Y.-Z. LI AND L. ZHANG
KODAI MATH. J.
35 (2012), 157–172

AN EMBEDDING THEOREM ON REDUCING SUBSPACE FRAME MULTIRESOLUTION ANALYSIS*

Yun-Zhang Li and Lin Zhang

Abstract

This paper addresses **FMRA** and **MRA** in the setting of reducing subspaces of $L^2(\mathbf{R}^d)$. We prove that an **FMRA** must be contained in some **MRA**, and obtain a sufficient condition for an **MRA** to contain no **FMRA** other than itself.

1. Introduction

An at most countable sequence $\{f_i\}_{i\in I}$ in a Hilbert space $\mathscr{H}$ is called a *frame* for $\mathscr{H}$ if there exist $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2 \tag{1.1}$$

for $f \in \mathscr{H}$, where A, B are called lower frame bound and upper frame bound, respectively. In particular, $\{f_i\}_{i\in I}$ is called a *tight frame* for $\mathscr{H}$ if $A = B$ in (1.1); and called a *Parseval frame* if $A = B = 1$ in (1.1). A frame for $\mathscr{H}$ is called a *Riesz basis* if it ceases to be a frame for $\mathscr{H}$ whenever an arbitrary element is removed. The fundamentals of frames can be found in [8], [12] and [23].

We denote by $\mathbf{Z}$, $\mathbf{Z}_+$, $\mathbf{N}$ the set of integers, the set of nonnegative integers and the set of positive integers, respectively. Given $d \in \mathbf{N}$, we denote by $\mathbf{T}^d = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ the d-dimensional torus, and by M^t the transpose of M for a $d \times d$ real matrix M. Throughout this paper, "measurable" always means "Lebesgue measurable", relations between two measurable sets in $\mathbf{R}^d$ such as equality, disjointness or inclusion, are always understood up to a set of measure zero, and similarly, equality or inequality between measurable functions is always understood in the "almost-everywhere" sense. For an arbitrary measurable set E in $\mathbf{R}^d$, we denote by E^P the $\mathbf{Z}^d$-periodization set of E: $E^P = \bigcup_{k\in\mathbf{Z}^d}(E + k)$,

---

2010 *Mathematics Subject Classification*. 42C40.

Key words and phrases. frame, refinable function, frame multiresolution analysis, reducing subspace.

* Supported by Beijing Natural Science Foundation (Grant No. 1122008), the Scientific Research Common Program of Beijing Municipal Commission of Education (Grant No. KM201110005030).

Received February 8, 2011; revised July 12, 2011.

by $|E|$ the measure of E, and by χ_E the characteristic function of E. For a measurable function f on $\mathbf{R}^d$, we define its support by

$$\operatorname{supp}(f) = \{x \in \mathbf{R}^d : f(x) \neq 0\}.$$

Then it is well-defined up to a set of measure zero. The Fourier transform of $f \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ is defined by

$$\hat{f}(\cdot) := \int_{\mathbf{R}^d} f(x) e^{-2\pi i \langle x, \cdot \rangle}\, dx$$

and extended to $L^2(\mathbf{R}^d)$ by the Plancheral theorem, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbf{R}^d$. For $f \in L^2(\mathbf{R}^d)$, we define its spectrum $\sigma(f)$ by

$$\sigma(f) = \left\{ x \in \mathbf{R}^d : \sum_{k \in \mathbf{Z}^d} |\hat{f}(\cdot + k)|^2 \neq 0 \right\},$$

and for a.e. $x \in \mathbf{R}^d$, define the vector $\hat{f}_{\|x}$ by

$$\hat{f}_{\|x} := (\hat{f}(x + k))_{k \in \mathbf{Z}^d}.$$

Given a measurable set G in $\mathbf{R}^d$, an at most countable collection $\{G_i : i \in I\}$ of measurable sets is called a *partition* of G if $G = \bigcup_{i \in I} G_i$, and $G_i \cap G_{i'} = \emptyset$ for $i, i' \in I$ with $i \neq i'$. Two measurable sets G and H in $\mathbf{R}^d$ are said to be $\mathbf{Z}^d$-*congruent* if there exists a partition $\{G_j : j \in \mathbf{Z}^d\}$ of G such that $\{G_j + j : j \in \mathbf{Z}^d\}$ is a partition of H. A $d \times d$ matrix M is called an *expansive matrix* if it is an integer matrix, and all its eigenvalues are greater than 1 in modulus. Let M be a $d \times d$ expansive matrix. Define the *dilation operator* D and *translation operator* T_k with $k \in \mathbf{Z}^d$ on $L^2(\mathbf{R}^d)$ by

$$Df(\cdot) := |\det M|^{1/2} f(M \cdot) \quad \text{and} \quad T_k f(\cdot) := f(\cdot - k)$$

for $f \in L^2(\mathbf{R}^d)$, respectively. Then they are both unitary operators. A measurable set S in $\mathbf{R}^d$ is called an *M-admissible scaling set* if S is $\mathbf{Z}^d$-congruent to $\mathbf{T}^d$, and $S \subset M^t S$. Given $f \in L^2(\mathbf{R}^d)$, we define

$$V_j(f) = \overline{\operatorname{span}}\{D^j T_k f : k \in \mathbf{Z}^d\} \tag{1.2}$$

for $j \in \mathbf{Z}$. f is called an *M-refinable function* if there exists a $\mathbf{Z}^d$-periodic measurable function m_f such that $\hat{f}(M^t \cdot) = m_f(\cdot)\hat{f}(\cdot)$, where m_f is called a symbol of f. Without specification, we always denote by m_f its symbol for an arbitrary M-refinable function f. f is called a *frame function/Parseval frame function/Riesz basis function/an orthonormal basis function* if $\{T_k f : k \in \mathbf{Z}^d\}$ is a frame/Parseval frame/Riesz basis/an orthonormal basis for $V_0(f)$; called an *M-refinable frame function/Parseval frame function/Riesz basis function/orthonormal basis function* if it is M-refinable and a frame function/Parseval frame function/Riesz basis function/an orthonormal basis function. A nonzero closed linear subspace X of $L^2(\mathbf{R}^d)$ is called a *reducing subspace* if $DX = X$ and

$T_k X = X$ for each $k \in \mathbf{Z}^d$. The following proposition provides us with a characterization of reducing subspace:

PROPOSITION 1.1 ([9, Theorem 1]). *Let M be a $d \times d$ expansive matrix. A nonzero closed subspace X of $L^2(\mathbf{R}^d)$ is a reducing subspace if and only if $X = FL^2(\Omega)$ for some measurable set Ω in $\mathbf{R}^d$ with the property $\Omega = M^t\Omega$, where*

$$FL^2(\Omega) := \{f \in L^2(\mathbf{R}^d) : \mathrm{supp}(\hat{f}) \subset \Omega\}.$$

By Proposition 1.1, a nonzero reducing subspace X of $L^2(\mathbf{R}^d)$ corresponds a set $\Omega \subset \mathbf{R}^d$ with nonzero measure for which

$$M^t\Omega = \Omega \quad \text{and} \quad X = FL^2(\Omega).$$

So, to be specific, we denote a reducing subspace by $FL^2(\Omega)$ instead of X. In particular, $L^2(\mathbf{R}^d)$ is a reducing subspace of $L^2(\mathbf{R}^d)$ for an arbitrary expansive matrix M, and $FL^2([0, \infty))$ (Hardy space) is also a reducing subspace of $L^2(\mathbf{R})$ for an arbitrary $2 \leq M \in \mathbf{N}$.

