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KODAI MATH. J.
35 (2012), 138–156

TEICHMÜLLER SPACE OF GENUS TWO BASED ON SCHMUTZ SCHALLER'S HYPERBOLIC POLYGONS

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Abstract

Following the idea of P. Schmutz Schaller, we shall consider a parametrization of the Teichmüller space $\mathscr{T}_2$ of compact Riemann surfaces of genus two. In the first part of this paper, we calculate the coordinates of 4 kinds of surface uniformized by Fuchsian groups whose fundamental regions can be the regular octagon. In the second part, we give a characterization of $\mathscr{T}_2$ in $\mathbf{R}^7$.

1. Introduction

Let $\mathscr{T}_g$ be the Teichmüller space of compact Riemann surfaces of genus $g \geq 2$. For an analysis of $\mathscr{T}_g$, it is useful to parametrize $\mathscr{T}_g$ in some way. Since a compact Riemann surface of genus $g \geq 2$ inherits a hyperbolic metric from the Poincaré metric $ds = 2|dz|/(1-|z|^2)$ of the universal covering surface $\mathbf{D} = \{z \in \mathbf{C}; |z| < 1\}$, a set of geodesic length functions is used as one of the tools to parametrize $\mathscr{T}_g$. It is known that at least $6g-5$ geodesic length functions are needed to parametrize $\mathscr{T}_g$, whereas $\mathscr{T}_g$ is homeomorphic to $\mathbf{R}^{6g-6}$ [3]. As a fundamental region of a Fuchsian group which uniformizes a compact Riemann surface of genus $g \geq 2$, P. Schmutz Schaller introduced in [4] a "canonical polygon", that is, a hyperbolic $4g$-gon such that the opposite sides have the same length and that some angle conditions are fulfilled. The Teichmüller space $\mathscr{T}_g$ was then described in terms of canonical polygons, and it was shown that $\mathscr{T}_2$ can be parametrized by 7 geodesic length functions, taken as homogeneous parameters. For a surface of genus two, the Weierstrass points play an important role to construct canonical polygons with equal opposite angles. In [2] T. Kuusalo and M. Näätänen found all the Weierstrass points of compact Riemann surfaces of genus two uniformized by Fuchsian groups whose fundamental regions can be regular polygons. In the present paper we shall study $\mathscr{T}_2$ based on the idea of P. Schmutz Schaller. In the first part, we construct canonical polygons for 4 kinds of genus 2-surface uniformized by Fuchsian groups whose fundamental

2000 *Mathematics Subject Classification*. 30F60, 30F10, 30F35.

This work was supported by Grant-in-Aid for Young Scientists (B) (No. 20740081), Japan Society for the Promotion of Science.

Received September 30, 2010; revised June 9, 2011.

regions can be the regular octagon, and present some generators satisfying a simple relation to the Fuchsian groups. As a result, we calculate the coordinates in $\mathbf{R}^7$ of the surfaces. In the second part, we give an equation and some inequalities to determine $\mathcal{T}_2$ as a subset of $\mathbf{R}^7$.

2. Canonical polygons

A canonical polygon introduced by P. Schmutz Schaller is defined as follows:

DEFINITION 2.1. Let $g \geq 2$. A hyperbolic polygon $P = P(g)$ with $4g$ sides is called a canonical polygon if it satisfies the following conditions. Let a_j $(j = 1, \dots, 4g)$ denote the sides of P, which are geodesic segments ordered clockwise, and let α_j denote the angle between a_j and a_{j+1} $(a_{4g+1} = a_1)$.

(I) a_j and a_{j+2g} have the same length for every j;
(II) $\sum_{j=1}^{4g} \alpha_j = 2\pi$;
(III) $0 < \alpha_j < \pi$ for every j;
(IV) $\alpha_1 = \alpha_{2g+1}$;
(V) $\sum_{j=1}^{g} \alpha_{2j-1} + \sum_{j=g+1}^{2g} \alpha_{2j} = \sum_{j=1}^{g} \alpha_{2j} + \sum_{j=g+1}^{2g} \alpha_{2j-1}$ $(= \pi)$.

We say that two canonical polygons $P(g)$ and $P'(g)$ are equivalent if and only if there exists an isometry which maps the j-th side $a_j(P(g))$ of $P(g)$ to the j-th side $a_j(P'(g))$ of $P'(g)$ for every j. Let $\mathcal{P}(g)$ be the set of all equivalence classes of such canonical polygons with $4g$ sides. For convenience we write the equivalence class of $P(g)$ by $P(g)$ itself. We give a topology to $\mathcal{P}(g)$. A sequence of canonical polygons $\{P_k(g)\}$ converges to $P(g)$ if and only if the length of $a_j(P_k(g))$ (resp. the angle $\alpha_j(P_k(g))$) converges to the length of $a_j(P(g))$ (resp. the angle $\alpha_j(P(g))$) for every j.

The Teichmüller space $\mathcal{T}_g$ is by definition the space of all marked compact Riemann surfaces of genus g obtained by pasting opposite sides of the canonical polygons $P(g)$, where the surfaces inherit a marking corresponding to a_j $(j = 1, \dots, 4g)$. We remark that any compact Riemann surface M of genus $g \geq 2$ is obtained by pasting opposite sides of some canonical polygon $P(g)$ as follows. Let c_1 and c_2 be simple closed geodesics on M with a unique intersection p as depicted in Figure 1. Then the two angles θ_1 and φ_1 are equal. We have simple geodesic loops $c_3, \dots, c_{2g}$ passing through p such that they have no intersection other than p as depicted in Figure 1 (the figure shows the case of $g = 3$). Cut M by $c_1, \dots, c_{2g}$ and put

$$\alpha_{2j-1} := \begin{cases} \theta_{2j-1} & (j = 1, \dots, g) \\ \varphi_{2(j-g)-1} & (j = g+1, \dots, 2g) \end{cases}$$

$$\alpha_{2j} := \begin{cases} \varphi_{2j} & (j = 1, \dots, g) \\ \theta_{2(j-g)} & (j = g+1, \dots, 2g) \end{cases},$$

then we obtain a canonical polygon for M.

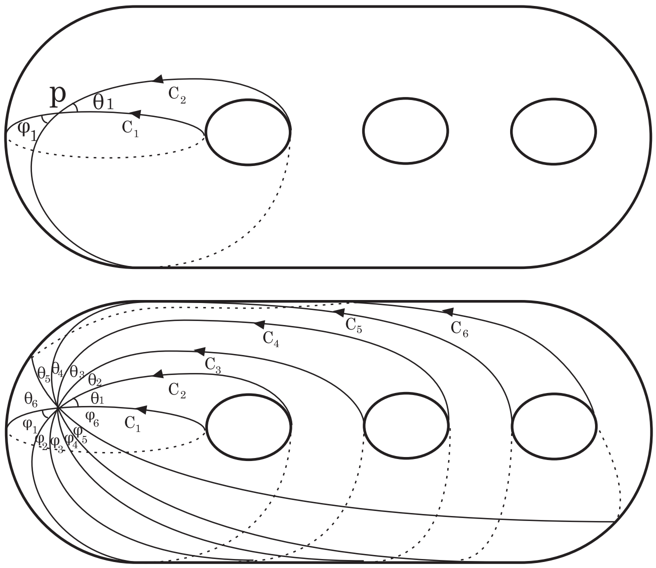

FIGURE 1. A surface of genus $g = 3$

Fix a surface $M_0 \in \mathscr{T}_g$. For every $M \in \mathscr{T}_g$ we can choose a marking-preserving homeomorphism $\phi_M : M_0 \to M$. Given a simple closed geodesic c in M_0, define a function $L_c : \mathscr{T}_g \to \mathbf{R}_+$ such as $L_c(M)$ is the length of the simple closed geodesic in M which is homotopic to $\phi_M(c)$. We call L_c the geodesic length function for c.

