G_{2,7} = G_{2,8} = 0$ the resulting system has no non-trivial solutions. This completes the proof of the non-inclusion $\overline{N_4} \not\subset \overline{N_2}$.

(4) • $\overline{N_6} \not\subset \overline{N_4}$.

In this case we truncate at the order $o_6 = 12$. We argue by contradiction. Assume that $\overline{N_6(12)} \subset \overline{N_4(12)}$. Let ϕ_{64} be a wedge with generic arc living in $N_4(12)$ and special arc mapping to the generic point of $N_6(12)$. The following equations hold on $N_4(12)$:

$$\mathbf{a}_1 = \mathbf{b}_1 = \mathbf{c}_1 = \mathbf{c}_2 = \mathbf{c}_3 = \mathbf{b}_2 = 0$$

$$f_{4,8} = \mathbf{c}_4^2 + \mathbf{a}_2^4 = 0$$

$$f_{4,9} = 2\mathbf{c}_4\mathbf{c}_5 + \mathbf{b}_3^3 + 4\mathbf{a}_2^3\mathbf{a}_3 = 0$$

$$f_{4,10} = \mathbf{c}_5^2 + 2\mathbf{c}_4\mathbf{c}_6 + 3\mathbf{b}_3^2\mathbf{b}_4 + 6\mathbf{a}_2^2\mathbf{a}_3^2 + 4\mathbf{a}_2^3\mathbf{a}_4 = 0$$

$$f_{4,11} = 2\mathbf{c}_5\mathbf{c}_6 + 2\mathbf{c}_4\mathbf{c}_7 + 3\mathbf{b}_3^2\mathbf{b}_5 + 3\mathbf{b}_3\mathbf{b}_4^2 + 12\mathbf{a}_2^2\mathbf{a}_3\mathbf{a}_4 + 4\mathbf{a}_2^3\mathbf{a}_5 + 4\mathbf{a}_2\mathbf{a}_3^3 = 0$$

We have $f_{4,8} = (\mathbf{c}_4 + i\mathbf{a}_2^2)(\mathbf{c}_4 - i\mathbf{a}_2^2)$. As can be seen from Figure 1, $\mathscr{P}_{4,12}$ is the distinguished ideal corresponding to the irreducible factor $\mathbf{c}_4 - i\mathbf{a}_2^2$. Let us use the notation

$$g_{2,4} := \mathbf{c}_4 - i\mathbf{a}_2^2;$$

we have $g_{2,4} = 0$ on $N_4(12)$. We have

$$a_{20} = b_{30} = c_{40} = c_{50} = 0; \tag{70}$$

we want to show that

$$\mathbf{a}_2(s) \mid \mathbf{b}_3(s) \tag{71}$$

$$\mathbf{a}_2^2(s) \mid \mathbf{c}_4(s) \tag{72}$$

$$\mathbf{a}_2(s) \mid \mathbf{c}_5(s) \tag{73}$$

in $L[[s]]$.

The equation $g_{2,4}=0$ implies (72).

Now, (71) and (73) are equivalent to saying that

$$\alpha_2 \leqslant \beta_3 \quad \text{and} \tag{74}$$

$$\alpha_2 \leqslant \gamma_5. \tag{75}$$

By (70), we have $\alpha_2 > 0$. Using (72), equations $f_{4,9}=0$ and $f_{4,10}=0$ yield

- $\beta_3 \geqslant \min\{\frac{2}{3}\alpha_2 + \frac{1}{3}\gamma_5, \alpha_2\}$
- $\gamma_5 \geqslant \min\{\alpha_2, \beta_3\}$.

We prove (74) and (75) by contradiction. Suppose at least one of (74) and (75) is false. Then both (74) and (75) are false by the above inequalities. Then

- $\beta_3 \geqslant \frac{2}{3}\alpha_2 + \frac{1}{3}\gamma_5$
- $\gamma_5 \geqslant \beta_3$.

Hence $\frac{2}{3}\gamma_5 \geqslant \frac{2}{3}\alpha_2$, so $\gamma_5 \geqslant \alpha_2$, a contradiction. This completes the proof of (71)–(73).

For the purposes of this non-inclusion, we will deviate slightly from our standard notation. Namely, we will write $b_3 = b_{3\alpha_2}$ and $c_5 = c_{5\alpha_2}$. The meaning of all the other symbols remains unchanged.

Then the first coefficients of the wedge have to satisfy:

$$c_4 - ia_2^2 = 0$$

$$2c_4c_5 + b_3^3 + 4a_2^3a_3 = 0$$

$$c_5^2 + 2c_4c_6 + 3b_3^2b_4 + 6a_2^2a_3^2 = 0$$

$$2c_5c_6 + 3b_3b_4^2 + 4a_2a_3^3 = 0$$

as well as

$$c_6^2 + b_4^3 + a_3^4 = 0. \tag{76}$$

Substituting c_4 for ia_2^2, the above system rewrites as

$$2ia_2^2c_5 + b_3^3 + 4a_2^3a_3 = 0$$

$$c_5^2 + (2ic_6 + 6a_3^2)a_2^2 + 3b_3^2b_4 = 0$$

$$2c_5c_6 + 3b_3b_4^2 + 4a_2a_3^3 = 0.$$

We view this system as a system of three homogeneous equations over L in three unknowns a_2, b_3, c_5. The coefficients of the system are polynomials in a_3, b_4, c_6, which are viewed as fixed elements of $K(N_6(12))$. Moreover, we must have $a_2 \neq 0$ by definition of a_2. As in the previous non-inclusion, to prove that this system has no non-zero solutions, it suffices to find specific values of a_3, b_4, c_6 in $\mathbf{k}$ satisfying (76), such that the resulting system of three equations has no non-zero solutions. We take $a_3 = 0$. Then

$$c_6^2 + b_4^3 = 0 \tag{77}$$

and our system becomes

$$2ia_2^2c_5 + b_3^3 = 0 \tag{78}$$

$$c_5^2 + 2ic_6a_2^2 + 3b_3^2b_4 = 0 \tag{79}$$

$$2c_5c_6 + 3b_3b_4^2 = 0. \tag{80}$$

We work in a finite extension of $K(N_6(12))$ which contains a square root of b_4; we pick and fix one of the two possible square roots and denote it by $b_4^{1/2}$. From (77) we obtain

$$c_6 = -b_4^{3/2}. \tag{81}$$

Substituting (81) into (80) and dividing through by $b_4^{3/2}$, we obtain

$$c_5 = \frac{3}{2}b_3b_4^{1/2}. \tag{82}$$

Substituting (82) into (78) yields

$$b_3^2 = -2ib_4^{1/2}a_2^2. \tag{83}$$

Finally, substituting (82) and (83) into (79), we obtain

$$\left(-\frac{27}{4} - 2 - 9\right)ib_4^{3/2}a_2^2 = 0. \tag{84}$$

Now, substitute suitable non-zero elements of $\mathbf{k}$ for $b_4^{1/2}$ and c_6 in such a way that (77) is satisfied. By (84), any solution of the resulting system of equations satisfies $a_2 = 0$. Then $b_3 = c_5 = 0$ from (78)–(80). Thus our system of equations has no non-zero solutions, as desired. This completes the proof of the non-inclusion $\overline{N_6} \not\subset \overline{N_4}$.

(5) • $\overline{N_6} \not\subset \overline{N_1}$.

