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DERIVATIVES OF ROTATION NUMBER OF ONE PARAMETER FAMILIES OF CIRCLE DIFFEOMORPHISMS

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Dedicated to Professor Kazuo Masuda on his 65-th birthday

Abstract

We consider the rotation number $\rho(t)$ of a diffeomorphism $f_t = R_t \circ f$, where R_t is the rotation by t and f is an orientation preserving C^∞ diffeomorphism of the circle S^1. We shall show that if $\rho(t)$ is irrational

$$\limsup_{t' \to t} (\rho(t') - \rho(t))/(t' - t) \geq 1.$$

1. Introduction

Let f be an orientation preserving C^∞ diffeomorphism of the circle $S^1 = \mathbf{R}/\mathbf{Z}$, and consider a one parameter family f_t, $t \in J = [-1/2, 1/2]$, of diffeomorphisms defined by $f_t = R_t \circ f$, where R_t denotes the rotation by t. Fix once and for all a lift $\tilde{f} : \mathbf{R} \to \mathbf{R}$ of f to the universal covering $\mathbf{R}$ of S^1. Then a lift $\tilde{f}_t$ of f_t is chosen as $\tilde{f}_t = T_t \circ \tilde{f}$, where T_t is the translation by t. The rotation number $\rho(t) \in \mathbf{R}$ of $\tilde{f}_t$ is a continuous and nondecreasing function of t. Define a closed set C by

$$C = J \setminus \mathrm{Int}(\rho^{-1}(\mathbf{Q})),$$

and assume for simplicity that $\rho(-1/2) = 0$, $\rho(1/2) = 1$ and C is contained in the interior of J.

V. I. Arnold [A] showed that $m(C) > 0$, where m denotes the Lebesgue measure. Denote by $\mathcal{N}$ the set of non Liouville numbers and define a Borel subset N of C by

$$N = \rho^{-1}(\mathcal{N}).$$

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M. R. Herman [H1] showed that ρ is an absolutely continuous function and that $m(N) > 0$, (under a much less restrictive condition on the one parameter family). A famous theorem of J.-C. Yoccoz says that if $t \in N$, then f_t is C^∞ conjugate to a rotation. Thus the result of M. R. Herman says that the set of the value t such that f_t is C^∞ conjugate to a rotation has positive Lebesgue measure. On the other hand it is known ([H2] p. 170 and [KH] p. 412) that for a generic value t in C the conjugacy of f_t to a rotation is a non absolutely continuous homeomorphism, provided that f is a real analytic diffeomorphism and f' is not constantly equal to 1. Nevertheless it is shown that $m(C \backslash N) = 0$ ([T]) and furthermore that $\dim_H(C \backslash N) = 0$ ([G]), where $\dim_H$ denotes the Hausdorff dimension. The purpose of this paper is to show a somewhat stronger result in this direction.

Theorem 1. *If $\rho(t)$ is irrational, then we have*

$$\limsup_{t' \to t} \frac{\rho(t') - \rho(t)}{t' - t} \geq 1.$$

Notice that the above theorem implies by the absolute continuity of ρ that $\rho^{-1}(B)$ is null if B is a null Borel set. As for the case $\rho(t)$ is rational, we have:

Theorem 2. *Assume that f is real analytic and f' is not constantly equal to* 1. *For $t \in C$ such that $\rho(t) \in \mathbf{Q}$, we have*

$$\limsup_{t' \to t} \frac{\rho(t') - \rho(t)}{t' - t} = \infty.$$

These phenomena can be found in the computer graphics of the derivative ρ' in [LV]. The plan of the paper is as follows. In Sect. 2, we prove a weaker version of Theorem 1 and apply it to a new proof of the result of [G]. In Sect. 3, we give an ellaboration of the argument of Sect. 2, which yields a proof of Theorem 1 for $\rho(t)$ a Liouville number, while the non Liouville case is treated in Sect. 4. Finally Sect. 5 is devoted to the proof of Theorem 2. Also we shall remark that it is necessary to consider lim sup instead of lim inf in Theorem 1.

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2. Weaker version of Theorem 1

The purpose of this section is to show the following proposition which is a weaker version of Theorem 1, and by applying it to prove that $\dim_H(C \backslash N) = 0$ ([G]).

A positive integer q is called a *closest return* of an irrational number α if for any $0 < j < q$, we have $|j\alpha|_{S^1} > |q\alpha|_{S^1}$, where $|x|_{S^1}$ is the distance of x and 0 in S^1. If $|q\alpha|_{S^1} = |q\alpha - p|$ for some integer p, the rational number p/q is called a *convergent* of α.