DEFINITION 1.1. Let M be a $d \times d$ expansive matrix, and let $FL^2(\Omega)$ be a reducing subspace of $L^2(\mathbf{R}^d)$. A sequence $\{V_j\}_{j \in \mathbf{Z}}$ of closed subspaces of $FL^2(\Omega)$ is called a frame multiresolution analysis (**FMRA**)/multiresolution analysis (**MRA**) associated with M for $FL^2(\Omega)$ if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for $j \in \mathbf{Z}$;
(ii) $\overline{\bigcup_{j \in \mathbf{Z}} V_j} = FL^2(\Omega)$ and $\bigcap_{j \in \mathbf{Z}} V_j = \{0\}$;
(iii) $V_j = D^j V_0$ for $j \in \mathbf{Z}$;
(iv) there exists $\varphi \in FL^2(\Omega)$ such that $\{T_k\varphi : k \in \mathbf{Z}^d\}$ is a frame/Reisz basis or an orthonormal basis for V_0.

This definition is a natural generalization of the ones in the setting of $L^2(\mathbf{R}^d)$. Similarly, we call φ a *scaling function* of the **FMRA**/**MRA**. From the definition, we know that φ is an M-refinable frame function/Reisz basis function or an orthonormal basis function satisfying

$$V_j = V_j(\varphi) \quad \text{for } j \in \mathbf{Z}. \tag{1.3}$$

So, we also say that φ generates the **FMRA**/**MRA**. We denote by W_j the orthogonal complement of V_j in V_{j+1} for $j \in \mathbf{Z}$. Then $W_j = D^j W_0$ for $j \in \mathbf{Z}$. A finite set $\Psi = \{\psi_1, \psi_2, \ldots, \psi_n\} \subset FL^2(\Omega)$ is called a *frame multiwavelet/multiwavelet* for $FL^2(\Omega)$ if $\{D^j T_k \psi_l : 1 \leq l \leq n, j \in \mathbf{Z}, k \in \mathbf{Z}^d\}$ is a frame/an orthonormal basis for $FL^2(\Omega)$; called an **FMRA** *frame multiwavelet* for $FL^2(\Omega)$ if it is a frame multiwavelet for $FL^2(\Omega)$, and $\Psi \subset W_0$. In particular, when Ψ consists of a single element ψ, we say that ψ is a *frame wavelet/wavelet/an* **FMRA** *frame wavelet* for $FL^2(\Omega)$, respectively.

Early study of frame wavelet in $FL^2([0, \infty))$ can be found in [14], [19] and [21], where $M = 2$. For a general reducing subspace $FL^2(\Omega)$ with a general expansive matrix M, Dai, Diao, Gu and D. Han in [9], [10] and [11] characterized the sets $E \subset \mathbf{R}^d$ with χ_E being a frame wavelet; B. Han in [13] discussed dual frame multiwavelets in $FL^2(\Omega)$. Ron and Shen in [18] investigated the construction of frame wavelets for $L^2(\mathbf{R}^d)$ in a general **MRA** setting. Benedetto and Li in [1] obtained an explicit expression of **FMRA** frame wavelets for $L^2(\mathbf{R})$ when $M = 2$. Motivated by the above works, under the setting of a general reducing subspace of $L^2(\mathbf{R}^d)$, Li and Zhou in [16] and [24] investigated unitary extension principles for the construction of frame multiwavelets and the construction of **FMRA** frame wavelets. In particular, when $M = 2$, Kim, Kim and Lim in [15] proved that an arbitrary **FMRA** for $L^2(\mathbf{R})$ must be contained in some **MRA**, and obtained a sufficient condition for an **MRA** to contain no **FMRA** other than itself. It is natural to ask that whether similar results hold under the setting of a general reducing subspace $FL^2(\Omega)$ of $L^2(\mathbf{R}^d)$. This problem is not trivial since the geometry of such general Ω and M^t acting on vectors in $\mathbf{R}^d$ can be quite complex. The present paper gives an affirmative answer to this problem.

Section 2 is an auxiliary one to Section 3. In this section, we prove that, for an arbitrary measurable set Ω in $\mathbf{R}^d$ with non-empty interior and $M^t\Omega = \Omega$, there exists a set $S \subset \Omega$ such that S is an M-admissible scaling set. Under the setting of a general reducing subspace $FL^2(\Omega)$ of $L^2(\mathbf{R}^d)$, we prove in Section 3 that an arbitrary **FMRA** must be contained in some **MRA** if Ω has non-empty interior, and obtain in Section 4 a sufficient condition for an **MRA** to contain no **FMRA** other than itself.

2. The existence of admissible scaling sets

This section is an auxiliary one to Section 3. We begin with the following notion of quasi-norm:

Definition 2.1. Given a $d \times d$ expansive matrix M, a nonnegative function ρ defined on $\mathbf{R}^d$ is called a quasi-norm associated with M if the following conditions hold:

(i) $\rho(x) = 0$ if and only if $x = 0$;

(ii) there exists a constant c_1 such that $\rho(x + y) \le c_1(\rho(x) + \rho(y))$ for $x, y \in \mathbf{R}^d$;

(iii) ρ is continuous on $\mathbf{R}^d$ and smooth on $\mathbf{R}^d \backslash \{0\}$;

(iv) there exists a constant $\delta > 1$ such that $\rho(M^t \cdot) = \delta\rho(\cdot)$;

(v) there exist positive constants c, α_1, α_2, β_1, β_2 such that

$$c^{-1}|x|^{\alpha_1} \le \rho(x) \le c|x|^{\beta_1} \quad \text{when } |x| \le 1,$$

$$c^{-1}|x|^{\alpha_2} \le \rho(x) \le c|x|^{\beta_2} \quad \text{when } |x| \ge 1,$$

where $|x|$ denotes the Euclidean norm of x.

The existence of such a quasi-norm can be found in [20]. It is easy to check that, for $\alpha > 0$, ρ^α is also a quasi-norm if ρ is a quasi-norm.

LEMMA 2.1. *Given a $d \times d$ expansive matrix M, let Ω be a measurable set in $\mathbf{R}^d$ having non-empty interior, and let $\Omega = M^t\Omega$. Then there exist $x_0 \in \mathbf{R}^d$ and $E \subset \Omega$ with $|E| > 0$ such that $\mathbf{T}^d + x_0 \subset \Omega$, and that*

$$E \subset \mathbf{T}^d \cap M^t E, \quad \Omega = \bigcup_{m=1}^{\infty} (M^t)^m E. \tag{2.1}$$

Proof. We use the notations in Definition 2.1. By the assumption that Ω has non-empty interior, there exist $y_0 \in \mathbf{R}^d$ and $\varepsilon_0 > 0$ such that

$$\{x \in \mathbf{R}^d : |x| < \varepsilon_0\} + y_0 \subset \Omega. \tag{2.2}$$

Since M is an expansive matrix, we have $\lim_{m\to\infty} \|(M^t)^{-m}\|^{1/m} = r < 1$. Take r_0 with $r < r_0 < 1$. Then there exists $m_0 \in \mathbf{N}$ such that $\|(M^t)^{-m_0}\| < r_0^{m_0} < \dfrac{2\varepsilon_0}{\sqrt{d}}$. So

$$\|(M^t)^{-m_0}x\| \le \|(M^t)^{-m_0}\| \, |x| < \varepsilon_0$$

for $x \in \mathbf{T}^d$, equivalently, $(M^t)^{-m_0}\mathbf{T}^d \subset \{x \in \mathbf{R}^d : |x| < \varepsilon_0\}$. Combined with (2.2), it follows that

$$(M^t)^{-m_0}\mathbf{T}^d + y_0 \subset \Omega.$$

Letting $x_0 = (M^t)^{m_0} y_0$, we have $\mathbf{T}^d + x_0 \subset \Omega$.