In this case we truncate at the order $o_6 = 12$. We argue by contradiction: suppose that $\overline{N_6}(12) \subset \overline{N_1}(12)$. Let ϕ_{61} be a wedge with the generic arc living in N_1 and the special arc mapping to the generic point of $N_6(12)$.

The following equations hold on $N_1(12)$:

$$\mathbf{a}_1 = \mathbf{b}_1 = \mathbf{c}_1 = \mathbf{c}_2 = 0$$

$$f_{1,6} = \mathbf{c}_3^2 + \mathbf{b}_2^3 = 0$$

$$f_{1,7} = 2\mathbf{c}_3\mathbf{c}_4 + 3\mathbf{b}_2^2\mathbf{b}_3 = 0$$

$$f_{1,8} = \mathbf{c}_4^2 + 2\mathbf{c}_3\mathbf{c}_5 + 3\mathbf{b}_2^2\mathbf{b}_4 + 3\mathbf{b}_2\mathbf{b}_3^2 + \mathbf{a}_2^4 = 0$$

$$f_{1,9} = 2\mathbf{c}_4\mathbf{c}_5 + 2\mathbf{c}_3\mathbf{c}_6 + \mathbf{b}_3^3 + 6\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4 + 3\mathbf{b}_2^2\mathbf{b}_5 + 4\mathbf{a}_2^3\mathbf{a}_3 = 0$$

$$f_{1,10} = \mathbf{c}_5^2 + 2\mathbf{c}_4\mathbf{c}_6 + 2\mathbf{c}_3\mathbf{c}_7 + 3\mathbf{b}_3^2\mathbf{b}_4 + 3\mathbf{b}_2^2\mathbf{b}_6 + 6\mathbf{b}_2\mathbf{b}_3\mathbf{b}_5 + 3\mathbf{b}_2\mathbf{b}_4^2 + 6\mathbf{a}_2^2\mathbf{a}_3^2 + 4\mathbf{a}_2^3\mathbf{a}_4 = 0$$

$$f_{1,11} = 2\mathbf{c}_5\mathbf{c}_6 + 2\mathbf{c}_4\mathbf{c}_7 + 2\mathbf{c}_3\mathbf{c}_8 + 3\mathbf{b}_2^2\mathbf{b}_7 + 3\mathbf{b}_3^2\mathbf{b}_5 + 3\mathbf{b}_3\mathbf{b}_4^2 + 6\mathbf{b}_2\mathbf{b}_3\mathbf{b}_6 + 6\mathbf{b}_2\mathbf{b}_4\mathbf{b}_5 + 4\mathbf{a}_2^3\mathbf{a}_5 + 12\mathbf{a}_2^2\mathbf{a}_3\mathbf{a}_4 + 4\mathbf{a}_2\mathbf{a}_3^3 = 0$$

The following equations come from the equations of $N_6(12)$:

$$a_{10} = a_{20} = b_{10} = b_{20} = b_{30} = c_{10} = c_{20} = c_{30} = c_{40} = c_{50} = 0.$$

We want to prove the following divisibility relations:

$$\mathbf{b}_2(s) \mid \mathbf{a}_2(s)^2 \tag{85}$$

$$\mathbf{b}_2(s) \mid \mathbf{b}_3(s)^2 \tag{86}$$

$$\mathbf{b}_2(s)^3 \mid \mathbf{c}_3(s)^2 \tag{87}$$

$$\mathbf{b}_2(s) \mid \mathbf{c}_4(s) \tag{88}$$

$$\mathbf{b}_2(s) \mid \mathbf{c}_5(s)^2. \tag{89}$$

To do this, it is sufficient to show that

$$\gamma_3 = \frac{3}{2}\beta_2 \tag{90}$$

$$\gamma_4 \geqslant \beta_2 \tag{91}$$

$$\alpha_2, \beta_3, \gamma_5 \geqslant \frac{1}{2}\beta_2. \tag{92}$$

We have $\beta_2 > 0$. The equality (90) is immediate from $f_{1,6} = 0$. (91) follows from $f_{1,6} = f_{1,7} = 0$ and (92). It remains to prove (92), which is equivalent to saying that

$$\min\{\alpha_2, \beta_3, \gamma_5\} \geqslant \frac{1}{2}\beta_2. \tag{93}$$

We prove (93) by contradiction. Let $M = \min\{\alpha_2, \beta_3, \gamma_5\}$ and assume that

$$M < \frac{1}{2}\beta_2. \tag{94}$$

Equations $f_{1,6} = f_{1,7} = 0$ can be interpreted as saying that $\dfrac{\mathbf{c}_3(s)}{\mathbf{b}_2(s)^{3/2}}$ and $\dfrac{\mathbf{c}_4(s)}{\mathbf{b}_2(s)^{1/2}\mathbf{b}_3(s)}$ is invertible in a suitable finite extension B of $L[[s]]$. Substituting $\mathbf{c}_3(s)$ and $\mathbf{c}_4(s)$ in $f_{1,8}$, $f_{1,9}$ and $f_{1,10}$ by suitable multiples of

$\mathbf{b}_2(s)^{3/2}$ and $\mathbf{b}_2(s)^{1/2}\mathbf{b}_3(s)$ by a unit of B, we obtain the following inequalities:

$$\alpha_2 \geqslant \frac{1}{4} \min\left\{\frac{3}{2}\beta_2 + \gamma_5, 2\beta_2, \beta_2 + 2\beta_3\right\} \tag{95}$$

$$\beta_3 \geqslant \frac{1}{3} \min\left\{\frac{1}{2}\beta_2 + \beta_3 + \gamma_5, \beta_2 + \beta_3, \frac{3}{2}\beta_2, 3\alpha_2\right\} \tag{96}$$

$$\gamma_5 \geqslant \frac{1}{2} \min\left\{\frac{1}{2}\beta_2 + \beta_3, 2\beta_3, \beta_2, 2\alpha_2\right\}. \tag{97}$$

Now, (94), (95) and the definition of M imply that

$$M < \alpha_2 \tag{98}$$

(indeed, if we had $M \geqslant \alpha_2$, we could use (94) and the definition of M to show that M is strictly less than each of the three quantities on the right hand side of (95), which would be a contradiction).

In a similar way, (94), (96), (98) and the definition of M imply that

$$M < \beta_3. \tag{99}$$

By (98) and (99), we have $M = \gamma_5$, which contradicts (97) (using (98) and (99) once again). This completes the proof of (85)–(89).