A compact Riemann surface of genus $g \geq 2$ is said to be hyperelliptic if it admits a conformal involution with precisely $2g + 2$ fixed points. These points are called Weierstrass points of the surface.

A hyperbolic surface is hyperelliptic if and only if it is uniformized by a Fuchsian group of which fundamental region can be a canonical polygon with equal opposite angles ($\alpha_j = \alpha_{2g+j}$ for $j = 1, \dots, 2g$) [4, Theorem 14]. For the case of hyperelliptic surfaces the condition (V) in Definition 2.1 is easily reduced to

$$\sum_{j=1}^{2g} \alpha_{i+j} = \pi \quad \text{for every } i. \tag{1}$$

3. Side-pairing mappings

Let T_j be the side-pairing mapping of a canonical polygon which maps a_j onto a_{j+2g}, where subscripts are taken modulo $4g$. We note that $T_j^{-1} = T_{j+2g}$. For our convenience, we shall adopt $\prod_{j=1}^{n} f_j$ as a composition of mappings $f_n \circ f_{n-1} \circ \cdots \circ f_1$.

PROPOSITION 3.1. *Let $P(g)$ be a canonical polygon. We denote by v_i the vertex of $P(g)$ between the sides a_i and a_{i+1}. Let Γ be the group generated by side-pairing mappings of $P(g)$. Then the element of Γ which maps v_i to v_{i+2k} is written by*

$$U_{i,i+2k} := \prod_{j=1}^{k} T_{i+2j+2g} T_{i+2j-1},$$

and the element of Γ which maps v_i to v_{i+2k+1} is written by

$$U_{i,i+2k+1} := T_{i+2k+1+2g} \prod_{j=1}^{k+g} T_{i+2j+2g} T_{i+2j-1}.$$

Proof. The following diagram shows that $U_{i,i+2k}$ maps v_i to v_{i+2k}.

$$v_i \xmapsto{T_{i+1}} v_{i+1+2g} \xmapsto{T_{i+2+2g}} v_{i+2} \longmapsto \cdots \longmapsto v_{i+2k-2} \xmapsto{T_{i+2k-1}} v_{i+2k-1+2g} \xmapsto{T_{i+2k+2g}} v_{i+2k}.$$

The following diagram shows that $U_{i,i+2k+1}$ maps v_i to v_{i+2k+1}.

$$v_i \xmapsto{T_{i+1}} v_{i+1+2g} \xmapsto{T_{i+2+2g}} v_{i+2} \longmapsto \cdots \longmapsto v_{i+2k-2+2g} \xmapsto{T_{i+2k-1+2g}} v_{i+2k-1}$$

$$\xmapsto{T_{i+2k}} v_{i+2k+2g} \xmapsto{T_{i+2k+1+2g}} v_{i+2k+1}.$$

□

PROPOSITION 3.2. *Let $P(g)$ be a canonical polygon with equal opposite angles. Let $U_{i-1,i}$ and $U_{i,i+2g}$ be the transformations as in Proposition 3.1. Then the side a_i with endpoints v_{i-1} and v_i is on the axis of $U_{i-1,i}$, and the geodesic segment b_i with endpoints v_i and v_{i+2g} is on the axis of $U_{i,i+2g}$.*

Our proof is given by a direct calculation of the mapping behaviour.

Proof. First, we shall prove that a_i is on the axis of $U_{i-1,i}$. By Proposition 3.1 we have

$$U_{i-1,i} = T_{i+2g} \prod_{j=1}^{g} T_{i-1+2j+2g} T_{i+2j-2}.$$

It is sufficient to show that a_i and its image $U_{i-1,i}(a_i)$ are connected at v_i with angle π. This fact is equivalent to the equality

$$\sum_{j=1}^{g} \alpha_{i-(2j-1)+2g} + \sum_{j=1}^{g} \alpha_{i-2j} = \sum_{j=1}^{g} \alpha_{i-(2j-1)} + \sum_{j=1}^{g} \alpha_{i-2j} = \sum_{j=1}^{2g} \alpha_{i-j} = \pi$$

and this follows from (1). See Figure 2.

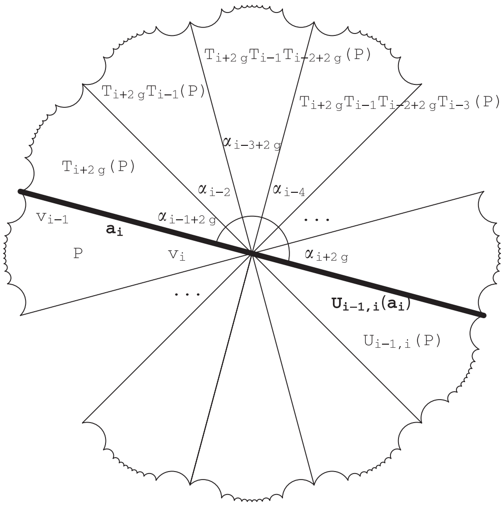

FIGURE 2. A tessellation around the vertex v_i

Next, we shall show that b_i and its image $U_{i,i+2g}(b_i)$ are connected at v_{i+2g} with angle π, where

$$U_{i,i+2g} = \prod_{j=1}^{g} T_{i+2j+2g}T_{i+2j-1}.$$

From Figure 3 and (1) it follows that

$$\sum_{j=1}^{g} \alpha_{i-(2j-1)} + \sum_{j=1}^{g} \alpha_{i-2j+2g} = \sum_{j=1}^{g} \alpha_{i-(2j-1)} + \sum_{j=1}^{g} \alpha_{i-2j} = \sum_{j=1}^{2g} \alpha_{i-j} = \pi.$$

Since b_i divides $P = P(g)$ into two isometric polygons, it follows that

$$\angle v_{i-1}v_iv_{i+2g} = \angle v_iv_{i+2g}v_{i-1+2g} (=: \theta).$$

Also

$$\angle v_{i-1}v_iv_{i+2g} = \angle U_{i,i+2g}(v_{i-1})v_{i+2g}U_{i,i+2g}(v_{i+2g})$$

holds by translation $U_{i,i+2g}$. Hence b_i and $U_{i,i+2g}(b_i)$ are connected at v_{i+2g} with angle π, and b_i is on the axis of $U_{i,i+2g}$. □

4. Examples

In this section we consider compact Riemann surfaces of genus two which are obtained by pasting sides of the regular octagon. We construct canonical