Now we turn to (2.1). Write $c_0 = c^{-1}\min(2^{-\alpha_1}, 2^{-\alpha_2})$, and define $E = \{x \in \mathbf{R}^d : \rho(x) < c_0\} \cap \Omega$. Then $E \subset \mathbf{T}^d$ by (v) in Definition 2.1, and

$$M^t E = \{x \in \mathbf{R}^d : \rho(x) < c_0\delta\} \cap \Omega$$

by (iv) in Definition 2.1 and the assumption that $\Omega = M^t\Omega$. So $E \subset \mathbf{T}^d \cap M^t E$. By Definition 2.1, we have

$$(M^t)^m(\{x \in \mathbf{R}^d : \rho(x) < c_0\}) = \{x \in \mathbf{R}^d : \rho(x) < c_0\delta^m\}$$

for $m \in \mathbf{N}$, and consequently,

$$\mathbf{R}^d = \bigcup_{m=1}^{\infty} (M^t)^m(\{x \in \mathbf{R}^d : \rho(x) < c_0\}).$$

It follows that

$$\Omega = \bigcup_{m=1}^{\infty} (\Omega \cap (M^t)^m(\{x \in \mathbf{R}^d : \rho(x) < c_0\})). \tag{2.3}$$

Recall that $\Omega = M^t\Omega$. We have $\Omega = (M^t)^m\Omega$, and thus

$$\Omega \cap (M^t)^m(\{x \in \mathbf{R}^d : \rho(x) < c_0\}) = (M^t)^m(\{x \in \mathbf{R}^d : \rho(x) < c_0\} \cap \Omega) = (M^t)^m E$$

for $m \in \mathbf{N}$. This implies that $\Omega = \bigcup_{m=1}^{\infty}(M^t)^m E$ by (2.3). Also observing that $|\Omega| > 0$ by the assumption that Ω has non-empty interior, we have $|E| > 0$. The proof is completed. $\square$

Theorem 2.1. *Given a $d \times d$ expansive matrix M, let Ω be a measurable set in $\mathbf{R}^d$ having non-empty interior, and let $\Omega = M^t\Omega$. Then there exists a measurable set $S \subset \Omega$ such that S is an M-admissible scaling set.*

Proof. Let τ be the translation projection of $\mathbf{R}^d$ into $\mathbf{T}^d$, that is $\tau(x) = x - k$ for $x \in \mathbf{T}^d + k$ with $k \in \mathbf{Z}^d$. Then, by Lemma 2.2 in [6], to every measurable set $\tilde{F}$ in $\mathbf{R}^d$ there corresponds a measurable subset $F \subset \tilde{F}$ such that $\tau(F) = \tau(\tilde{F})$ and $\tau|_F$ is injective. Define a sequence $\{E_n\}_{n\in\mathbf{N}}$ of measurable subsets of Ω in the following way:

$E_1 = E$, E is as in Lemma 2.1;

$\tilde{E}_{n+1} = (M^tE_n)\backslash(\bigcup_{i=1}^{n} E_i^P)$, E_{n+1} is a measurable subset of $\tilde{E}_{n+1}$ such that $\tau(E_{n+1}) = \tau(\tilde{E}_{n+1})$ and $\tau|_{E_{n+1}}$ is injective for $n \in \mathbf{N}$.

Next we prove that $S = \bigcup_{n=1}^{\infty} E_n$ is as desired. By Lemma 2.1 and the construction of $\{E_n\}_{n\in\mathbf{N}}$,

$$E_1 \subset M^tE_1, \quad E_n \subset \tilde{E}_n \subset M^tE_{n-1} \quad \text{for } n > 1, \tag{2.4}$$

and

$$E_m \cap E_n^P = \emptyset \quad \text{for } m, n \in \mathbf{N} \text{ with } m > n. \tag{2.5}$$

It follows that $S \subset M^tS$, and that S is $\mathbf{Z}^d$-congruent to a subset of $\mathbf{T}^d$. So, to finish the proof, it suffices to prove that

$$\mathbf{T}^d + x_0 \subset S^P, \tag{2.6}$$

where x_0 is as in Lemma 2.1. Indeed, if (2.6) holds, then

$$1 = |\mathbf{T}^d| = |\tau(\mathbf{T}^d + x_0)| \le |\tau(S^P)| = |S| \le 1,$$

which implies that $|S| = 1$. Therefore, S is $\mathbf{Z}^d$-congruent to $\mathbf{T}^d$, and thus S is as desired. Now we turn to (2.6). Write $\mathbf{S}^d = \mathbf{T}^d + x_0$, and $F_{m,n,l} = ((M^t)^{-(m-1)}\mathbf{S}^d) \cap M^t(E_n + l)$ for $m, n \in \mathbf{N}$ and $l \in \mathbf{Z}^d$. Then $(F_{m,n,l} - M^tl)\backslash(\bigcup_{i=1}^{n} E_i^P) \subset \tilde{E}_{n+1} \subset E_{n+1}^P \subset S^P$. Thus by the definition of S, $F_{m,n,l} - M^tl \subset S^P$, which implies that $\mathbf{S}^d \cap (M^t)^m(E_n + l) \subset (M^t)^{m-1}S^P$ for $m, n \in \mathbf{N}$ and $l \in \mathbf{Z}^d$. Again by the definition of S, it follows that

$$\mathbf{S}^d \cap (M^t)^m S^P \subset (M^t)^{m-1}S^P \quad \text{for } m \in \mathbf{N}. \tag{2.7}$$

Now we claim that

$$\mathbf{S}^d \cap (M^t)^m S^P \subset S^P \quad \text{for each } m \in \mathbf{N}. \tag{2.8}$$

It holds for $m = 1$ by (2.7). Let us proceed by induction from m to $m+1$. Suppose (2.8) holds for m. We have $\mathbf{S}^d \cap (M^t)^{m+1}S^P \subset (M^t)^m S^P$ by (2.7). It follows that $\mathbf{S}^d \cap (M^t)^{m+1} S^P \subset \mathbf{S}^d \cap (M^t)^m S^P$, and thus $\mathbf{S}^d \cap (M^t)^{m+1} S^P \subset S^P$ by the induction assumption. Therefore, (2.8) holds. Also observe that

$$\mathbf{S}^d = \bigcup_{m=1}^{\infty} (\mathbf{S}^d \cap (M^t)^m E) \subset \bigcup_{m=1}^{\infty} (\mathbf{S}^d \cap (M^t)^m S^P)$$

by Lemma 2.1 and the definition of S. We therefore have (2.6). The proof is completed. □

3. Embedding of FMRA into MRA

This section is devoted to an embedding theorem of **FMRA** into **MRA**. Before proceeding, we introduce some necessary lemmas. The first one is borrowed from [1], [7] and [8], and the second one from [24, Proposition 7], some variations of which can be found in other literatures.