Replacing s by s^2 in the parametrization of the wedge, we may assume, without loss of generality, that β_2 is even. The first coefficients of the wedge must satisfy the following equations (as above we change the notation by $c_4 = c_{4,\beta_2}$, $c_5 = c_{5,\beta_2/2}$, $b_3 = b_{3,\beta_2/2}$ and $a_2 = a_{2,\beta_2/2}$):

$$\begin{gathered}
c_3^2 + b_2^3 = 0 \\
2c_3c_4 + 3b_2^2b_3 = 0 \\
c_4^2 + 2c_3c_5 + 3b_2^2b_4 + 3b_2b_3^2 + a_2^4 = 0 \\
2c_4c_5 + 2c_3c_6 + b_3^3 + 6b_2b_3b_4 + 4a_2^3a_3 = 0 \\
c_5^2 + 2c_4c_6 + 3b_3^2b_4 + 3b_2b_4^2 + 6a_2^2a_3^2 = 0 \\
2c_5c_6 + 3b_3b_4^2 + 4a_2a_3^3 = 0
\end{gathered}$$

as well as

$$c_6^2 + b_4^3 + a_3^4 = 0. \tag{100}$$

We view this system as a system of six homogeneous equations over L in six unknowns a_2, b_2, b_3, c_3, c_4, c_5. The coefficients of the system are polynomials in a_3, b_4, c_6, which are viewed as fixed elements of $K(N_6(12))$. As in the previous non-inclusion, to prove that this system has no non-zero solutions, it suffices to find specific values of a_3, b_4, c_6 in $\mathbf{k}$ satisfying (76), such that the resulting system of six equations has no non-zero solutions. In this case, we take $a_3 = 0$, $c_6 = 1$ and b_4 a non-real root of $z^3 = -1$. We obtain:

$$c_3^2 + b_2^3 = 0$$
$$2c_3c_4 + 3b_2^2b_3 = 0$$
$$c_4^2 + 2c_3c_5 + 3(1/2 - \sqrt{3}/2i)b_2^2 + 3b_2b_3^2 + a_2^4 = 0$$
$$2c_4c_5 + 2c_3 + b_3^3 + 6(1/2 - \sqrt{3}/2i)b_2b_3 = 0$$
$$c_5^2 + 2c_4 + 3(1/2 - \sqrt{3}/2i)b_3^2 + 3(1/2 - \sqrt{3}/2i)^2b_2 = 0$$
$$2c_5 + 3(1/2 - \sqrt{3}/2i)^2b_3 = 0$$

Then we ask Maple to solve it and the solution that Maple gives is: $\{c5 = 0, b3 = 0, a2 = 0, c4 = 0, c3 = 0, b2 = 0\}$, so that the unique solution is the zero one.

(6) • $\overline{N_6} \not\subset \overline{N_2}$.

In this case we truncate at the order $o_6 = 12$. We argue by contradiction: suppose that $\overline{N_6}(12) \subset \overline{N_2}(12)$. Let ϕ_{62} be a wedge with the generic arc living in N_2 and the special arc mapping to the generic point of $N_6(12)$. The following equations vanish on $N_2(12)$:

$$\mathbf{b}_1 = \mathbf{c}_1 = 0$$
$$g_{2,2} = \mathbf{c}_2 - i\mathbf{a}_1^2 = 0$$
$$\bar{f}_{2,3} = \mathbf{c}_3 - 2i\mathbf{a}_1\mathbf{a}_2 = 0$$
$$f_{2,6} = \mathbf{c}_3^2 + 2\mathbf{c}_2\mathbf{c}_4 + \mathbf{b}_2^3 + 4\mathbf{a}_1^3\mathbf{a}_3 + 6\mathbf{a}_1^2\mathbf{a}_2^2 = 0$$
$$f_{2,7} = 2\mathbf{c}_3\mathbf{c}_4 + 2\mathbf{c}_2\mathbf{c}_5 + 3\mathbf{b}_2^2\mathbf{b}_3 + 4\mathbf{a}_1^3\mathbf{a}_4 + 12\mathbf{a}_1^2\mathbf{a}_2\mathbf{a}_3 + 4\mathbf{a}_2^3\mathbf{a}_1 = 0$$
$$\vdots$$
$$f_{2,11} = 3\mathbf{b}_2^2\mathbf{b}_7 + \cdots + 2\mathbf{c}_2\mathbf{c}_9 + \cdots + 4\mathbf{a}_2^3\mathbf{a}_5 + \cdots = 0;$$

We write it in the following way:

$$\mathbf{b}_1 = \mathbf{c}_1 = 0$$
$$g_{2,2} = \mathbf{c}_2 - i\mathbf{a}_1^2 = 0$$
$$\bar{f}_{2,3} = \mathbf{c}_3 - 2i\mathbf{a}_1\mathbf{a}_2 = 0$$
$$f_{2,6} = \mathbf{b}_2^3 + 2i\mathbf{a}_1^2(\mathbf{c}_4 - i\mathbf{a}_2^2 - 2i\mathbf{a}_1\mathbf{a}_3) = 0$$
$$f_{2,7} = 3\mathbf{b}_2^2\mathbf{b}_3 + 2i\mathbf{a}_1^2(\mathbf{c}_5 - 2i\mathbf{a}_2\mathbf{a}_3 - 2i\mathbf{a}_1\mathbf{a}_4) + 4i\mathbf{a}_1\mathbf{a}_2(\mathbf{c}_4 - i\mathbf{a}_2^2 - 2i\mathbf{a}_1\mathbf{a}_3) = 0$$
$$f_{2,8} = 3\mathbf{b}_2^2\mathbf{b}_4 + 3\mathbf{b}_2\mathbf{b}_3^2 + 2i\mathbf{a}_1^2(\mathbf{c}_6 - i\mathbf{a}_3^2 - 2i\mathbf{a}_2\mathbf{a}_4 - 2i\mathbf{a}_1\mathbf{a}_5)$$
$$+ 4i\mathbf{a}_1\mathbf{a}_2(\mathbf{c}_5 - 2i\mathbf{a}_2\mathbf{a}_3 - 2i\mathbf{a}_1\mathbf{a}_4) + (\mathbf{c}_4 + i\mathbf{a}_2^2 + 2i\mathbf{a}_1\mathbf{a}_3)(\mathbf{c}_4 - i\mathbf{a}_2^2 - 2i\mathbf{a}_1\mathbf{a}_3) = 0$$

$$f_{2,9} = \mathbf{b}_3^3 + 3\mathbf{b}_2^2\mathbf{b}_5 + 6\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4 + 2i\mathbf{a}_1^2(\mathbf{c}_7 - 2i\mathbf{a}_3\mathbf{a}_4 - 2i\mathbf{a}_2\mathbf{a}_5 - 2i\mathbf{a}_1\mathbf{a}_6)$$
$$+ 4i\mathbf{a}_1\mathbf{a}_2(\mathbf{c}_6 - i\mathbf{a}_3^2 - 2i\mathbf{a}_2\mathbf{a}_4 - 2i\mathbf{a}_1\mathbf{a}_5)$$
$$+ (\mathbf{c}_4 + i\mathbf{a}_2^2 + 2i\mathbf{a}_1\mathbf{a}_3)(\mathbf{c}_5 - 2i\mathbf{a}_2\mathbf{a}_3 - 2i\mathbf{a}_1\mathbf{a}_4)$$
$$+ (\mathbf{c}_5 + 2i\mathbf{a}_2\mathbf{a}_3 + 2i\mathbf{a}_1\mathbf{a}_4)(\mathbf{c}_4 - i\mathbf{a}_2^2 - 2i\mathbf{a}_1\mathbf{a}_3) = 0$$