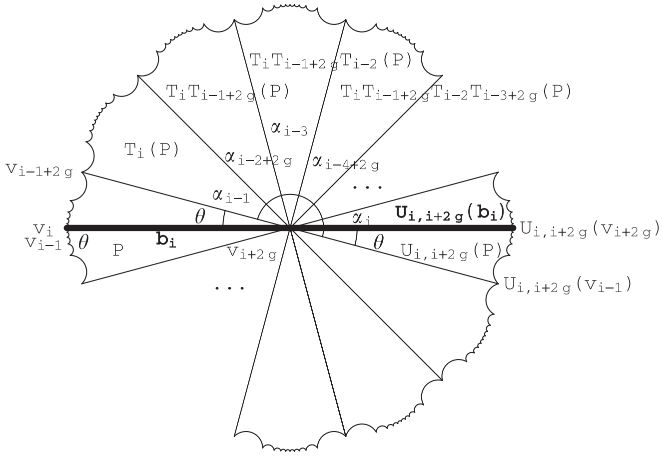

FIGURE 3. A tessellation around the vertex v_{i+2g}

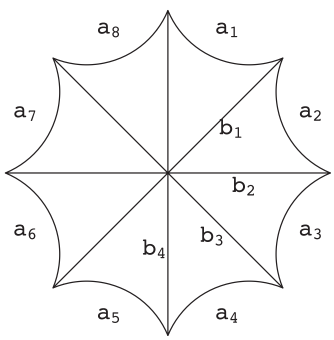

FIGURE 4. a_i and b_j for P_0

polygons as fundamental regions of Fuchsian groups uniformizing the surfaces and obtain some generators with a simple relation for the Fuchsian groups. Furthermore, we calculate the coordinates in $\mathbf{R}^7$ of the surfaces by the use of 7 geodesic length functions. In what follows we suppose that the regular octagon is located in such a way that the center is the origin and one vertex is on the real axis in the unit disc model $\mathbf{D}$.

In order to construct geodesic length functions we take a surface M_0 which is obtained by pasting the opposite sides of the regular octagon P_0 with a marking a_i $(i = 1, \dots, 8)$ and with the segment b_j between two opposite vertices as is depicted in Figure 4. Both a_i and b_j are geodesic segments in the hyperbolic plane. Since the corresponding curves in M_0, denoted by A_i and B_j respectively,

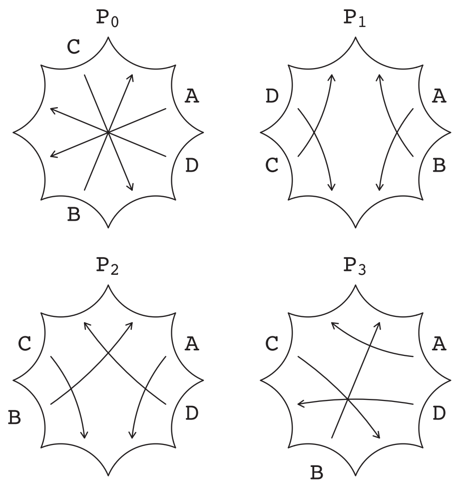

FIGURE 5. 4 kinds of side-pairing pattern

are simple closed geodesics, we shall use A_i $(i = 1, 2, 3)$ and B_j $(j = 1, 2, 3, 4)$ for 7 geodesic length functions

$$(L_{A_1}, L_{A_2}, L_{A_3}, L_{B_1}, L_{B_2}, L_{B_3}, L_{B_4}) : \mathscr{T}_2 \to \mathbf{R}_+^7$$

according to [4, Theorem 16]. In this paper we adopt

$$\left(\cosh L_{A_1}, \cosh L_{A_2}, \cosh L_{A_3}, \cosh \frac{L_{B_1}}{2}, \cosh \frac{L_{B_2}}{2}, \cosh \frac{L_{B_3}}{2}, \cosh \frac{L_{B_4}}{2}\right)$$

to calculate the coordinate of $M \in \mathscr{T}_2$ for the simplicity of representation.

4.1. Side-pairing patterns for the octagon

There are 4 kinds of side-pairing pattern for the regular octagon to make a compact surface of genus two as depicted in Figure 5, including P_0. For each polygon in Figure 5, the real axis is taken to be horizontal. The set of generators A, B, C, D of each case fulfills the following relation.

$$P_0 : ABCDA^{-1}B^{-1}C^{-1}D^{-1} = 1$$

$$P_1 : ABA^{-1}B^{-1}CDC^{-1}D^{-1} = 1$$

$$P_2 : ABCB^{-1}DA^{-1}D^{-1}C^{-1} = 1$$

$$P_3 : ABCDA^{-1}C^{-1}D^{-1}B^{-1} = 1$$

By simple calculation we can represent the generators by matrices.

PROPOSITION 4.1. *Put* $a = 1 + \sqrt{2}$ *and* $b = \sqrt{2 + 2\sqrt{2}}$. *The matrix representations of the generators* A, B, C, D *for each case are listed as follows.*

$$P_0 : A = \begin{pmatrix} -a & be^{\pi i/8} \\ be^{-\pi i/8} & -a \end{pmatrix}, \qquad B = \begin{pmatrix} a & be^{3\pi i/8} \\ be^{-3\pi i/8} & a \end{pmatrix},$$

$$C = \begin{pmatrix} -a & be^{5\pi i/8} \\ be^{-5\pi i/8} & -a \end{pmatrix}, \qquad D = \begin{pmatrix} a & be^{7\pi i/8} \\ be^{-7\pi i/8} & a \end{pmatrix}.$$

$$P_1 : A = \begin{pmatrix} ae^{5\pi i/4} & be^{3\pi i/8} \\ be^{-3\pi i/8} & ae^{-5\pi i/4} \end{pmatrix}, \quad B = \begin{pmatrix} ae^{-\pi i/4} & be^{5\pi i/8} \\ be^{-5\pi i/8} & ae^{\pi i/4} \end{pmatrix},$$

$$C = \begin{pmatrix} ae^{\pi i/4} & be^{3\pi i/8} \\ be^{-3\pi i/8} & ae^{-\pi i/4} \end{pmatrix}, \quad D = \begin{pmatrix} ae^{3\pi i/4} & be^{5\pi i/8} \\ be^{-5\pi i/8} & ae^{-3\pi i/4} \end{pmatrix}.$$

$$P_2 : A = \begin{pmatrix} ae^{5\pi i/4} & be^{3\pi i/8} \\ be^{-3\pi i/8} & ae^{-5\pi i/4} \end{pmatrix}, \quad B = \begin{pmatrix} ae^{\pi i/8} & be^{\pi i/4} \\ be^{-\pi i/4} & ae^{-\pi i/8} \end{pmatrix},$$

$$C = \begin{pmatrix} ae^{3\pi i/4} & be^{5\pi i/8} \\ be^{-5\pi i/8} & ae^{-3\pi i/4} \end{pmatrix}, \quad D = \begin{pmatrix} ae^{-\pi i/8} & be^{3\pi i/4} \\ be^{-3\pi i/4} & ae^{\pi i/8} \end{pmatrix}.$$

$$P_3 : A = \begin{pmatrix} ae^{3\pi i/4} & be^{-\pi i/8} \\ be^{\pi i/8} & ae^{-3\pi i/4} \end{pmatrix}, \quad B = \begin{pmatrix} a & be^{3\pi i/8} \\ be^{-3\pi i/8} & a \end{pmatrix},$$

$$C = \begin{pmatrix} ae^{7\pi i/8} & be^{3\pi i/4} \\ be^{-3\pi i/4} & ae^{-7\pi i/8} \end{pmatrix}, \quad D = \begin{pmatrix} ae^{\pi i/8} & -b \\ -b & ae^{-\pi i/8} \end{pmatrix}.$$