LEMMA 3.1. *A function f in $L^2(\mathbf{R}^d)$ is a frame function with frame bounds A and B (Parseval frame function) if and only if*

$$A \le \sum_{k \in \mathbf{Z}^d} |\hat{f}(\cdot + k)|^2 \le B \quad \Big(\sum_{k \in \mathbf{Z}^d} |\hat{f}(\cdot + k)|^2 = 1\Big)$$

on $\sigma(f)$; is a Riesz basis function with Riesz bounds A and B (an orthonormal basis function) if and only if

$$A \le \sum_{k \in \mathbf{Z}^d} |\hat{f}(\cdot + k)|^2 \le B \quad \Big(\sum_{k \in \mathbf{Z}^d} |\hat{f}(\cdot + k)|^2 = 1\Big)$$

on $\mathbf{R}^d$.

LEMMA 3.2. *Let M be a $d \times d$ expansive matrix. For an M-refinable function f, define $\tilde{f}$ via its Fourier transform by*

$$\hat{\tilde{f}}(x) := \begin{cases} \dfrac{\hat{f}(\cdot)}{\left(\sum_{k \in \mathbf{Z}^d} |\hat{f}(\cdot + k)|^2\right)^{1/2}} & \textit{on } \operatorname{supp}(\hat{f}); \\ 0 & \textit{otherwise}. \end{cases}$$

Then $\tilde{f}$ is an M-refinable Parseval frame function, and $V_0(f) = V_0(\tilde{f})$.

LEMMA 3.3. *Given a $d \times d$ expansive matrix M, let S be an M-admissible scaling set, and let f be an M-refinable Parseval frame function with $\lim_{j \to \infty} |\hat{f}((M^t)^{-j} \cdot)| = 1$ on S. Then there exists an M-refinable orthonormal basis function g such that $V_0(f) \subset V_0(g)$ and $\operatorname{supp}(\hat{g}) \subset \bigcup_{j=0}^{\infty} (M^t)^j \operatorname{supp}(\hat{f})$.*

Proof. Define $F_0 = S \cap \sigma(f)$, and

$$F_j = \{x \in S : \hat{f}_{\|(M^t)^{-j}x} \neq 0, \text{ and } \hat{f}_{\|(M^t)^{-m}x} = 0, 0 \leq m < j\}$$

for $j \in \mathbf{N}$. Then $S = \bigcup_{j=0}^{\infty} F_j$, and the union here is a disjoint one by the fact that $\lim_{j \to \infty} |\hat{f}((M^t)^{-j} \cdot)| = 1$ on S. For an invertible integer matrix A, we define $P_A : l^2(\mathbf{Z}^d) \to l^2(\mathbf{Z}^d)$ by

$$(P_A a)(k) := \begin{cases} a(l), & k = Al, \ l \in \mathbf{Z}^d; \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

for $a \in l^2(\mathbf{Z}^d)$. Now we define $g \in L^2(\mathbf{R}^d)$ via its Fourier transform by

$$\hat{g}_{\|x} := P_{(M^t)^j} (\hat{f}((M^t)^{-j}(x+k)))_{k \in \mathbf{Z}^d} \quad \text{for } x \in F_j, \ j \in \mathbf{Z}_+,$$

that is

$$\begin{cases} \hat{g}_{\|x}((M^t)^j k) = \hat{f}_{\|(M^t)^{-j}x}(k), & \text{if } k \in \mathbf{Z}^d; \\ \hat{g}_{\|x}(k) = 0, & \text{if } k \notin (M^t)^j \mathbf{Z}^d \end{cases} \tag{3.2}$$

for $x \in F_j$ and $j \in \mathbf{Z}_+$. Then g is well-defined since S is a disjoint union of F_j with $j \in \mathbf{Z}_+$, and S is $\mathbf{Z}^d$-congruent to $\mathbf{T}^d$ (this implies that $\{S + k : k \in \mathbf{Z}^d\}$ is a partition of $\mathbf{R}^d$). Also recall that f is an M-refinable Parseval frame function. From Lemma 3.1, it follows that

$$\|\hat{g}_{\|x}\|_{l^2(\mathbf{Z}^d)} = \|\hat{f}_{\|(M^t)^{-j}x}\|_{l^2(\mathbf{Z}^d)} = 1$$

for a.e. $x \in F_j$ and $j \in \mathbf{Z}_+$, and thus g is an orthonormal basis function. By (3.2) and the fact that $S = \bigcup_{j=0}^{\infty} F_j$ and $\{S + k : k \in \mathbf{Z}^d\}$ is a partition of $\mathbf{R}^d$, we have

$$\begin{aligned} \operatorname{supp}(\hat{g}) &= \bigcup_{j=0}^{\infty} \{x + (M^t)^j k : x \in F_j, k \in \mathbf{Z}^d, \text{ and } \hat{f}_{\|(M^t)^{-j}x}(k) \neq 0\} \\ &\subset \bigcup_{j=0}^{\infty} (M^t)^j \operatorname{supp}(\hat{f}). \end{aligned}$$

Since S is $\mathbf{Z}^d$-congruent to $\mathbf{T}^d$, we have $F_0^P = \sigma(f)$. So, for a.e. $x \in \sigma(f)$, there exists $k_0 \in \mathbf{Z}^d$ such that $x + k_0 \in F_0$. It follows that $\hat{f}_{\|x+k_0} = \hat{g}_{\|x+k_0}$, which implies that $\hat{f}(\cdot) = \chi_{\sigma(f)}(\cdot)\hat{g}(\cdot)$. It leads to $V_0(f) \subset V_0(g)$. So to finish the proof we only remain to prove that g is M-refinable, equivalently, there exists a measurable function m_g on S such that

$$(\hat{g}(M^t x + M^t k))_{k \in \mathbf{Z}^d} = m_g(x) \hat{g}_{\|x} \tag{3.3}$$

for a.e. $x \in S$.

Since f is M-refinable, we have

$$(\hat{f}(M^t x + M^t k))_{k \in \mathbf{Z}^d} = m_f(x) \hat{f}_{\|x} \quad \text{for a.e. } x \in S.$$

Observe that S can be written as the following disjoint union:

$$S = (F_0 \cap (M^t)^{-1}\sigma(f)) \cup (F_0 \cap (M^t)^{-1}(S\backslash F_0)^P) \\ \cup ((S\backslash F_0) \cap (M^t)^{-1}\sigma(f)) \cup ((S\backslash F_0) \cap (M^t)^{-1}(S\backslash F_0)^P).$$

Next we divide four cases to define m_g.

CASE 1. On $F_0 \cap (M^t)^{-1}\sigma(f)$.

Observe that $\{S + k : k \in \mathbf{Z}^d\}$ is a partition of $\mathbf{R}^d$. For a.e. $x \in F_0 \cap (M^t)^{-1}\sigma(f)$, $\hat{g}_{\|x} = \hat{f}_{\|x}$, there exists $l_x \in \mathbf{Z}^d$ such that $M^t x - l_x \in S$ and thus $M^t x - l_x \in F_0$. It follows that

$$\begin{aligned} \hat{g}(M^t x + M^t k) &= \hat{g}(M^t x - l_x + M^t k + l_x) \\ &= \hat{f}(M^t x + M^t k) \\ &= m_f(x)\hat{f}(x + k) \\ &= m_f(x)\hat{g}(x + k). \end{aligned}$$

Take $m_g(x) = m_f(x)$ for a.e. $x \in F_0 \cap (M^t)^{-1}\sigma(f)$, then

$$(\hat{g}(M^t x + M^t k))_{k \in \mathbf{Z}^d} = m_g(x)\hat{g}_{\|x}.$$

CASE 2. On $F_0 \cap (M^t)^{-1}(S\backslash F_0)^P$.