$$f_{2,10} = 3\mathbf{b}_2^2\mathbf{b}_6 + 3\mathbf{b}_3^2\mathbf{b}_4 + 3\mathbf{b}_4^2\mathbf{b}_2 + 6\mathbf{b}_2\mathbf{b}_3\mathbf{b}_5$$
$$+ 2i\mathbf{a}_1^2(\mathbf{c}_8 - i\mathbf{a}_4^2 - 2i\mathbf{a}_3\mathbf{a}_5 - 2i\mathbf{a}_2\mathbf{a}_6 - 2i\mathbf{a}_1\mathbf{a}_7)$$
$$+ 4i\mathbf{a}_1\mathbf{a}_2(\mathbf{c}_7 - 2i\mathbf{a}_3\mathbf{a}_4 - 2i\mathbf{a}_2\mathbf{a}_5 - 2i\mathbf{a}_1\mathbf{a}_6)$$
$$+ (\mathbf{c}_4 + i\mathbf{a}_2^2 + 2i\mathbf{a}_1\mathbf{a}_3)(\mathbf{c}_6 - i\mathbf{a}_3^2 - 2i\mathbf{a}_2\mathbf{a}_4 - 2i\mathbf{a}_1\mathbf{a}_5)$$
$$+ (\mathbf{c}_5 + 2i\mathbf{a}_2\mathbf{a}_3 + 2i\mathbf{a}_1\mathbf{a}_4)(\mathbf{c}_5 - 2i\mathbf{a}_2\mathbf{a}_3 - 2i\mathbf{a}_1\mathbf{a}_4)$$
$$+ (\mathbf{c}_6 + i\mathbf{a}_3^2 + 2i\mathbf{a}_2\mathbf{a}_4 + 2i\mathbf{a}_1\mathbf{a}_5)(\mathbf{c}_4 - i\mathbf{a}_2^2 - 2i\mathbf{a}_1\mathbf{a}_3) = 0$$

$$f_{2,11} = 3\mathbf{b}_2^2\mathbf{b}_7 + 6\mathbf{b}_2\mathbf{b}_3\mathbf{b}_6 + 6\mathbf{b}_2\mathbf{b}_4\mathbf{b}_5 + 3\mathbf{b}_3\mathbf{b}_4^2$$
$$+ 3\mathbf{b}_3^2\mathbf{b}_5 + 2i\mathbf{a}_1^2(\mathbf{c}_9 - 2i\mathbf{a}_4\mathbf{a}_5 - 2i\mathbf{a}_3\mathbf{a}_6 - 2i\mathbf{a}_2\mathbf{a}_7 - 2i\mathbf{a}_1\mathbf{a}_8)$$
$$+ 4i\mathbf{a}_1\mathbf{a}_2(\mathbf{c}_8 - i\mathbf{a}_4^2 - 2i\mathbf{a}_3\mathbf{a}_5 - 2i\mathbf{a}_2\mathbf{a}_6 - 2i\mathbf{a}_1\mathbf{a}_7)$$
$$+ (\mathbf{c}_4 + i\mathbf{a}_2^2 + 2i\mathbf{a}_1\mathbf{a}_3)(\mathbf{c}_7 - 2i\mathbf{a}_3\mathbf{a}_4 - 2i\mathbf{a}_2\mathbf{a}_5 - 2i\mathbf{a}_1\mathbf{a}_6)$$
$$+ (\mathbf{c}_5 + 2i\mathbf{a}_2\mathbf{a}_3 + 2i\mathbf{a}_1\mathbf{a}_4)(\mathbf{c}_6 - i\mathbf{a}_3^2 - 2i\mathbf{a}_2\mathbf{a}_4 - 2i\mathbf{a}_1\mathbf{a}_5)$$
$$+ (\mathbf{c}_6 + i\mathbf{a}_3^2 + 2i\mathbf{a}_2\mathbf{a}_4 + 2i\mathbf{a}_1\mathbf{a}_5)(\mathbf{c}_5 - 2i\mathbf{a}_2\mathbf{a}_3 - 2i\mathbf{a}_1\mathbf{a}_4)$$
$$+ (\mathbf{c}_7 + 2i\mathbf{a}_3\mathbf{a}_4 + 2i\mathbf{a}_2\mathbf{a}_5 + 2i\mathbf{a}_1\mathbf{a}_6)(\mathbf{c}_4 - i\mathbf{a}_2^2 - 2i\mathbf{a}_1\mathbf{a}_3) = 0.$$

The following equations come from the equations of $N_6(12)$:

$$a_{10} = a_{20} = b_{10} = b_{20} = b_{30} = c_{10} = c_{20} = c_{30} = c_{40} = c_{50} = 0.$$

In this case, because of the number of variables, the computation is more difficult than for the other cases. We want to compute or at least bound below the rational numbers

(101) $$\alpha_2' := \frac{\alpha_2}{\alpha_1}$$

(102) $$\beta_2' := \frac{\beta_2}{\alpha_1}$$

(103) $$\beta_3' := \frac{\beta_3}{\alpha_1}$$

(104) $$\gamma_4' := \frac{\gamma_4}{\alpha_1}$$

(105) $$\gamma_5' := \frac{\gamma_5}{\alpha_1}.$$

We use the following dichotomy.

If $\alpha_2 \geqslant \frac{1}{2}\alpha_1$ then we have

$$\beta_2 \geqslant \alpha_1 \tag{106}$$

$$\beta_3 \geqslant \frac{1}{2}\alpha_1 \tag{107}$$

$$\gamma_4 \geqslant \alpha_1 \tag{108}$$

$$\gamma_5 \geqslant \frac{1}{2}\alpha_1. \tag{109}$$

We try to construct a wedge as usual. Replacing s by s^2 in the parametrization of the wedge, we may assume, without loss of generality, that α_1 is even. We deviate from our standard notation (only for the purposes of the case $\alpha_2 \geqslant \frac{1}{2}\alpha_1$), in that we put $a_2 = a_{2,\alpha_1/2}$ $b_2 = b_{2,\alpha_1}$, $b_3 = b_{3,\alpha_1/2}$, $c_4 = c_{4,\alpha_1}$, $c_5 = c_{5,\alpha_1/2}$.

In this case the first equations are of the form

$$b_2^3 + 2ia_1^2(c_4 - ia_2^2 - 2ia_1a_3) = 0$$

$$3b_2^2b_3 + 2ia_1^2(c_5 - 2ia_2a_3) + 4ia_1a_2(c_4 - ia_2^2 - 2ia_1a_3) = 0$$

$$3b_2^2b_4 + 3b_2b_3^2 + 2ia_1^2(c_6 - ia_3^2) + 4ia_1a_2(c_5 - 2ia_2a_3)$$
$$+ (c_4 + ia_2^2 + 2ia_1a_3)(c_4 - ia_2^2 - 2ia_1a_3) = 0$$

$$b_3^3 + 6b_2b_3b_4 + 4ia_1a_2(c_6 - ia_3^2) + (c_4 + ia_2^2 + 2ia_1a_3)(c_5 - 2ia_2a_3)$$
$$+ (c_5 + 2ia_2a_3)(c_4 - ia_2^2 - 2ia_1a_3) = 0$$

$$3b_4^2b_2 + 3b_3^2b_4 + (c_4 + ia_2^2 + 2ia_1a_3)(c_6 - ia_3^2)$$
$$+ (c_5 + 2ia_2a_3)(c_5 - 2ia_2a_3) + (c_6 + ia_3^2)(c_4 - ia_2^2 - 2ia_1a_3) = 0$$

$$3b_3b_4^2 + (c_5 + 2ia_2a_3)(c_6 - ia_3^2) + (c_6 + ia_3^2)(c_5 - 2ia_2a_3) = 0.$$

Thanks to XMaple, taking in this case $b_4 = 0$ and $a_3 = 1$ one can show that the above system of equations, combined with the first equation of N_6,

$$a_3^4 + b_4^3 + c_6^2 = 0,$$

has no non-zero solutions, so the wedge cannot be constructed.