4.2. Surface M_0

We shall calculate the coordinate of the surface M_0 derived from P_0. Since $L_{A_i}(M_0)$ and $L_{B_j}(M_0)$ are equal to the lengths $\ell(a_i)$ and $\ell(b_j)$, respectively, it is easy to verify that the coordinate of M_0 is

$$\left(\cosh \ell(a_1), \cosh \ell(a_2), \cosh \ell(a_3), \cosh \frac{\ell(b_1)}{2}, \cosh \frac{\ell(b_2)}{2}, \cosh \frac{\ell(b_3)}{2}, \cosh \frac{\ell(b_4)}{2}\right)$$
$$= (5+4\sqrt{2}, 5+4\sqrt{2}, 5+4\sqrt{2}, 3+2\sqrt{2}, 3+2\sqrt{2}, 3+2\sqrt{2}, 3+2\sqrt{2}).$$

4.3. Surface M_1

Let M_1 be the surface derived from P_1 with the side-pairing mappings A, B, C, D (Figure 5). By [2] we can specify the Weierstrass points of M_1 completely, and they are represented in P_1 as follows:

$$w_1 = ke^{\pi i/4}, \quad w_2 = k, \quad w_3 = \overline{w_1}, \quad w_4 = -w_1, \quad w_5 = -k, \quad w_6 = -\overline{w_1},$$

where $k = \sqrt[4]{2} - \sqrt{\sqrt{2}-1}$. The point w_1 is the intersection of two hyperbolic lines

$$\{z \in \mathbf{D} : |z - 2\sqrt{\sqrt{2}-1}e^{\pi i/8}| = \sqrt{4\sqrt{2}-5}\},$$

$$\{z \in \mathbf{D} : |z - \sqrt[4]{2}e^{\pi i/4}| = \sqrt{\sqrt{2}-1}\},$$

which are the axes of hyperbolic transformations B and BA, respectively.

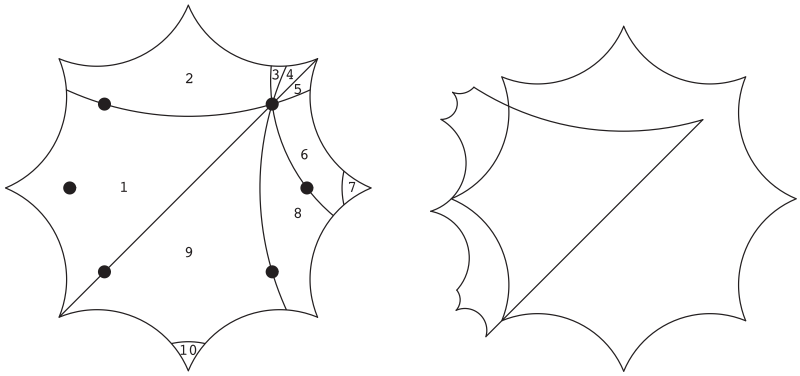

FIGURE 6. Left is P_1 cut along the geodesics through the Weierstrass points ($\bullet$); right is the contours of P_1 and a canonical polygon.

The point w_2 is the intersection of two hyperbolic lines

$$\{z \in \mathbf{D} : |z - 2\sqrt{\sqrt{2} - 1}e^{-\pi i/8}| = \sqrt{4\sqrt{2} - 5}\},$$
$$\{z \in \mathbf{D} : |z - 2\sqrt{\sqrt{2} - 1}e^{\pi i/8}| = \sqrt{4\sqrt{2} - 5}\},$$

which are the axes of two hyperbolic transformations A and B, respectively. The other Weierstrass points are obtained by the symmetricity of the side-pairings.

In order to change P_1 into a canonical polygon P_1' with equal opposite angles, i.e., $\alpha_i = \alpha_{i+4}$, we decompose P_1 into 10 pieces by simple closed geodesics through two Weierstrass points (Figure 6). Arranging the pieces according to the side-pairings, we obtain the required one (Figure 7), where we give P_1' a marking a_i' $(i = 1, \dots, 8)$ and geodesic segments b_j' $(j = 1, \dots, 4)$ as depicted in Figure 8. We remark that Figure 8 shows the same canonical polygon rotated and translated to emphasize the symmetricity, where one piece is so small that it is hard to recognize it.

Considering the pattern of pieces in P_1' carefully, we have the following.

PROPOSITION 4.2. *A Fuchsian group with the fundamental region P_1' is generated by the elements*

$$A' := D^{-1}B^{-1}C : a_2' \to a_6', \quad B' := D : a_5' \to a_1',$$
$$C' := C^{-1} : a_8' \to a_4', \qquad D' := D^{-1}AB^{-1}C : a_3' \to a_7'$$

with the relation $A'B'C'D'A'^{-1}B'^{-1}C'^{-1}D'^{-1} = 1$.

Calculating the hyperbolic lengths of $a_1', \dots, b_4'$, we shall obtain the coordinate of the surface M_1 with a marking derived from P_1'. By Propositions 3.2

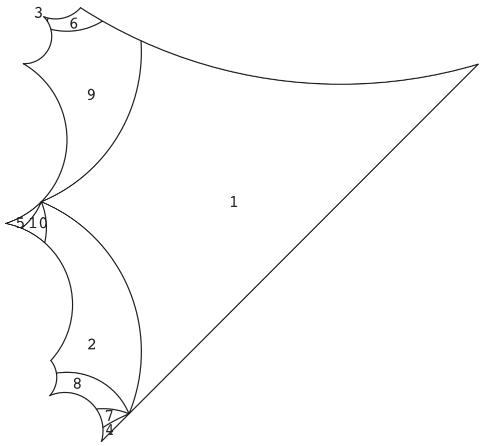

FIGURE 7. A canonical polygon obtained from pasting pieces

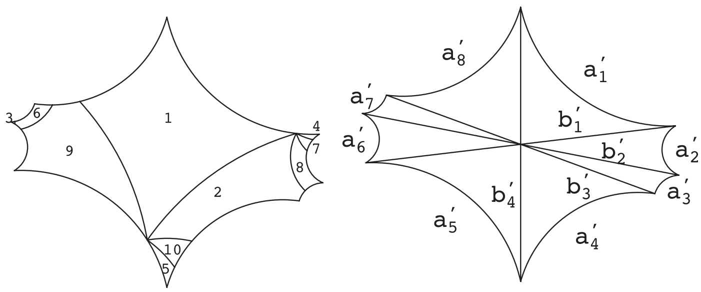

FIGURE 8. P_1' and its marking

and 4.2, the side a_1' is on the axis of $A'^{-1}D'C' = C^{-1}BAB^{-1}$. Hence the length is equal to the translation length $\mathrm{Tl}(C^{-1}BAB^{-1})$, which is calculated by

$$\cosh \frac{1}{2}\ \mathrm{Tl}(C^{-1}BAB^{-1}) = \frac{1}{2}|\mathrm{trace}(C^{-1}BAB^{-1})|.$$

Using matrix representations in Proposition 4.1, we have $\cosh(\ell(a_1')/2) = 3 + 2\sqrt{2}$.