(a) On $F_0 \cap (M^t)^{-1}(S\backslash F_0)$.

For a.e. $x \in F_0 \cap (M^t)^{-1}(S\backslash F_0)$, $M^t x \in F_1$. So $\hat{g}(M^t x + M^t k) = \hat{f}(x + k) = \hat{g}(x + k)$ for $k \in \mathbf{Z}^d$. Take $m_g(x) = 1$ for a.e. $x \in F_0 \cap (M^t)^{-1}(S\backslash F_0)$, then

$$(\hat{g}(M^t x + M^t k))_{k \in \mathbf{Z}^d} = m_g(x)\hat{g}_{\|x}.$$

(b) On $F_0 \cap (M^t)^{-1}(\bigcup_{0 \neq l \in \mathbf{Z}^d}(S\backslash F_0 + l))$.

Since S is an M-admissible scaling set, we have

$$F_0 \cap (M^t)^{-1}(S\backslash F_0 + l) \subset S \cap (S + (M^t)^{-1}l) = \emptyset$$

for $0 \neq l \in M^t\mathbf{Z}^d$. It follows that

$$F_0 \cap (M^t)^{-1}\left(\bigcup_{0 \neq l \in \mathbf{Z}^d}(S\backslash F_0 + l)\right) = F_0 \cap (M^t)^{-1}\left(\bigcup_{l \in \mathbf{Z}^d\backslash M^t\mathbf{Z}^d}(S\backslash F_0 + l)\right).$$

For an arbitrary $k \in \mathbf{Z}^d$ and a.e. $x \in F_0 \cap (M^t)^{-1}(S\backslash F_0 + l)$ with $l \in \mathbf{Z}^d\backslash M^t\mathbf{Z}^d$, we have $M^t x - l \in F_{j(x)}$ for some $j(x) \in \mathbf{N}$, and $M^t k + l \notin (M^t)^{j(x)}\mathbf{Z}^d$. It follows that

$$\hat{g}(M^t x + M^t k) = \hat{g}(M^t x - l + (M^t k + l)) = 0.$$

Take $m_g(x) = 0$ for a.e. $x \in F_0 \cap (M^t)^{-1}(\bigcup_{0 \neq l \in \mathbf{Z}^d}(S\backslash F_0 + l))$, then

$$(\hat{g}(M^t x + M^t k))_{k \in \mathbf{Z}^d} = 0 = m_g(x)\hat{g}_{\|x}.$$

CASE 3. On $(S\backslash F_0)\cap(M^t)^{-1}\sigma(f)$.

(a) On $(S\backslash F_0)\cap(M^t)^{-1}F_0$.

At this time, we have $M^t x\in F_0$ for a.e. $x\in(S\backslash F_0)\cap(M^t)^{-1}F_0$. So

$$\hat{g}(M^t x+M^t k)=\hat{f}(M^t x+M^t k)=m_f(x)\hat{f}(x+k)=0$$

for $k\in\mathbf{Z}^d$.

(b) On $(S\backslash F_0)\cap(M^t)^{-1}(\bigcup_{0\neq l\in\mathbf{Z}^d}(F_0+l))$.

For an arbitrary $k\in\mathbf{Z}^d$ and a.e. $x\in(S\backslash F_0)\cap(M^t)^{-1}(F_0+l)$ with $0\neq l\in\mathbf{Z}^d$, we have $M^t x-l\in F_0$. It follows that

$$\begin{aligned}\hat{g}(M^t x+M^t k)&=\hat{g}(M^t x-l+(M^t k+l))\\&=\hat{f}(M^t x-l+(M^t k+l))\\&=\hat{f}(M^t x+M^t k)\\&=m_f(x)\hat{f}(x+k)\\&=0.\end{aligned}$$

Combining (a) and (b), and taking $m_g(x)=0$ for a.e. $x\in(S\backslash F_0)\cap(M^t)^{-1}\sigma(f)$ leads to

$$(\hat{g}(M^t x+M^t k))_{k\in\mathbf{Z}^d}=m_g(x)\hat{g}_{\|x}.$$

CASE 4. On $(S\backslash F_0)\cap(M^t)^{-1}(S\backslash F_0)^P$.

(a) On $(S\backslash F_0)\cap(M^t)^{-1}(S\backslash F_0)$.

For a.e. $x\in(S\backslash F_0)\cap(M^t)^{-1}(S\backslash F_0)$, there exists $j(x)\in\mathbf{N}$ such that $x\in F_{j(x)}$ and $M^t x\in F_{j(x)+1}$. It follows that

$$\hat{g}(M^t x+M^t(M^t)^{j(x)}k)=\hat{f}_{\|(M^t)^{-j(x)}x}(k)=\hat{g}(x+(M^t)^{j(x)}k)$$

for $k\in\mathbf{Z}^d$, and that

$$\hat{g}(M^t x+M^t k)=0=\hat{g}(x+k)$$

for $k\in\mathbf{Z}^d\backslash(M^t)^{j(x)}\mathbf{Z}^d$. Take $m_g(x)=1$ for a.e. $x\in(S\backslash F_0)\cap(M^t)^{-1}(S\backslash F_0)$, then

$$(\hat{g}(M^t x+M^t k))_{k\in\mathbf{Z}^d}=m_g(x)\hat{g}_{\|x}$$

for $k\in\mathbf{Z}^d$.

(b) On $(S\backslash F_0)\cap(M^t)^{-1}(\bigcup_{0\neq l\in\mathbf{Z}^d}(S\backslash F_0+l))$.

Since S is an M-admissible scaling set, we have

$$(S\backslash F_0)\cap(M^t)^{-1}(S\backslash F_0+l)\subset S\cap(S+(M^t)^{-1}l)=\emptyset$$

for $0\neq l\in M^t\mathbf{Z}^d$. It follows that

$$(S\backslash F_0)\cap(M^t)^{-1}\left(\bigcup_{0\neq l\in\mathbf{Z}^d}(S\backslash F_0+l)\right)=(S\backslash F_0)\cap(M^t)^{-1}\left(\bigcup_{l\in\mathbf{Z}^d\backslash M^t\mathbf{Z}^d}(S\backslash F_0+l)\right).$$

For a.e. $x \in (S \backslash F_0) \cap (M^t)^{-1}(S \backslash F_0 + l)$ with $l \in \mathbf{Z}^d \backslash M^t \mathbf{Z}^d$, we have $M^t x - l \in F_{j(x)}$ for some $j(x) \in \mathbf{N}$, and $M^t k + l \notin (M^t)^{j(x)} \mathbf{Z}^d$ for $k \in \mathbf{Z}^d$, which implies that

$$\hat{g}(M^t x + M^t k) = \hat{g}(M^t x - l + (M^t k + l)) = 0.$$

Take $m_g(x) = 0$ for a.e. $x \in (S \backslash F_0) \cap (M^t)^{-1}(\bigcup_{0 \neq l \in \mathbf{Z}^d}(S \backslash F_0 + l))$, then

$$(\hat{g}(M^t x + M^t k))_{k \in \mathbf{Z}^d} = 0 = m_g(x) \hat{g}_{\|x}.$$

Collecting Case 1 to Case 4, we obtain (3.3) by defining

$$m_g(x) := \begin{cases} m_f(x) & \text{if } x \in F_0 \cap (M^t)^{-1} \sigma(f); \\ 1 & \text{if } x \in (M^t)^{-1}(S \backslash F_0)); \\ 0 & \text{otherwise} \end{cases}$$

on S. The proof is completed. □

The following lemma is from [24, Proposition 6], a special case of which is obtained in [17, Theorem 1].