PROPOSITION 2.1. *Suppose $\rho(t)$ is irrational, p/q a convergent of $\rho(t)$, $t' \in \rho^{-1}(p/q)$ the point nearest to t. Then we have*

$$\frac{\rho(t') - \rho(t)}{t' - t} \geq e^{-M},$$

where $M = \|(\log f')'\|_{C^0}$.

To begin with let us prepare the following lemma.

LEMMA 2.2. *If $\rho(0)$ is irrational, then for any nonnegative integers i, j, we have $\mu((f^i)' \circ f^j) \geq 1$, where μ is the unique f-invariant probability measure on S^1.*

Proof. By the downward concavity of log, it suffices to show

$$\mu(\log(f^i)' \circ f^j) = 0.$$

Since μ is f-invariant, this is equivalent to

$$\mu(\log(f^i)') = 0.$$

Again since μ is f-invariant and

$$\log(f^i)' = \sum_{\nu=0}^{i-1} \log f' \circ f^\nu,$$

this follows from

$$\mu(\log f') = 0.$$

By the unique ergodicity of f, we have a uniform convergence

$$n^{-1} \log(f^n)' = n^{-1} \sum_{k=0}^{n-1} \log f' \circ f^k \to \mu(\log f') = a.$$

But if $a > 0$ and if n is sufficiently large we have

$$(f^n)' > \exp \frac{an}{2} > 1,$$

and if $a < 0$, then

$$(f^n)' < \exp \frac{an}{2} < 1.$$

In any case these contradict

$$\int_{S^1} (f^n)'(x)\, dx = 1. \qquad \square$$

Proof of Proposition 2.1. Assume that t in Proposition 2.1 is 0. The rotation number of $\tilde{f}$, $\rho(0) = \alpha$, is irrational by the hypothesis.

Given $x, y \in S^1$, denote

$$[x, y] = \{z \in S^1 \mid x \preceq z \preceq y\},$$

where $\preceq$ is the positive cyclic order of S^1, and by $y - x$ the length of $[x, y]$.

Assume that p/q is a convergent of α, and, to fix the idea, that $q\alpha - p < 0$. Thus we have $\alpha < p/q$, and shall estimate the value of $t > 0$ such that $\rho(t) = p/q$.

Now since q is a closest return, the intervals $R_\alpha^j[0, -q\alpha + p]$, $0 \le j \le q - 1$, are mutually disjoint, where R_α denotes the rotation by α. The diffeomorphism f is topologically conjugate to R_α by an orientation preserving homeomorphism which maps a given point x to 0. This implies that if we set $L(x) = [x, f^{-q}(x)]$, then

(2.1) $$f^j L(x) \text{ are mutually disjoint for } 0 \le j \le q - 1,$$

for any $x \in S^1$. Let t be the smallest positive value such that $\rho(t) = p/q$. Our aim is to estimate the value of $(p/q - \alpha)/t$ from below.

For $0 \le s \le t$ consider the point $f_s^q(x)$. For $s = 0$, this is just $f^q(x)$ and as $s \to t$ the point $f_s^q(x)$ increases from $f^q(x)$ towards $f_t^q(x)$ on the interval

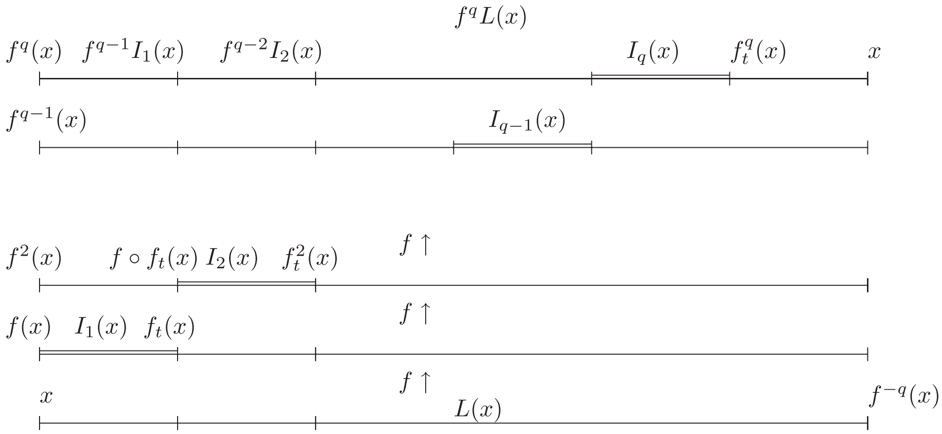
$$f^q L(x) = [f^q(x), x].$$

Thus for some x, $f_t^q(x) = x$, while for any x $f_t^q(x)$ lies on $f^q L(x)$, since t is the smallest value such that $\rho(t) = p/q$. This shows that the point $f_t^i(x)$ lies on the interval $f^i L(x)$ for $1 \le i \le q$. See the figure.