The lengths of the other sides are obtained in the same way.

$$\cosh \frac{\ell(a_2')}{2} = \cosh \frac{\mathrm{Tl}(D'^{-1}C'^{-1}B'^{-1})}{2} = \cosh \frac{\mathrm{Tl}(C^{-1}BBA^{-1}B^{-1}C)}{2}$$
$$= \frac{1}{2}|\mathrm{trace}(BA^{-1})|,$$

$$\cosh\frac{\ell(a_3')}{2} = \cosh\frac{\mathrm{Tl}(C'B'A')}{2} = \cosh\frac{\mathrm{Tl}(C^{-1}B^{-1}C)}{2} = \frac{1}{2}|\mathrm{trace}(B)|,$$

$$\cosh\frac{\ell(b_1')}{2} = \cosh\frac{\mathrm{Tl}(B'^{-1}C'^{-1}D'^{-1}A')}{2} = \cosh\frac{\mathrm{Tl}(D^{-1}BA^{-1}B^{-1}C)}{2}$$
$$= \frac{1}{2}|\mathrm{trace}(D^{-1}BA^{-1}B^{-1}C)|,$$

$$\cosh\frac{\ell(b_2')}{2} = \cosh\frac{\mathrm{Tl}(A'B'C'D')}{2} = \cosh\frac{\mathrm{Tl}(D^{-1}AB^{-1}B^{-1}C)}{2}$$
$$= \frac{1}{2}|\mathrm{trace}(D^{-1}AB^{-1}B^{-1}C)|,$$

$$\cosh\frac{\ell(b_3')}{2} = \cosh\frac{\mathrm{Tl}(D'A'^{-1}B'^{-1}C'^{-1})}{2} = \cosh\frac{\mathrm{Tl}(D^{-1}AC)}{2} = \frac{1}{2}|\mathrm{trace}(D^{-1}AC)|,$$

$$\cosh\frac{\ell(b_4')}{2} = \cosh\frac{\mathrm{Tl}(C'^{-1}D'^{-1}A'B')}{2} = \cosh\frac{\mathrm{Tl}(A^{-1}DC)}{2} = \frac{1}{2}|\mathrm{trace}(A^{-1}DC)|.$$

Calculating these traces, we have the coordinate of M_1 as follows:

$$\left(\cosh\ell(a_1'), \cosh\ell(a_2'), \cosh\ell(a_3'), \cosh\frac{\ell(b_1')}{2}, \cosh\frac{\ell(b_2')}{2}, \cosh\frac{\ell(b_3')}{2}, \cosh\frac{\ell(b_4')}{2}\right)$$
$$= \left(33+24\sqrt{2}, 11+8\sqrt{2}, 2+2\sqrt{2}, 7+\frac{11}{\sqrt{2}}, 15+\frac{21}{\sqrt{2}}, 3+\frac{5}{\sqrt{2}}, 3+\frac{3}{\sqrt{2}}\right).$$

Remark. There are many ways to construct a canonical polygon by decomposing P_1 into pieces and arranging them. What we have done here is a single example.

4.4. Surface M_2

Let M_2 be the surface derived from the polygon P_2 with the side-pairing mappings A, B, C, D (Figure 5). We shall take a Weierstrass point, $w_1 = (\sqrt{2\sqrt{2}} - \sqrt{2\sqrt{2}-1})i$, which is the intersection of two axes

$$\{z \in \mathbf{D} : |z - 2\sqrt[4]{2}e^{\pi i/4}| = \sqrt{4\sqrt{2}-1}\},$$
$$\{z \in \mathbf{D} : |z - 2\sqrt[4]{2}e^{3\pi i/4}| = \sqrt{4\sqrt{2}-1}\},$$

of hyperbolic elements D and B, respectively (cf. [2]). Cut P_2 by geodesics through w_1 as depicted in Figure 9 and paste those pieces to make a canonical polygon P_2' with equal opposite angles (Figure 9), which is equipped with a marking a_j' $(j = 1, \ldots, 8)$ such as in Figure 10. In Figure 10, the polygon is translated to emphasize the symmetricity.

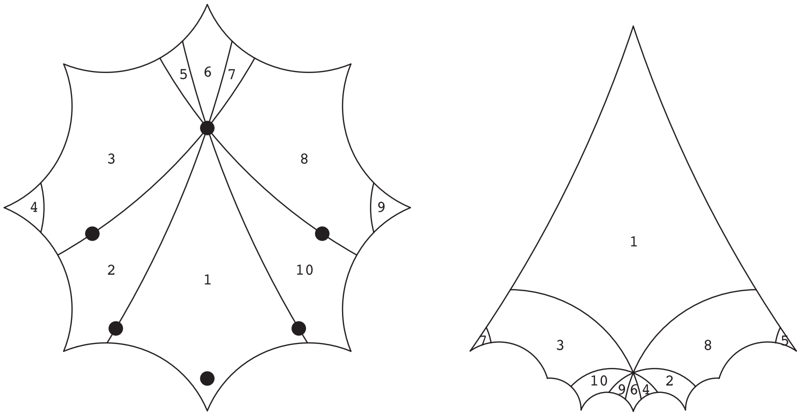

FIGURE 9. Left is P_2 cut along the geodesics through the Weierstrass points ($\bullet$); right is a canonical polygon obtained from pasting pieces.

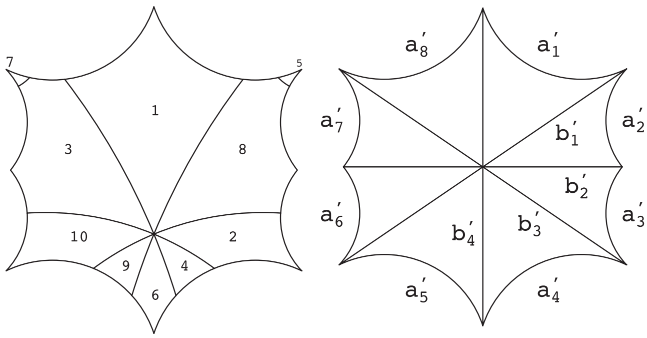

FIGURE 10. P_2' and its marking

Considering the pattern of pieces in P_2' carefully, we have the following.

PROPOSITION 4.3. *A Fuchsian group with the fundamental region P_2' is generated by the elements*

$$A' := CDA^{-1} : a_2' \to a_6', \quad B' := D^{-1}C^{-1} : a_5' \to a_1',$$
$$C' := AB : a_8' \to a_4', \qquad D' := CB^{-1}A^{-1} : a_3' \to a_7'$$

with the relation $A'B'C'D'A'^{-1}B'^{-1}C'^{-1}D'^{-1} = 1$.