Lemma 3.4. *Given a $d \times d$ expansive matrix M, let φ be an M-refinable function. Then $\overline{\bigcup_{j \in \mathbf{Z}} V_j(\varphi)} = FL^2(\Omega)$, where $\Omega = \bigcup_{j \in \mathbf{Z}} (M^t)^j \operatorname{supp}(\hat{\varphi})$.*

By [5, Theorem 1.1], $\bigcap_{j \in \mathbf{Z}} V_j(\varphi) = \{0\}$ for $\varphi \in L^2(\mathbf{R}^d)$ (not necessarily M-refinable). This result is obtained by [2, Corollary 4.14] when $M = 2I_d$, where I_d denotes the $d \times d$ identity matrix. So by Lemma 3.4 and Definition 1.1, we have the following lemma:

Lemma 3.5. *Let M be a $d \times d$ expansive matrix, and let $FL^2(\Omega)$ be a reducing subspace of $L^2(\mathbf{R}^d)$. Then, for $\varphi \in FL^2(\Omega)$, $\{V_j(\varphi)\}_{j \in \mathbf{Z}}$ is an* **FMRA** *(***MRA***) associated with M if and only if*
(i) $\Omega = \bigcup_{j \in \mathbf{Z}} (M^t)^j \operatorname{supp}(\hat{\varphi})$;
(ii) *φ is an M-refinable frame function (Riesz basis function or orthonormal basis function).*

Lemma 3.6. *Given a $d \times d$ expansive matrix M, let φ be an M-refinable Parseval frame function. Then $\lim_{j \to \infty} |\hat{\varphi}((M^t)^{-j} \cdot)| = 1$ a.e. on $\bigcup_{j \in \mathbf{Z}} (M^t)^j \operatorname{supp}(\hat{\varphi})$.*

Proof. Write $\Omega = \bigcup_{j \in \mathbf{Z}} (M^t)^j \operatorname{supp}(\hat{\varphi})$. Then $\{V_j(\varphi)\}_{j \in \mathbf{Z}}$ is an **FMRA** for $FL^2(\Omega)$ associated with M, and $\{D^j T_k \varphi : k \in \mathbf{Z}^d\}$ is a Parseval frame for $V_j(\varphi)$ for each $j \in \mathbf{Z}$ by Lemma 3.4 and Lemma 3.5. So, by Proposition 5.3.5 in [8], the orthogonal projection operator P_j of $FL^2(\Omega)$ onto $V_j(\varphi)$ for each $j \in \mathbf{Z}$ has the form

$$P_j f = \sum_{k \in \mathbf{Z}^d} \langle f, D^j T_k \varphi \rangle D^j T_k \varphi, \tag{3.4}$$

and

$$\lim_{j\to\infty}\|P_jf-f\|=0. \tag{3.5}$$

for $f\in FL^2(\Omega)$. Take $f_1\in FL^2(\Omega)$ via its Fourier transform by $\hat{f}_1=\chi_{[-1,1]^d\cap\Omega}$. Since $\{D^jT_k\varphi : k\in\mathbf{Z}^d\}$ is a Parseval frame for $V_j(\varphi)$, we have

$$\|P_jf_1\|^2=\sum_{k\in\mathbf{Z}^d}|\langle P_jf_1, D^jT_k\varphi\rangle|^2=\sum_{k\in\mathbf{Z}^d}|\langle f_1, P_jD^jT_k\varphi\rangle|^2=\sum_{k\in\mathbf{Z}^d}|\langle f_1, D^jT_k\varphi\rangle|^2.$$

Observe that

$$\langle f_1, D^jT_k\varphi\rangle=|\det M|^{j/2}\int_{(M^t)^{-j}([-1,1]^d\cap\Omega)}\overline{\hat\varphi(x)}e^{2\pi i\langle k,x\rangle}\,dx$$

by the Plancheral theorem. It follows that

$$\|P_jf_1\|^2=|\det M|^j\sum_{k\in\mathbf{Z}^d}\left|\int_{(M^t)^{-j}([-1,1]^d\cap\Omega)}\overline{\hat\varphi(x)}e^{2\pi i\langle k,x\rangle}\,dx\right|^2. \tag{3.6}$$

However, $(M^t)^{-j}([-1,1]^d\cap\Omega)\subset\mathbf{T}^d$ for j large enough since M is expansive. So, from (3.6) we have

$$\|P_jf_1\|^2=|\det M|^j\int_{(M^t)^{-j}([-1,1]^d\cap\Omega)}|\hat\varphi(x)|^2\,dx=\int_{[-1,1]^d\cap\Omega}|\hat\varphi((M^t)^{-j}x)|^2\,dx$$

for j large enough. Letting $j\to\infty$, we have

$$|[-1,1]^d\cap\Omega|=\lim_{j\to\infty}\int_{[-1,1]^d\cap\Omega}|\hat\varphi((M^t)^{-j}x)|^2\,dx \tag{3.7}$$

by (3.5).

Since φ is an M-refinable Parseval frame function, we have

$$1\geq\sum_{k\in\mathbf{Z}^d}|\hat\varphi(M^t\cdot+M^tk)|^2=|m_\varphi(\cdot)|^2\sum_{k\in\mathbf{Z}^d}|\hat\varphi(\cdot+k)|^2=|m_\varphi(\cdot)|^2 \tag{3.8}$$

a.e. on $\sigma(\varphi)$ by Lemma 3.1. By M-refinable property of φ and Lemma 3.4, to a.e. $x\in\Omega$ there corresponds $l_x\in\mathbf{Z}$ such that

$$0<|\hat\varphi((M^t)^{l_x}x)|=\left(\prod_{k=-j}^{l_x-1}|m_\varphi((M^t)^kx)|\right)|\hat\varphi((M^t)^{-j}x)|$$

for $j\in\mathbf{Z}$ with $-j<l_x$. So $(M^t)^kx\in\sigma(\varphi)$ for $k\in\mathbf{Z}$ with $k\leq l_x$, and

$$|\hat\varphi((M^t)^{-j}x)|=|m_\varphi((M^t)^{-j-1}x)|\,|\hat\varphi((M^t)^{-j-1}x)|\leq|\hat\varphi((M^t)^{-j-1}x)|\leq 1$$

for $j \in \mathbf{Z}$ with $-j \le l_x$ by (3.8). It follows that $\{|\hat{\varphi}((M^t)^{-j}\cdot)|\}$ converges increasingly to some nonnegative function $\alpha(\cdot)$ with $\alpha(\cdot) \le 1$ a.e. on Ω, which implies that

$$|[-1,1]^d \cap \Omega| = \int_{[-1,1]^d \cap \Omega} (\alpha(x))^2\,dx$$

by (3.7). Therefore, we have

$$\alpha(\cdot) = 1 \tag{3.9}$$

a.e. on $[-1,1]^d \cap \Omega$. By the definition of α, $\alpha(\cdot) = \alpha((M^t)^{-j}\cdot)$ for $j \in \mathbf{Z}$. Also by and the expansive property of M, we obtain that $\alpha(\cdot) = 1$ a.e. on Ω. The proof is completed. □