From now on we shall assume that $\alpha_2 < \frac{1}{2}\alpha_1$. One always has $\beta_2 \geqslant \frac{2}{3}\alpha_1$ thanks to the equation $f_{2,6}$.

For each equation, let us write the μ-adic orders of monomials appearing in it, which can possibly be the lowest for this equation:

$$f_{2,6} : 3\beta_2, 2\alpha_1 + \gamma_4, 2\alpha_1 + 2\alpha_2$$

$$f_{2,7} : 2\beta_2 + \beta_3, 2\alpha_1 + \gamma_5, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \gamma_4, \alpha_1 + 3\alpha_2$$

$$f_{2,8} : 2\beta_2, \beta_2 + 2\beta_3, 2\alpha_1, \alpha_1 + \alpha_2 + \gamma_5, \alpha_1 + 2\alpha_2, 2\gamma_4, 4\alpha_2$$
$$f_{2,9} : 3\beta_3, 2\beta_2, \beta_2 + \beta_3, \gamma_5 + \gamma_4, 2\alpha_2 + \gamma_5, \gamma_4 + \alpha_2, 3\alpha_2$$
$$f_{2,10} : \beta_2, 2\beta_3, \gamma_4, 2\alpha_2, 2\gamma_5$$
$$f_{2,11} : \beta_2, \beta_3, \gamma_4, \alpha_2, \gamma_5$$

Note: Here we have used the fact (easy to prove) that the following four expressions have μ-adic value equal to zero: $\mathbf{c}_6 - i\mathbf{a}_3^2 - 2i\mathbf{a}_2\mathbf{a}_4 - 2i\mathbf{a}_1\mathbf{a}_5$, $\mathbf{c}_6 + i\mathbf{a}_3^2 + 2i\mathbf{a}_2\mathbf{a}_4 + 2i\mathbf{a}_1\mathbf{a}_5$, $\mathbf{c}_7 - 2i\mathbf{a}_3\mathbf{a}_4 - 2i\mathbf{a}_2\mathbf{a}_5 - 2i\mathbf{a}_1\mathbf{a}_6$, $\mathbf{c}_7 + 2i\mathbf{a}_3\mathbf{a}_4 + 2i\mathbf{a}_2\mathbf{a}_5 + 2i\mathbf{a}_1\mathbf{a}_6$.

Suppose that $\beta_3 \geqslant \frac{1}{2}\alpha_1 (> \alpha_2)$.

Now, if $\gamma_4 \leqslant \alpha_2$ then from the equation $f_{2,11}$ we see that $\gamma_5 \geqslant \gamma_4$ (otherwise the term with μ-adic value γ_5 would be the only dominant term). But then the term of value γ_4 is the only dominant term in $f_{2,10}$, a contradiction; so

$$\gamma_4 > \alpha_2.$$

Then the μ-adic values of possible dominant terms are:

$$f_{2,6} : 3\beta_2, 2\alpha_1 + \gamma_4, 2\alpha_1 + 2\alpha_2$$
$$f_{2,7} : 2\beta_2 + \beta_3, 2\alpha_1 + \gamma_5, \alpha_1 + \alpha_2 + \gamma_4, \alpha_1 + 3\alpha_2$$
$$f_{2,8} : 2\beta_2, \beta_2 + 2\beta_3, \alpha_1 + \alpha_2 + \gamma_5, 2\gamma_4, 4\alpha_2$$
$$f_{2,9} : 3\beta_3, 2\beta_2, \beta_2 + \beta_3, 3\alpha_2, \gamma_5 + \gamma_4, 2\alpha_2 + \gamma_5, \gamma_4 + \alpha_2$$
$$f_{2,10} : \beta_2, \gamma_4, 2\alpha_2, 2\gamma_5$$
$$f_{2,11} : \alpha_2, \gamma_5.$$

From $f_{2,11}$ we have $\alpha_2 = \gamma_5$.

- First case:

  Suppose that $\gamma_4 > 2\alpha_2$.

  Then the dominant values are:

$$f_{2,6} : 3\beta_2, 2\alpha_1 + 2\alpha_2$$
$$f_{2,7} : 2\beta_2 + \beta_3, \alpha_1 + 3\alpha_2$$
$$f_{2,8} : 2\beta_2, \beta_2 + 2\beta_3, 4\alpha_2$$
$$f_{2,9} : 2\beta_2, \beta_2 + \beta_3, 2\alpha_2 + \gamma_5 = 3\alpha_2$$
$$f_{2,10} : \beta_2, 2\alpha_2, 2\gamma_5$$
$$f_{2,11} : \alpha_2, \gamma_5$$

So by $f_{2,6}$ we have: $3\beta_2 = 2\alpha_1 + 2\alpha_2$.

And by $f_{2,7}$ we have:

$$2\beta_2 + \beta_3 = \alpha_1 + 3\alpha_2.$$

The two equations imply that $\beta_3 < \frac{1}{2}\alpha_1$, a contradiction.

- Second case:

  Suppose that $\gamma_4 < 2\alpha_2$.

  Then the possible dominant values are:

$$\begin{aligned} f_{2,6} &: 3\beta_2, 2\alpha_1 + \gamma_4 \\ f_{2,7} &: 2\beta_2 + \beta_3, \alpha_1 + \gamma_4 + \alpha_2 \\ f_{2,8} &: 2\beta_2, \beta_2 + 2\beta_3, 2\gamma_4 \\ f_{2,9} &: 2\beta_2, \beta_2 + \beta_3, \alpha_2 + \gamma_4 \\ f_{2,10} &: \beta_2, \gamma_4 \\ f_{2,11} &: \alpha_2, \gamma_5 \end{aligned}$$

So by $f_{2,6}$ and $f_{2,10}$ we have: $\beta_2 = \alpha_1 = \gamma_4$, a contradiction (as $\alpha_2 < \frac{1}{2}\alpha_1$).

- Last case: $\gamma_4 = 2\alpha_2$.

  The dominant values are:

$$\begin{aligned} f_{2,6} &: 3\beta_2 = 2\alpha_1 + \gamma_4 = 2\alpha_1 + 2\alpha_2 \\ f_{2,7} &: \alpha_2 + \gamma_4 = \alpha_1 + 3\alpha_2 \\ f_{2,8} &: 2\gamma_4 = 4\alpha_2 \\ f_{2,9} &: \gamma_5 + \gamma_4 = 2\alpha_2 + \gamma_5 = \gamma_4 + \alpha_2 = 3\alpha_2 \\ f_{2,10} &: \gamma_4 = 2\alpha_2 = 2\gamma_5 \\ f_{2,11} &: \alpha_2 = \gamma_5 \end{aligned}$$

Then the first equations of the wedge are:

$$\bar{f}_{2,6} = b_2^3 + 2ia_1^2(c_4 - ia_2^2) = 0$$
$$\bar{f}_{2,7} = 4ia_1a_2(c_4 - ia_2^2) = 0$$
$$\bar{f}_{2,8} = (c_4 + ia_2^2)(c_4 - ia_2^2) = 0$$
$$\bar{f}_{2,9} = (c_4 + ia_2^2)(c_5 - 2ia_2a_3) + (c_5 + 2ia_2a_3)(c_4 - ia_2^2) = 0$$
$$\bar{f}_{2,10} = (c_4 + ia_2^2)(c_6 - ia_3^2) + (c_5 + 2ia_2a_3)(c_5 - 2ia_2a_3) + (c_6 + ia_3^2)(c_4 - ia_2^2) = 0$$
$$\bar{f}_{2,11} = (c_5 + 2ia_2a_3)(c_6 - ia_3^2) + (c_6 + ia_3^2)(c_5 - 2ia_2a_3) = 0.$$
$$f_{6,12} = a_3^4 + b_4^3 + c_6^2 = 0$$