The coordinate of M_2 with a marking derived from P_2' is obtained from Propositions 3.2 and 4.3. The lengths of $a_1', \dots, b_4'$ are calculated as follows:

$$\cosh\frac{\ell(a_1')}{2} = \cosh\frac{\mathrm{Tl}(A'^{-1}D'C')}{2} = \cosh\frac{\mathrm{Tl}(AD^{-1})}{2} = \frac{1}{2}|\mathrm{trace}(AD^{-1})|,$$

$$\cosh\frac{\ell(a_2')}{2} = \cosh\frac{\mathrm{Tl}(D'^{-1}C'^{-1}B'^{-1})}{2} = \cosh\frac{\mathrm{Tl}(ADA^{-1})}{2} = \frac{1}{2}|\mathrm{trace}(D)|$$

$$\cosh\frac{\ell(a_3')}{2} = \cosh\frac{\mathrm{Tl}(C'B'A')}{2} = \cosh\frac{\mathrm{Tl}(ABA^{-1})}{2} = \frac{1}{2}|\mathrm{trace}(B)|,$$

$$\cosh\frac{\ell(b_1')}{2} = \cosh\frac{\mathrm{Tl}(B'^{-1}C'^{-1}D'^{-1}A')}{2} = \cosh\frac{\mathrm{Tl}(CDDA^{-1})}{2} = \frac{1}{2}|\mathrm{trace}(CDDA^{-1})|,$$

$$\cosh\frac{\ell(b_2')}{2} = \cosh\frac{\mathrm{Tl}(A'B'C'D')}{2} = \cosh\frac{\mathrm{Tl}(CA^{-1})}{2} = \frac{1}{2}|\mathrm{trace}(CA^{-1})|,$$

$$\cosh\frac{\ell(b_3')}{2} = \cosh\frac{\mathrm{Tl}(D'A'^{-1}B'^{-1}C'^{-1})}{2} = \cosh\frac{\mathrm{Tl}(CB^{-1}B^{-1}A^{-1})}{2} = \frac{1}{2}|\mathrm{trace}(ABBC^{-1})|,$$

$$\cosh\frac{\ell(b_4')}{2} = \cosh\frac{\mathrm{Tl}(C'^{-1}D'^{-1}A'B')}{2} = \cosh\frac{\mathrm{Tl}(DA^{-1}D^{-1}C^{-1})}{2} = \frac{1}{2}|\mathrm{trace}(CDAD^{-1})|.$$

By Proposition 4.1 the coordinate of M_2 is

$$\left(\cosh\ell(a_1'), \cosh\ell(a_2'), \cosh\ell(a_3'), \cosh\frac{\ell(b_1')}{2}, \cosh\frac{\ell(b_2')}{2}, \cosh\frac{\ell(b_3')}{2}, \cosh\frac{\ell(b_4')}{2}\right)$$
$$= \left(16 + \frac{23}{\sqrt{2}}, 4 + \frac{7}{\sqrt{2}}, 4 + \frac{7}{\sqrt{2}}, 6 + \frac{9}{\sqrt{2}}, 2 + \sqrt{2}, 6 + \frac{9}{\sqrt{2}}, 3 + 2\sqrt{2}\right).$$

4.5. Surface M_3

We shall consider the surface M_3 derived from the polygon P_3 in Figure 5 in the same fashion. Take a Weierstrass point $w_1 = (\sqrt{4\sqrt{2}-4} - \sqrt{4\sqrt{2}-5})e^{3\pi i/8}$, which is the intersection of two axes

$$\{z \in \mathbf{D} : |z - 2\sqrt{\sqrt{2}-1}e^{3\pi i/8}| = \sqrt{4\sqrt{2}-5}\},$$
$$\{z = x + iy \in \mathbf{D} : y = (1+\sqrt{2})x\},$$

of hyperbolic elements A and B, respectively (cf. [2]). Cut P_3 by geodesics through w_1 and paste those pieces to make a canonical polygon P_3' with equal opposite angles (Figure 11), which is equipped with a marking a_j' (Figure 12). The polygon in Figure 12 is rotated and translated.

Considering the pattern of pieces in P_3' carefully, we have the following.

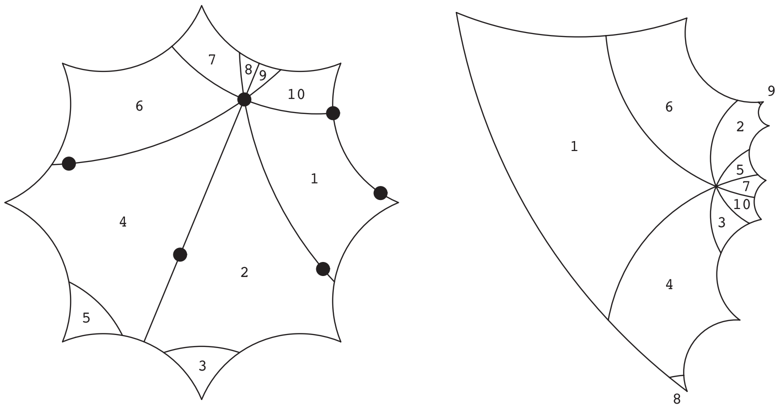

FIGURE 11. Left is P_3 cut along the geodesics through the Weierstrass points (•); right is a canonical polygon obtained from pasting pieces.

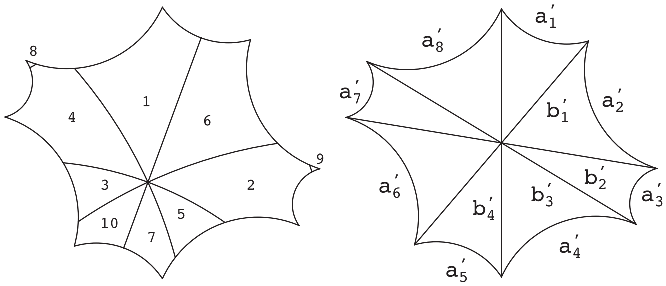

FIGURE 12. P_3' and its marking

PROPOSITION 4.4. *A Fuchsian group with the fundamental region P_3' is generated by the elements*

$$A' := D^{-1}A : a_2' \to a_6', \qquad B' := BCD : a_5' \to a_1',$$
$$C' := A^{-1}C^{-1} : a_8' \to a_4', \quad D' := D^{-1}CA : a_3' \to a_7'$$

with the relation $A'B'C'D'A'^{-1}B'^{-1}C'^{-1}D'^{-1} = 1$.

The coordinate of M_3 with a marking derived from P_3' is obtained from Propositions 3.2 and 4.4. The lengths of $a_1', \dots, b_4'$ are calculated as follows:

$$\cosh\frac{\ell(a_1')}{2} = \cosh\frac{\mathrm{Tl}(A'^{-1}D'C')}{2} = \cosh\frac{\mathrm{Tl}(A^{-1})}{2} = \frac{1}{2}|\mathrm{trace}(A)|,$$
$$\cosh\frac{\ell(a_2')}{2} = \cosh\frac{\mathrm{Tl}(D'^{-1}C'^{-1}B'^{-1})}{2} = \cosh\frac{\mathrm{Tl}(A^{-1}C^{-1}B^{-1}A)}{2} = \frac{1}{2}|\mathrm{trace}(BC)|,$$

$$\cosh\frac{\ell(a_3')}{2} = \cosh\frac{\mathrm{Tl}(C'B'A')}{2} = \cosh\frac{\mathrm{Tl}(A^{-1}C^{-1}BCA)}{2} = \frac{1}{2}|\mathrm{trace}(B)|,$$

$$\cosh\frac{\ell(b_1')}{2} = \cosh\frac{\mathrm{Tl}(B'^{-1}C'^{-1}D'^{-1}A')}{2} = \cosh\frac{\mathrm{Tl}(D^{-1}C^{-1}B^{-1}A)}{2}$$
$$= \frac{1}{2}|\mathrm{trace}(A^{-1}BCD)|,$$

$$\cosh\frac{\ell(b_2')}{2} = \cosh\frac{\mathrm{Tl}(A'B'C'D')}{2} = \cosh\frac{\mathrm{Tl}(D^{-1}BCA)}{2} = \frac{1}{2}|\mathrm{trace}(D^{-1}BCA)|,$$

$$\cosh\frac{\ell(b_3')}{2} = \cosh\frac{\mathrm{Tl}(D'A'^{-1}B'^{-1}C'^{-1})}{2} = \cosh\frac{\mathrm{Tl}(D^{-1}B^{-1}CA)}{2}$$
$$= \frac{1}{2}|\mathrm{trace}(D^{-1}B^{-1}CA)|,$$

$$\cosh\frac{\ell(b_4')}{2} = \cosh\frac{\mathrm{Tl}(C'^{-1}D'^{-1}A'B')}{2} = \cosh\frac{\mathrm{Tl}(ABCD)}{2} = \frac{1}{2}|\mathrm{trace}(ABCD)|.$$