THEOREM 3.1. *Let M be a $d \times d$ expansive matrix, and let $FL^2(\Omega)$ be a reducing subspace of $L^2(\mathbf{R}^d)$ with Ω having non-empty interior. Assume that $\{V_j\}_{j\in\mathbf{Z}}$ is an* **FMRA** *for $FL^2(\Omega)$ associated with M. Then there exists an* **MRA** *$\{\mathscr{V}_j\}_{j\in\mathbf{Z}}$ for $FL^2(\Omega)$ associated with M such that $V_j \subset \mathscr{V}_j$ for $j \in \mathbf{Z}$.*

Proof. By Lemma 3.2, Lemma 3.5 and Lemma 3.6, we may as well suppose that φ is a scaling function of the **FMRA** $\{V_j\}_{j\in\mathbf{Z}}$ such that φ is an M-refinable Parseval frame function, that $\Omega = \bigcup_{j\in\mathbf{Z}} (M^t)^j \operatorname{supp}(\hat{\varphi})$, and that $\lim_{j\to\infty} |\hat{\varphi}((M^t)^{-j}\cdot)| = 1$ a.e. on Ω. By Theorem 2.1, there exists a measurable set $S \subset \Omega$ such that S is an M-admissible scaling set. So, by Lemma 3.3, there exists an M-refinable orthonormal basis function $\tilde{\varphi}$ such that $V_0 \subset V_0(\tilde{\varphi})$, and that $\operatorname{supp}(\hat{\tilde{\varphi}}) \subset \bigcup_{j=0}^{\infty} (M^t)^j \operatorname{supp}(\hat{\varphi}) \subset \Omega$, which implies that

$$\tilde{\Omega} \subset \Omega, \tag{3.10}$$

where $\tilde{\Omega} = \bigcup_{j\in\mathbf{Z}} (M^t)^j \operatorname{supp}(\hat{\tilde{\varphi}})$. By Lemma 3.4 and Lemma 3.5, $\{\mathscr{V}_j\}_{j\in\mathbf{Z}}$ is an **FMRA** for $FL^2(\tilde{\Omega})$ associated with M, where $\mathscr{V}_j = V_j(\tilde{\varphi})$ for $j \in \mathbf{Z}$. Also observing that $\mathscr{V}_j = D^j \mathscr{V}_0$, we have $V_j \subset \mathscr{V}_j$ for $j \in \mathbf{Z}$. Since $\varphi \in V_0 \subset \mathscr{V}_0$, there exists a $\mathbf{Z}^d$-periodic measurable function m such that $\hat{\varphi}(\cdot) = m(\cdot)\hat{\tilde{\varphi}}(\cdot)$ by Theorem 2.14 in [3], which implies that $\Omega = \bigcup_{j\in\mathbf{Z}} (M^t)^j \operatorname{supp}(\hat{\varphi}) \subset \tilde{\Omega}$ by Lemma 3.5. This together with (3.10) leads to $\tilde{\Omega} = \Omega$. Therefore $\{\mathscr{V}_j\}_{j\in\mathbf{Z}}$ is an **MRA** for $FL^2(\Omega)$ associated with M. The proof is completed. □

Remark 3.1. In Theorem 3.1, Ω is required to have non-empty interior. It is unresolved whether this condition can be removed. This question is well-posed by following arguments, which is equivalent to the existence of $\Omega \subset \mathbf{R}^d$ with empty interior such that $\Omega = M^t\Omega$ and $|\Omega| > 0$. It is well-known that there exists a compact set in $\mathbf{R}^d$ with positive measure which has empty interior. Let K be such a set. Since a translation of K does not change such properties of K as compactness, nowhere density and measure, we may as well assume that $0 \notin K$. By Definition 2.1, we have

$$\mathbf{R}^d \backslash \{0\} = \bigcup_{j\in\mathbf{Z}} \{x \in \mathbf{R}^d : \delta^j \le \rho(x) < \delta^{j+1}\},$$

and thus

$$K = \bigcup_{j \in \mathbf{Z}} (K \cap \{x \in \mathbf{R}^d : \delta^j \le \rho(x) < \delta^{j+1}\}).$$

Combined with $|K| > 0$, it follows that

$$|K \cap \{x \in \mathbf{R}^d : \delta^{j_0} \le \rho(x) < \delta^{j_0+1}\}| > 0$$

for some $j_0 \in \mathbf{Z}$. Write

$$K_0 = K \cap \{x \in \mathbf{R}^d : \delta^{j_0} \le \rho(x) < \delta^{j_0+1}\},$$

and define $\Omega = \bigcup_{j \in \mathbf{Z}} (M^t)^j K_0$. Then $\Omega = M^t\Omega$, $|\Omega| > 0$, and Ω has empty interior.

4. MRAs containing no genuine FMRA

Theorem 3.1 shows that an arbitrary **FMRA** must be embedded into some **MRA** in the setting of reducing subspaces $FL^2(\Omega)$ of $L^2(\mathbf{R}^d)$ with Ω having non-empty interior. Observe that an arbitrary **MRA** is always an **FMRA**. It is natural to ask the following question: what **MRA** contains no **FMRA** other than itself? This section gives a sufficient condition to this question in the setting of a general reducing subspace $FL^2(\Omega)$ of $L^2(\mathbf{R}^d)$ (Ω not necessarily has non-empty interior).

Lemma 4.1. *Given a $d \times d$ expansive matrix M and an M-refinable orthonormal basis function g, let f be an M-refinable Parseval frame function such that $f \in V_0(g)$. Then*

$$\chi_{\sigma(f)}(\cdot)|m_f(\cdot)|^2 = \chi_{(M^t)^{-1}\sigma(f)}(\cdot)|m_g(\cdot)|^2.$$

Proof. By Lemma 3.1,

$$\chi_{\sigma(f)}(\cdot) = \sum_{k \in \mathbf{Z}^d} |\hat{f}(\cdot + k)|^2 \quad \text{and} \quad 1 = \sum_{k \in \mathbf{Z}^d} |\hat{g}(\cdot + k)|^2. \tag{4.1}$$

Since $f \in V_0(g)$, there exists a $\mathbf{Z}^d$-periodic function λ such that $\hat{f}(\cdot) = \lambda(\cdot)\hat{g}(\cdot)$. Combined with (4.1), it follows that

$$|\lambda(\cdot)| = \chi_{\sigma(f)}(\cdot). \tag{4.2}$$

However, by refinable property of f and g, we have

$$m_f(\cdot)\lambda(\cdot)\hat{g}(\cdot) = m_f(\cdot)\hat{f}(\cdot) = \hat{f}(M^t\cdot) = \lambda(M^t\cdot)\hat{g}(M^t\cdot) = \lambda(M^t\cdot)m_g(\cdot)\hat{g}(\cdot).$$