By definitions, we are looking for solutions with a_1, a_2, b_2 different from 0. It is easy to see that this is impossible already from the equations $\bar{f}_{2,6}$ and $\bar{f}_{2,7}$. Indeed, since $a_1, a_2 \neq 0$ we have $c_4 - ia_2^2 = 0$ by $\bar{f}_{2,7}$. Then $\bar{f}_{2,6}$ shows that $b_2 = 0$. This completes the proof of the non-existence of the wedge in the case $\beta_3 \geqslant \frac{1}{2}\alpha_1$.

Thus we will assume from now on that $\beta_3 < \frac{1}{2}\alpha_1$. Then $\beta_2 > \beta_3$ and the possible dominant values are:

$$f_{2,6} : 3\beta_2, 2\alpha_1 + \gamma_4, 2\alpha_1 + 2\alpha_2$$
$$f_{2,7} : 2\beta_2 + \beta_3, 2\alpha_1 + \gamma_5, \alpha_1 + \alpha_2 + \gamma_4, \alpha_1 + 3\alpha_2$$
$$f_{2,8} : 2\beta_2, \beta_2 + 2\beta_3, \alpha_1 + \alpha_2 + \gamma_5, 2\gamma_4, 4\alpha_2$$
$$f_{2,9} : 3\beta_3, \beta_2 + \beta_3, \gamma_5 + \gamma_4, 2\alpha_2 + \gamma_5, \gamma_4 + \alpha_2, 3\alpha_2$$
$$f_{2,10} : \beta_2, 2\beta_3, \gamma_4, 2\alpha_2, 2\gamma_5$$
$$f_{2,11} : \beta_3, \gamma_4, \alpha_2, \gamma_5$$

- Suppose that $\gamma_5 \geqslant \frac{1}{2}\alpha_1$.
  - If

    (110) $$\gamma_4 \leqslant \alpha_2$$

    then $\beta_3 \leqslant \alpha_2$ by $f_{2,11}$. Hence γ_4 becomes the only dominant value in $f_{2,10}$ which is not possible. Thus $\gamma_4 > \alpha_2$, which implies that $\beta_3 = \alpha_2$ by $f_{2,11}$.
  - If

    (111) $$\gamma_4 < 2\alpha_2,$$

    then $\beta_2 = \gamma_4$ by $f_{2,10}$ and hence $\beta_2 = \alpha_1$ by $f_{2,6}$. Thus $\gamma_4 = \alpha_1$, which contradicts (111) and the fact that $\alpha_2 < \frac{1}{2}\alpha_1$.
  - If $\gamma_4 > 2\alpha_2$ then $\beta_2 = 2\alpha_2$ by $f_{2,10}$ and hence $\alpha_2 = \frac{2}{5}\alpha_1$, $\beta_2 = \frac{4}{5}\alpha_1$ by $f_{2,6}$. Using the fact that $\gamma_5 > 0$, we see that in $f_{2,7}$, $2\beta_2 + \alpha_2$ is the only dominant value, a contradiction.
  - The remaining case is $\gamma_4 = 2\alpha_2$. The dominant values are:

$$f_{2,6} : 3\beta_2 \geqslant 2\alpha_1 + \gamma_4 = 2\alpha_1 + 2\alpha_2$$
$$f_{2,7} : 2\beta_2 + \beta_3 \geqslant \alpha_1 + \alpha_2 + \gamma_4 = \alpha_1 + 3\alpha_2$$
$$f_{2,8} : 2\beta_2, \beta_2 + 2\beta_3, 2\gamma_4 = 4\alpha_2$$
$$f_{2,9} : 3\beta_3, \beta_2 + \beta_3, \gamma_4 + \alpha_2 = 3\alpha_2$$
$$f_{2,10} : \beta_2 \geqslant 2\beta_3 = \gamma_4 = 2\alpha_2$$
$$f_{2,11} : \beta_3 = \alpha_2$$

    If $\beta_2 \leqslant 2\alpha_2$ then by $f_{2,6}$ we would have $\beta_2 \geqslant \alpha_1$ and hence $\alpha_2 \geqslant \frac{1}{2}\alpha_1$, a contradiction. Thus $\beta_2 > 2\alpha_2$.

CLAIM. *The only dominant values in $f_{2,7}$ are*

$$\alpha_1 + \alpha_2 + \gamma_4 = \alpha_1 + 3\alpha_2$$

Proof of Claim. If not, we would have

$$2\beta_2 + \beta_3 = 2\beta_2 + \alpha_2 = \alpha_1 + 3\alpha_2$$

Then $2\beta_2 = \alpha_1 + 2\alpha_2 \leqslant 3\beta_2 - \alpha_1$ (by $f_{2,6}$), thus $\beta_2 \geqslant \alpha_1$.

We obtain $2\beta_2 + \beta_3 \geqslant 2\alpha_1 + \beta_3 = 2\alpha_1 + \alpha_2 > \alpha_1 + 3\alpha_2$ a contradiction. This proves the Claim.

Then the first two equations of the wedge are

$$\bar{f}_{2,6} = b_2^3 + 2ia_1^2(c_4 - ia_2^2) = 0$$
$$\bar{f}_{2,7} = 4ia_1a_2(c_4 - ia_2^2) = 0$$

so there are no solutions with $b_2 \neq 0$, $a_2 \neq 0$, $a_1 \neq 0$, contradiction.

• Thus $\gamma_5 < \frac{1}{2}\alpha_1$.

First of all, we claim that γ_4 cannot be dominant in $f_{2,11}$. Indeed, suppose it was, in other words, suppose that $\gamma_4 \leqslant \min\{\beta_3, \alpha_2, \gamma_5\}$. In particular,

(112) $$\gamma_4 < \frac{1}{2}\alpha_1.$$

Then by $f_{2,10}$ we have

(113) $$\beta_2 = \gamma_4.$$

But then by $f_{2,6}$ we have

(114) $$\beta_2 = \alpha_1,$$

which contradicts (112) and (114). This proves that

(115) $$\gamma_4 > \min\{\beta_3, \alpha_2, \gamma_5\}.$$

We continue to study the possible dominant values in $f_{2,11}$. There are two cases to consider.

– First case: $\gamma_5 = \beta_3 < \alpha_2$.