By Proposition 4.1 the coordinate of M_3 is

$$\left(\cosh\ell(a_1'), \cosh\ell(a_2'), \cosh\ell(a_3'), \cosh\frac{\ell(b_1')}{2}, \cosh\frac{\ell(b_2')}{2}, \cosh\frac{\ell(b_3')}{2}, \cosh\frac{\ell(b_4')}{2}\right)$$
$$= \left(2+2\sqrt{2}, 12+\frac{17}{\sqrt{2}}, 5+4\sqrt{2}, 2+\frac{3}{\sqrt{2}}, 6+4\sqrt{2}, 6+4\sqrt{2}, 2+\frac{3}{\sqrt{2}}\right).$$

There are three other regular polygons which can be fundamental regions of Fuchsian groups uniformizing some compact Riemann surfaces of genus two: 10-gon, 12-gon, and 18-gon. There are 6, 6, and 8 side-pairing patterns for these polygons respectively. For all these patterns, the Weierstrass points are revealed in [2]. Hence we can make canonical polygons by a process of cut-and-paste and calculate the coordinates of the surfaces derived from them.

5. A characterization of the Teichmüller space of genus two

We shall give a characterization of the Teichmüller space of genus two in terms of the coordinates described in the preceding sections. In this section we denote the length of a segment a by the same notation a itself.

LEMMA 5.1. *Let $\triangle(a,b,c)$ be the hyperbolic triangle with sides a, b, and c. Put $x = \cosh a$, $y = \cosh b$, and $z = \cosh c$. Then the triangle inequality for $\triangle(a,b,c)$ is written in terms of x, y, and z as follows*:

$$|yz - x| < \sqrt{(y^2-1)(z^2-1)},$$

which is equivalent to $x^2 + y^2 + z^2 - 1 < 2xyz$, and also to $x + y + z + 1 < \sqrt{2(x+1)(y+1)(z+1)}$.

Proof. The triangle inequality $|b-c| < a < b+c$ is written by $\cosh|b-c| < \cosh a < \cosh(b+c)$. It is equivalent to $|\cosh a - \cosh b \cosh c| < \sinh b \sinh c$, the required one. The others are also obtained by simple calculations. □

LEMMA 5.2. *Let $\xi_i \in (-1,1)$, $i = 1,2,3$. Then the inequality $\sum_{i=1}^{3} \arccos \xi_i < \pi$ is equivalent to*

$$\sum_{i=1}^{3} \xi_i > 1 + \sqrt{2} \prod_{i=1}^{3} \sqrt{1-\xi_i}.$$

Proof. The inequality $\arccos \xi_1 + \arccos \xi_2 < \pi - \arccos \xi_3$ is equivalent to $\cos((\arccos \xi_1 + \arccos \xi_2)/2) > \cos((\pi - \arccos \xi_3)/2)$ because $(\arccos \xi_1 + \arccos \xi_2)/2$ and $(\pi - \arccos \xi_3)/2$ are in $(0,\pi)$. Since

$$\cos\left(\frac{\arccos \xi_i}{2}\right) = \sqrt{\frac{1+\xi_i}{2}}, \quad \sin\left(\frac{\arccos \xi_i}{2}\right) = \sqrt{\frac{1-\xi_i}{2}},$$

the inequality is equivalent to

$$\sqrt{\frac{1+\xi_1}{2}}\sqrt{\frac{1+\xi_2}{2}} - \sqrt{\frac{1-\xi_1}{2}}\sqrt{\frac{1-\xi_2}{2}} > \sqrt{\frac{1-\xi_3}{2}}.$$

Squaring both sides of $\sqrt{(1+\xi_1)(1+\xi_2)} > \sqrt{(1-\xi_1)(1-\xi_2)} + \sqrt{2(1-\xi_3)}$, we have the required one. □

LEMMA 5.3. *Let $\xi_i \in (0,1)$, $i = 1,2,3,4$. Then the equality $\sum_{i=1}^{4} \arccos \xi_i = \pi$ is equivalent to*

$$\sum_{i=1}^{4} \xi_i^2 = 2 - 2\prod_{i=1}^{4} \xi_i + 2\prod_{i=1}^{4} \sqrt{1-\xi_i^2}.$$

Proof. The equality $\arccos \xi_1 + \arccos \xi_2 = \pi - (\arccos \xi_3 + \arccos \xi_4)$ is equivalent to $\cos(\arccos \xi_1 + \arccos \xi_2) = \cos(\pi - (\arccos \xi_3 + \arccos \xi_4))$ because $\arccos \xi_1 + \arccos \xi_2$ and $\pi - (\arccos \xi_3 + \arccos \xi_4)$ are in $(0,\pi)$. Hence

$$\xi_1\xi_2 + \xi_3\xi_4 = \sqrt{1-\xi_1^2}\sqrt{1-\xi_2^2} + \sqrt{1-\xi_3^2}\sqrt{1-\xi_4^2}.$$

Squiring both sides of this equality, we have the required one. □

As is discussed in the preceding sections, the coordinate $(x_1,\dots,x_7) \in \mathbf{R}^7$ of a marked compact Riemann surface of genus two is calculated by $x_i = \cosh a_i$ $(i = 1,2,3)$ and $x_{i+3} = \cosh(b_i/2)$ $(i = 1,2,3,4)$, where $x_i > 1$ for $i = 1\dots,7$.

THEOREM 5.4. *The Teichmüller space of compact Riemann surfaces of genus two is realized as the subset of $\mathbf{R}^7_{>1} = \{(x_1,\dots,x_7) \in \mathbf{R}^7 \mid x_i > 1 \ (i = 1,\dots,7)\}$ satisfying the following conditions*:

(i) $|X| < 1$, $|Y| < 1$, *and* $|Z| < 1$,
(ii) $X + Y + Z - 1 > \sqrt{2(1-X)(1-Y)(1-Z)}$,
(iii) $A^2 + B^2 + C^2 + D^2 + 2ABCD - 2 = 2\sqrt{(1-A^2)(1-B^2)(1-C^2)(1-D^2)}$,

where

$$X := \frac{x_4x_7 - x_1}{\sqrt{(x_4^2-1)(x_7^2-1)}}, \quad Y := \frac{x_4x_5 - x_2}{\sqrt{(x_4^2-1)(x_5^2-1)}}, \quad Z := \frac{x_5x_6 - x_3}{\sqrt{(x_5^2-1)(x_6^2-1)}},$$

$$A := \frac{x_1 + x_4 + x_7 + 1}{\sqrt{2(x_1+1)(x_4+1)(x_7+1)}}, \quad B := \frac{x_2 + x_4 + x_5 + 1}{\sqrt{2(x_2+1)(x_4+1)(x_5+1)}},$$

$$C := \frac{x_3 + x_5 + x_6 + 1}{\sqrt{2(x_3+1)(x_5+1)(x_6+1)}}, \quad D := \frac{x_6 + x_7 + x_8 + 1}{\sqrt{2(x_6+1)(x_7+1)(x_8+1)}},$$

and

$$x_8 := x_6x_7 + \sqrt{x_6^2-1}\sqrt{x_7^2-1}(XYZ - X\sqrt{1-Y^2}\sqrt{1-Z^2} - Y\sqrt{1-Z^2}\sqrt{1-X^2} - Z\sqrt{1-X^2}\sqrt{1-Y^2}).$$

Remark. By Lemma 5.1, (i) can be replaced by $A < 1$, $B < 1$, and $C < 1$. In the proof of this theorem, $x_1, \dots, x_7 > 1$ and $a_1, a_2, a_3, b_1, b_2, b_3, b_4 > 0$ are related by $x_i = \cosh a_i$ $(i = 1, 2, 3)$ and $x_{i+3} = \cosh(b_i/2)$ $(i = 1, 2, 3, 4)$.