So

$$|\lambda(\cdot)m_f(\cdot)|^2 \sum_{k \in \mathbf{Z}^d} |\hat{g}(\cdot + k)|^2 = |\lambda(M^t\cdot)m_g(\cdot)|^2 \sum_{k \in \mathbf{Z}^d} |\hat{g}(\cdot + k)|^2,$$

which implies that

$$\chi_{\sigma(f)}(\cdot)|m_f(\cdot)|^2 = \chi_{(M^t)^{-1}\sigma(f)}|m_g(\cdot)|^2$$

by (4.1) and (4.2). The proof is completed. □

LEMMA 4.2. *Given a $d \times d$ expansive matrix M and an M-refinable orthonormal basis function g with* $\mathrm{supp}(m_g) = \mathbf{R}^d$, *let f be a nonzero M-refinable frame function such that $f \in V_0(g)$. Then $V_0(f) = V_0(g)$.*

Proof. Write $C = \mathbf{T}^d \backslash \sigma(f)$ and define $T : \mathbf{T}^d \to \mathbf{T}^d$ by

$$Tx = M^t x (\mathrm{mod}\, \mathbf{Z}^d) \quad \text{for } x \in \mathbf{T}^d.$$

Then $C \neq \mathbf{T}^d$, and T is ergodic by Theorem 0.15 and Corollary 1.10.1 in [22]. By Lemma 3.2, we may as well suppose that f is an M-refinable Parseval frame function. Then $\chi_{\sigma(f)}(\cdot)|m_f(\cdot)|^2 = \chi_{(M^t)^{-1}\sigma(f)}(\cdot)|m_g(\cdot)|^2$ by Lemma 4.1, which implies that

$$(M^t)^{-1}\sigma(f) \subset \sigma(f) \tag{4.3}$$

since $\mathrm{supp}(m_g) = \mathbf{R}^d$. It follows that

$$C \subset \mathbf{R}^d \backslash (M^t)^{-1}\sigma(f) = (M^t)^{-1}(C^P),$$

and thus $C \subset T^{-1}(C)$. Also observing that $|T^{-1}(C)| = |C|$ since T is ergodic, we obtain that $C = T^{-1}(C)$. So $C = \emptyset$ since $C \neq \mathbf{T}^d$, and consequently, $\sigma(f) = \mathbf{R}^d$. Again by [4, Corollary 2.4] and (4.2), we have $V_0(f) = V_0(g)$. The proof is completed. □

As an immediate consequence of Lemma 3.2, Lemma 3.5 and Lemma 4.2, we obtain the following theorem:

THEOREM 4.1. *Let M be a $d \times d$ expansive matrix, and let $FL^2(\Omega)$ be a reducing subspace of $L^2(\mathbf{R}^d)$. Assume that $\{\mathscr{V}_j\}_{j \in \mathbf{Z}}$ is an* **MRA** *of $FL^2(\Omega)$ associated with M, that φ is a scaling function of the* **MRA**, *and that* $\mathrm{supp}(m_\varphi) = \mathbf{R}^d$. *Then there exists no* **FMRA** *$\{V_j\}_{j \in \mathbf{Z}}$ other than $\{\mathscr{V}_j\}_{j \in \mathbf{Z}}$ such that $V_j \subset \mathscr{V}_j$ for $j \in \mathbf{Z}$.*

Acknowledgements. The authors would like to thank the anonymous referee for his/her suggestions on Lemma 2.1 and the readability of the paper.

REFERENCES

[1] J. J. BENEDETTO AND S. LI, The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal. **5** (1998), 389–427.

[2] C. DE BOOR, R. DEVORE AND A. RON, On the construction of multivariate (pre)-wavelets, Constr. Approx. **9** (1993), 123–166.

[3] C. DE BOOR, R. DEVORE AND A. RON, Approximation from shift-invariant subspaces of $L_2(\mathbf{R}^d)$, Trans. Amer. Math. Soc. **341** (1994), 787–806.

[4] C. DE BOOR, R. DEVORE AND A. RON, The structure of finitely generated shift-invariant spaces in $L_2(\mathbf{R}^d)$, J. Funct. Anal. **119** (1994), 37–78.

[5] M. Bownik, Intersection of dilates of shift-invariant spaces, Proc. Amer. Math. Soc. **137** (2009), 563–572.
[6] M. Bownik, Z. Rzeszotnik and D. Speegle, A characterization of dimension functions of wavelets, Appl. Comput. Harmon. Anal. **10** (2001), 71–92.
[7] D. R. Chen, On the splitting trick and wavelet frame packets, SIAM J. Math. Anal. **31** (2000), 726–739.
[8] O. Christensen, An introduction to frames and Riesz bases, Birkhäuser, Boston, 2003.
[9] X. Dai, Y. Diao, Q. Gu and D. Han, Frame wavelets in subspaces of $L^2(\mathbf{R}^d)$, Proc. Amer. Math. Soc. **130** (2002), 3259–3267.
[10] X. Dai, Y. Diao and Q. Gu, Subspaces with normalized tight frame wavelets in **R**, Proc. Amer. Math. Soc. **130** (2001), 1661–1667.
[11] X. Dai, Y. Diao, Q. Gu and D. Han, The existence of subspace wavelet sets, J. Comput. Appl. Math. **155** (2003), 83–90.
[12] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. **72** (1952), 341–366.
[13] B. Han, On dual wavelet tight frames, Appl. Comput. Harmon. Anal. **4** (1997), 380–413.
[14] E. Hernández and G. Weiss, A first course on wavelets, CRC Press, Boca Raton, FL, 1996.
[15] H. O. Kim, R. Y. Kim and J. K. Lim, On the spectrums offrame multiresolution analyses, J. Math. Anal. Appl. **305** (2005), 528–545.
[16] Y.-Z. Li and F.-Y. Zhou, GMRA-based construction of framelets in reducing subspces of $L^2(\mathbf{R}^d)$, Int. J. Wavelets Multiresolut. Inf. Process. **9** (2011), 237–268.
[17] Q.-F. Lian and Y.-Z. Li, Reducing subspace frame multiresolution analysis and frame wavelets, Commun. Pure Appl. Anal. **6** (2007), 741–756.
[18] A. Ron and Z. Shen, Affine systems in $L^2(\mathbf{R}^d)$: the analysis of the analysis operator, J. Funct. Anal. **148** (1997), 408–447.
[19] K. Seip, Regular sets of sampling and interpolation for weighted Bergman spaces, Proc. Amer. Math. Soc. **117** (1993), 213–220.
[20] E. M. Stein and S. Wainger, Problems in harmonic analysis related to curvature, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.
[21] H. Volkmer, Frames of wavelets in Hardy space, Anal. **15** (1995), 405–421.
[22] P. Walters, An introduction to ergodic theory, Springer-Verlag, New York, 1982.
[23] R. M. Young, An introduction to nonharmonic fourier series, Academic Press, New York, 1980.
[24] F.-Y. Zhou and Y.-Z. Li, Multivariate FMRAs and FMRA frame wavelets for reducing subspaces of $L^2(\mathbf{R}^d)$, Kyoto J. Math. **50** (2010), 83–99.

Yun-Zhang Li
College of Applied Sciences
Beijing University of Technology
Beijing 100124
P. R. China
E-mail: yzlee@bjut.edu.cn

Lin Zhang
College of Applied Sciences
Beijing University of Technology
Beijing 100124
P. R. China
E-mail: zhanglin@emails.bjut.edu.cn