The possible dominant values are:

$$f_{2,6} : 3\beta_2, 2\alpha_1 + \gamma_4, 2\alpha_1 + 2\alpha_2$$
$$f_{2,7} : 2\beta_2 + \beta_3, 2\alpha_1 + \gamma_5, \alpha_1 + \alpha_2 + \gamma_4, \alpha_1 + 3\alpha_2$$
$$f_{2,8} : 2\beta_2, \beta_2 + 2\beta_3, \alpha_1 + \alpha_2 + \gamma_5, 2\gamma_4, 4\alpha_2$$
$$f_{2,9} : 3\beta_3, \beta_2 + \beta_3, \gamma_5 + \gamma_4, 2\alpha_2 + \gamma_5$$
$$f_{2,10} : \beta_2, 2\beta_3, \gamma_4, 2\gamma_5$$
$$f_{2,11} : \beta_3, \gamma_5$$

1) If $\gamma_4 \leqslant 2\gamma_5$ then $\gamma_4 < 2\alpha_2 < \alpha_1$,

$$3\beta_2 = 2\alpha_1 + \gamma_4 \tag{116}$$

by $f_{2,6}$. Hence

$$\beta_2 < \alpha_1, \tag{117}$$

so

$$2\beta_2 + \beta_3 = \alpha_1 + \alpha_2 + \gamma_4$$

by $f_{2,7}$. Thus

$$\beta_3 + \alpha_1 = \alpha_2 + \beta_2. \tag{118}$$

By (117) and (116) we have

$$\gamma_4 = 3\beta_2 - 2\alpha_1 < \beta_2. \tag{119}$$

Then $3\beta_3 = \gamma_4 + \gamma_5$ (that is, $\gamma_4 = 2\beta_3$) by $f_{2,9}$ and by $f_{2,8}$ we obtain

$$\beta_2 = \gamma_4,$$

contradicting (119).

2) Thus $\gamma_4 > 2\gamma_5$.

We have $\beta_2 > 2\gamma_5$, because otherwise $3\beta_2$ would be the only dominant value in $f_{2,6}$. Then the unique dominant value in $f_{2,9}$ is $3\beta_3$, a contradiction. This completes the proof in the first case.

– Second case: Thus $\alpha_2 \leqslant \gamma_5$ and $\alpha_2 \leqslant \beta_3$.

Then $\gamma_4 > \alpha_2$ by (115).

So the possible dominant values are:

$$f_{2,6} : 3\beta_2, 2\alpha_1 + \gamma_4, 2\alpha_1 + 2\alpha_2$$
$$f_{2,7} : 2\beta_2 + \beta_3, \alpha_1 + \alpha_2 + \gamma_4, \alpha_1 + 3\alpha_2$$
$$f_{2,8} : 2\beta_2, \beta_2 + 2\beta_3, 2\gamma_4, 4\alpha_2$$
$$f_{2,9} : 3\beta_3, \beta_2 + \beta_3, \gamma_5 + \gamma_4, 2\alpha_2 + \gamma_5, \gamma_4 + \alpha_2, 3\alpha_2$$
$$f_{2,10} : \beta_2, 2\beta_3, \gamma_4, 2\alpha_2, 2\gamma_5$$
$$f_{2,11} : \beta_3, \alpha_2, \gamma_5$$

1) If

$$\gamma_4 < 2\alpha_2 \tag{120}$$

then

$$\gamma_4 = \beta_2 \tag{121}$$

by $f_{2,10}$. From $f_{2,6}$ we obtain the equality (116), which implies

$$\beta_2 = \alpha_1, \tag{122}$$

which contradicts (120) and (121).

2) Suppose $\gamma_4 > 2\alpha_2$. By looking at the dominant terms of $f_{2,6}$ and $f_{2,7}$ we obtain again the equality (118). If $\beta_2 \leqslant 2\alpha_2$ then by $f_{2,6}$ we would have $\alpha_2 \geqslant \frac{1}{2}\alpha_1$, which is false. Hence $\beta_2 > 2\alpha_2$. Then by $f_{2,6}$ we have $\beta_2 < \alpha_1$ and now (118) implies $\beta_3 > \alpha_2$. Then the only possible dominant value in $f_{2,8}$ is $4\alpha_2$, a contradiction.

3) So $\gamma_4 = 2\alpha_2$.

Then

$$\beta_2 > 2\alpha_2 \tag{123}$$

(if not $3\beta_2$ would be the only dominant value in $f_{2,6}$). Using $f_{2,6}$ and $f_{2,7}$ we see that

$$\begin{aligned} 2\beta_2 + \beta_3 &\geqslant \frac{4}{3}(\alpha_1 + \alpha_2) + \alpha_2 \\ &= \alpha_1 + 2\alpha_2 + \frac{1}{3}(\alpha_1 + \alpha_2) > \alpha_1 + 3\alpha_2. \end{aligned} \tag{124}$$

Let us do another trichotomy:

– A) Suppose $\alpha_2 = \gamma_5 < \beta_3$

The possible dominant values are:

$$\begin{aligned} f_{2,6} &: 3\beta_2 \geqslant 2\alpha_1 + \gamma_4 = 2\alpha_1 + 2\alpha_2 \\ f_{2,7} &: \alpha_1 + \alpha_2 + \gamma_4 = \alpha_1 + 3\alpha_2 \\ f_{2,8} &: 2\gamma_4 = 4\alpha_2 \\ f_{2,9} &: \gamma_5 + \gamma_4 = 2\alpha_2 + \gamma_5 = \gamma_4 + \alpha_2 = 3\alpha_2 \\ f_{2,10} &: \gamma_4 = 2\alpha_2 = 2\gamma_5 \\ f_{2,11} &: \alpha_2 = \gamma_5 \end{aligned}$$

If $3\beta_2 > 2\alpha_1 + 2\alpha_2$ then the first equations of any wedge with $b_4 \neq 0$ are:

$$\begin{aligned} \bar{f}_{2,6} &= 2ia_1^2(c_4 - ia_2^2) = 0 \\ \bar{f}_{2,7} &= 4ia_1a_2(c_4 - ia_2^2) = 0 \\ \bar{f}_{2,8} &= (c_4 + ia_2^2)(c_4 - ia_2^2) = 0 \\ \bar{f}_{2,9} &= (c_4 + ia_2^2)(c_5 - 2ia_2a_3) + (c_5 + 2ia_2a_3)(c_4 - ia_2^2) = 0 \\ \bar{f}_{2,10} &= (c_4 + ia_2^2)(c_6 - ia_3^2) + (c_5 + 2ia_2a_3)(c_5 - 2ia_2a_3) + (c_6 + ia_3^2)(c_4 - ia_2^2) = 0 \\ \bar{f}_{2,11} &= (c_5 + 2ia_2a_3)(c_6 - ia_3^2) + (c_6 + ia_3^2)(c_5 - 2ia_2a_3) = 0. \\ f_{6,12} &= a_3^4 + b_4^3 + c_6^2 = 0. \end{aligned}$$

Thus $c_4 - ia_2^2 = 0$ and the last four equations become:

$$\bar{f}_{2,9} = (c_4 + ia_2^2)(c_5 - 2ia_2a_3) = 0$$

$$\bar{f}_{2,10} = (c_4 + ia_2^2)(c_6 - ia_3^2) + (c_5 + 2ia_2a_3)(c_5 - 2ia_2a_3) = 0$$

$$\bar{f}_{2,11} = (c_5 + 2ia_2a_3)(c_6 - ia_3^2) + (c_6 + ia_3^2)(c_5 - 2ia_2a_3) = 0.$$

$$f_{6,12} = a_3^4 + b_4^3 + c_6^2 = 0$$

Since $b_4 \neq 0$, we have $c_6 - ia_3^2 \neq 0$ and $c_6 - ia_3^2 \neq 0$. As well, $c_5 \neq 0$, $a_2 \neq 0$, $c_4 \neq 0$ which is incompatible with the above equations.