Proof. We shall show that the conditions (i), (ii), and (iii) are necessary for the point $(x_1, \dots, x_7)$ to represent a point of the Teichmüller space. From the canonical polygon of the surface, we have 4 triangles $\triangle(a_1, b_4/2, b_1/2)$, $\triangle(a_2, b_1/2, b_2/2)$, $\triangle(a_3, b_2/2, b_3/2)$, and $\triangle(a_4, b_3/2, b_4/2)$, where a_4 denotes the fourth side of the canonical polygon and we put $x_8 = \cosh a_4$. For each triangle, let $\delta_i \in (0, \pi)$ be the angle opposite to side a_i. Then it follows that

$$\cos\delta_1 = \frac{\cosh\frac{b_1}{2}\cosh\frac{b_4}{2} - \cosh a_1}{\sinh\frac{b_1}{2}\sinh\frac{b_4}{2}} = \frac{x_4x_7 - x_1}{\sqrt{(x_4^2-1)(x_7^2-1)}} =: X,$$

$$\cos\delta_2 = \frac{\cosh\frac{b_1}{2}\cosh\frac{b_2}{2} - \cosh a_2}{\sinh\frac{b_1}{2}\sinh\frac{b_2}{2}} = \frac{x_4x_5 - x_2}{\sqrt{(x_4^2-1)(x_5^2-1)}} =: Y,$$

$$\cos\delta_3 = \frac{\cosh\frac{b_2}{2}\cosh\frac{b_3}{2} - \cosh a_3}{\sinh\frac{b_2}{2}\sinh\frac{b_3}{2}} = \frac{x_5x_6 - x_3}{\sqrt{(x_5^2-1)(x_6^2-1)}} =: Z,$$

$$\cos\delta_4 = \frac{\cosh\frac{b_3}{2}\cosh\frac{b_4}{2} - \cosh a_4}{\sinh\frac{b_3}{2}\sinh\frac{b_4}{2}} = \frac{x_6x_7 - x_8}{\sqrt{(x_6^2-1)(x_7^2-1)}},$$

and that $|X| < 1$, $|Y| < 1$, and $|Z| < 1$. Since $\sum_{i=1}^{4} \delta_i = \pi$,

$$\begin{aligned}\cos\delta_4 &= \cos(\pi - (\delta_1 + \delta_2 + \delta_3)) \\ &= -XYZ + X\sqrt{1-Y^2}\sqrt{1-Z^2} \\ &\quad + Y\sqrt{1-Z^2}\sqrt{1-X^2} + Z\sqrt{1-X^2}\sqrt{1-Y^2}.\end{aligned}$$

Hence x_8 is expressed in terms of x_6, x_7, X, Y, and Z as in the statement of the theorem. Also, $\sum_{i=1}^{3} \delta_i < \pi$ induces (ii) by Lemma 5.2.

From an elementary hyperbolic geometry it follows that the area $|\triangle|$ of $\triangle = \triangle(a,b,c)$ satisfies

$$\cos\frac{|\triangle|}{2} = \frac{x+y+z+1}{\sqrt{2(x+1)(y+1)(z+1)}}$$

in terms of x, y, z in Lemma 5.1. Since the hyperbolic area of the canonical polygon is 4π, the total area of the 4 triangles is a half of it, 2π. Therefore

$$\begin{aligned}&\arccos\left(\frac{x_1+x_4+x_7+1}{\sqrt{2(x_1+1)(x_4+1)(x_7+1)}}\right) + \arccos\left(\frac{x_2+x_4+x_5+1}{\sqrt{2(x_2+1)(x_4+1)(x_5+1)}}\right) \\ &\quad + \arccos\left(\frac{x_3+x_5+x_6+1}{\sqrt{2(x_3+1)(x_5+1)(x_6+1)}}\right) \\ &\quad + \arccos\left(\frac{x_6+x_7+x_8+1}{\sqrt{2(x_6+1)(x_7+1)(x_8+1)}}\right) = \pi.\end{aligned}$$

If we put

$$A = \frac{x_1+x_4+x_7+1}{\sqrt{2(x_1+1)(x_4+1)(x_7+1)}}, \quad B = \frac{x_2+x_4+x_5+1}{\sqrt{2(x_2+1)(x_4+1)(x_5+1)}},$$

$$C = \frac{x_3+x_5+x_6+1}{\sqrt{2(x_3+1)(x_5+1)(x_6+1)}}, \quad D = \frac{x_6+x_7+x_8+1}{\sqrt{2(x_6+1)(x_7+1)(x_8+1)}},$$

then Lemma 5.1 implies $A, B, C, D \in (0,1)$. Hence (iii) holds by Lemma 5.3.

Conversely, suppose that $(x_1, \dots, x_7) \in \mathbf{R}^7_{>1}$ satisfies the conditions (i), (ii), and (iii). Lemma 5.1 shows that $|X| < 1$ in (i) means a triangle inequality in terms of x_1, x_4, x_7. Hence there exists a triangle $\triangle_A$ with length a_1, $b_1/2$, and $b_4/2$. Similarly we obtain triangles $\triangle_B$ and $\triangle_C$ from B and C, respectively. Denote by δ_i the opposite angle of side a_i for each triangle ($i = 1,2,3$). Then (ii) implies $\delta_1 + \delta_2 + \delta_3 < \pi$ by Lemma 5.2. Put $\delta_4 = \pi - (\delta_1 + \delta_2 + \delta_3)$, then we can construct a triangle $\triangle_D$ with sides $b_3/2$ and $b_4/2$ and with angle δ_4 between $b_3/2$ and $b_4/2$. Denote by a_4 the side of $\triangle_D$ opposite to δ_4. Then $\cosh a_4$ is equal to x_8 in the assumption. Pasting these 4 triangles and their copies suitably in order $\triangle_A$, $\triangle_B$, $\triangle_C$, $\triangle_D$, $\triangle_A$, $\triangle_B$, $\triangle_C$, $\triangle_D$, we can uniquely make an octagon P with the same opposite sides and with the same opposite angles. Since Lemma 5.3 shows that (iii) is equivalent to that the area of P is 4π, the total angle of the vertices α_i of P is equal to 2π. In addition, the inequalities $0 < \alpha_i < \pi$ are trivial. As a result we see that P is a canonical polygon. Therefore $(x_1, \dots, x_7)$ corresponds to a marked compact Riemann surface of genus two. □

Acknowledgements. The author is very grateful to the referee for his/her helpful comments and suggestions.

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