Thus $3\beta_2 = 2\alpha_1 + 2\alpha_2$. The first equations of the wedge are:

$$\bar{f}_{2,6} = b_2^3 + 2ia_1^2(c_4 - ia_2^2) = 0$$

$$\bar{f}_{2,7} = 4ia_1a_2(c_4 - ia_2^2) = 0$$

Thus as $a_1 \neq 0$ and $a_2 \neq 0$, we have $c_4 - ia_2^2 = 0$ and then $b_2 = 0$ (not allowed by definition).

Thus the case A) is impossible and $\alpha_2 = \beta_3$.

– B) $\alpha_2 = \beta_3 < \gamma_5$.

Using (123) and (124), we see that the possible dominant values are:

$$f_{2,6} : 3\beta_2 \geqslant 2\alpha_1 + \gamma_4 = 2\alpha_1 + 2\alpha_2$$

$$f_{2,7} : \alpha_1 + \alpha_2 + \gamma_4 = \alpha_1 + 3\alpha_2$$

$$f_{2,8} : 2\gamma_4 = 4\alpha_2$$

$$f_{2,9} : 3\beta_3 = \gamma_4 + \alpha_2 = 3\alpha_2$$

$$f_{2,10} : \gamma_4 = 2\alpha_2 = 2\beta_3$$

$$f_{2,11} : \alpha_2 = \beta_3$$

Suppose that $3\beta_2 = 2\alpha_1 + \gamma_4 = 2\alpha_1 + 2\alpha_2$, then the first equations of the wedge are:

$$\bar{f}_{2,6} = b_2^3 + 2ia_1^2(c_4 - ia_2^2) = 0 \tag{125}$$

$$\bar{f}_{2,7} = 4ia_1a_2(c_4 - ia_2^2) = 0 \tag{126}$$

Since $a_1 \neq 0$ and $a_2 \neq 0$, we obtain $c_4 - ia_2^2 = b_2 = 0$, which gives the desired contradiction.

Therefore $3\beta_2 > 2\alpha_1 + \gamma_4 = 2\alpha_1 + 2\alpha_2$. The first equations are:

$$\bar{f}_{2,6} = 2ia_1^2(c_4 - ia_2^2) = 0$$
$$\bar{f}_{2,7} = 4ia_1a_2(c_4 - ia_2^2) = 0$$
$$\bar{f}_{2,8} = (c_4 + ia_2^2)(c_4 - ia_2^2) = 0$$
$$\bar{f}_{2,9} = b_3^3 + 4a_2^3a_3 = 0$$
$$\bar{f}_{2,10} = (c_4 + ia_2^2)(c_6 - ia_3^2) + (c_4 - ia_2^2)(c_6 + ia_3^2) + 3b_3^2b_4 = 0$$
$$\bar{f}_{2,11} = 4a_2a_3^3 + 3b_3b_4^2 = 0.$$
$$f_{6,12} = a_3^4 + b_4^3 + c_6^2 = 0$$

First we see that $c_4 - ia_2^2 = 0$ and the equations become:

$$c_4 - ia_2^2 = 0$$
$$\bar{f}_{2,9} = b_3^3 + 4a_2^3a_3 = 0$$
$$\bar{f}_{2,10} = 2ia_2^2(c_6 - ia_3^2) + 3b_3^2b_4 = 0$$
$$\bar{f}_{2,11} = 4a_2a_3^3 + 3b_3b_4^2 = 0.$$
$$f_{6,12} = a_3^4 + b_4^3 + c_6^2 = 0$$

By XMaple (one can also do it by hand, as in $f_{2,11}$ the equation is linear in b_3 and a_2), these equations imply that $b_3 = 0$, which is not allowed by definition of b_3. Thus case B) is also impossible and the only remaining case to consider is

– C)

$$\alpha_2 = \beta_3 = \gamma_5 < \frac{1}{2}\alpha_1, \quad \frac{2}{3}\alpha_1 \leqslant \beta_2 \quad \text{and} \quad \gamma_4 = 2\alpha_2.$$

Using (123) and (124), we see that the possible dominant values are:

$$f_{2,6} : 3\beta_2 \geqslant 2\alpha_1 + \gamma_4 = 2\alpha_1 + 2\alpha_2$$
$$f_{2,7} : \alpha_1 + \alpha_2 + \gamma_4 = \alpha_1 + 3\alpha_2$$
$$f_{2,8} : 2\gamma_4 = 4\alpha_2$$
$$f_{2,9} : 3\beta_3 = \gamma_4 + \alpha_2 = 3\alpha_2 = \gamma_4 + \gamma_5 = 2\alpha_2 + \gamma_5$$
$$f_{2,10} : \gamma_4 = 2\alpha_2 = 2\beta_3 = 2\gamma_5$$
$$f_{2,11} : \alpha_2 = \beta_3 = \gamma_5$$

If $3\beta_2 = 2\alpha_1 + 2\alpha_2$ then the first two equations of the wedge are (125) and (126). We obtain the same contradiction as before: since $a_1 \neq 0$

and $a_2 \neq 0$, we have $c_4 - ia_2^2 = 0$ and then $b_2 = 0$ (not allowed by definition).

Finally, it remains to solve the case when

$$3\beta_2 > 2\alpha_1 + 2\alpha_2.$$

The equations of the wedge are:

$$\begin{aligned}
\bar{f}_{2,6} &= 2ia_1^2(c_4 - ia_2^2) = 0\\
\bar{f}_{2,7} &= 4ia_1a_2(c_4 - ia_2^2) = 0\\
\bar{f}_{2,8} &= (c_4 + ia_2^2)(c_4 - ia_2^2) = 0\\
\bar{f}_{2,9} &= (c_4 + ia_2^2)(c_5 - 2ia_2a_3) + b_3^3 = 0\\
\bar{f}_{2,10} &= (c_4 + ia_2^2)(c_6 - ia_3^2) + (c_5 + 2ia_2a_3)(c_5 - 2ia_2a_3) + 3b_3^2b_4 = 0\\
\bar{f}_{2,11} &= (c_5 + 2ia_2a_3)(c_6 - ia_3^2) + (c_6 + ia_3^2)(c_5 - 2ia_2a_3) + 3b_3b_4^2 = 0.\\
f_{6,12} &= a_3^4 + b_4^3 + c_6^2 = 0
\end{aligned}$$

As by definition $a_1 \neq 0$, we have $c_4 - ia_2^2 = 0$ and the last four equations become:

$$\begin{aligned}
\bar{f}_{2,9} &= 2ia_2^2(c_5 - 2ia_2a_3) + b_3^3 = 0\\
\bar{f}_{2,10} &= 2ia_2^2(c_6 - ia_3^2) + (c_5 + 2ia_2a_3)(c_5 - 2ia_2a_3) + 3b_3^2b_4 = 0\\
\bar{f}_{2,11} &= (c_5 + 2ia_2a_3)(c_6 - ia_3^2) + (c_6 + ia_3^2)(c_5 - 2ia_2a_3) + 3b_3b_4^2 = 0.\\
f_{6,12} &= a_3^4 + b_4^3 + c_6^2 = 0
\end{aligned}$$

By XMaple, one obtains $b_3 = 0$, which is not allowed by definition of b_3. So in this last case one cannot construct the wedge either. This completes the proof of the last non-inclusion. □

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