For each such i, consider the interval

$$I_i(x) = [f \circ f_t^{i-1}(x), f_t^i(x)] \subset f^i L(x).$$

Since $f_t^i(x) = f \circ f_t^{i-1}(x) + t$, these intervals have length t. Notice that the rightmost point of $I_i(x)$ is mapped by f to the leftmost point of $I_{i+1}(x)$.



The images

$$f^{q-1}I_1(x), f^{q-2}I_2(x)\dots, fI_{q-1}(x), I_q(x)$$

form a sequence of consecutive intervals towards the right. Their union

$$\bigcup_{i=1}^{q} f^{q-i}I_i(x) = [f^q(x), f_t^q(x)]$$

is contained in $f^qL(x) = [f^q(x), x]$ for any x.

Let $\tau_i(x)$ be the length of $f^{q-i}I_i(x)$. By the Denjoy distortion lemma and (2.1), we get

$$\tau_i(x) \geq e^{-M}(f^{q-i})' \circ f^i(x)t,$$

where $M = \|(\log f')'\|_{C^0}$. Summing them up we get for any $x \in S^1$,

$$x - f^q(x) \geq e^{-M} t \sum_{i=1}^{q} (f^{q-i})' \circ f^i(x).$$

Now let us evaluate the both hand sides by the invariant measure μ of f. It is well known that the evaluation of the left term yields -1 times the rotation number of $\tilde{f}^q$, i.e.

$$\mu(\mathrm{Id} - f^q) = p - q\alpha.$$

Therefore Lemma 2.2 implies that

$$(p/q - \alpha)/t \geq e^{-M},$$

as is desired. □

Now let us start the proof of Graczyk's Theorem ([G]). First we need the following easy lemma.

Lemma 2.3. *Assume α is irrational, $q > 1$, and for some $d > 3$*

$$|\alpha - p/q| < 1/q^d.$$

Then p/q is a convergent of α. □

Let

$$J_d(p/q) = \rho^{-1}((p/q - 1/q^d, p/q + 1/q^d)) - \rho^{-1}(p/q).$$

Also let

$$D = \{d \in (3, \infty) \mid q^d \notin \mathbf{Q}, \forall q \in \mathbf{N} \backslash \{1\}\}.$$

Since $(3, \infty) \backslash D$ is countable, D is dense in $(3, \infty)$.

Corollary 2.4. *If $q > 1$ and $d \in D$, then*

$$m(J_d(p/q)) \leq 2e^M q^{-d},$$

where m denotes the Lebesgue measure.

Proof. Apply Proposition 2.1 to the irrational numbers $\alpha = p/q - 1/q^d$ and $\alpha = p/q + 1/q^d$. □

Now the preimage L of the set of Liouville numbers by ρ can be described as

$$L = \bigcap_{d \in D} \bigcap_{q_0} \bigcup_{q \geq q_0} J_d(p/q),$$

where in the union p runs over the integers $0 < p < q$, coprime to p. Given any $\alpha > 0$, if $d > 2/\alpha$, we have

$$\sum_{q \geq q_0} m(J_d(p/q))^\alpha \leq \sum_{q=q_0}^{\infty} 2^\alpha e^{\alpha M} q \cdot q^{-\alpha d} \to 0 \quad (q_0 \to \infty),$$

which concludes that $\dim_H(L) = 0$.

3. Proof of Theorem 1 for Liouville $\rho(t)$

Let q_n be the n-th closest return of the irrational number $\alpha = \rho(0)$. Then the sequence $\{q_n\alpha\}$ converges to 0 in S^1, changing signs alternately. The closest returns satisfy

$$q_{n+1} = a_{n+1}q_n + q_{n-1},$$

where a_{n+1} is the $(n+1)$-st denominator of the continued franction of α.

Here we assume that α is a Liouville number. Thus the sequence $\{a_{n+1}\}$ is unbounded. It is no loss of generality to assume that there is a subsequence n_i such that

$$q_{n_i}\alpha \uparrow 0 \text{ in } S^1 \quad \text{and} \quad a_{n_i+1} \to \infty.$$

For simplicity we shall write n_i as n in what follows, and have in mind that

$$q_n\alpha \prec 0 \prec -q_n\alpha \prec q_{n-1}\alpha$$

and that a_{n+1} is as large as desired. All the notations of the previous section are used by replacing p/q with p_n/q_n.

Consider the first return map S of the rotation R_α^{-1} on the interval $[q_n\alpha, q_{n-1}\alpha]$. We have

$$S = \begin{cases} R_\alpha^{-q_n} \text{ on } [q_n\alpha, (q_n + q_{n-1})\alpha] & \text{sending it to } [0, q_{n-1}\alpha] \\ R_\alpha^{-q_{n-1}} \text{ on } [(q_n + q_{n-1})\alpha, q_{n-1}\alpha] & \text{sending it to } [q_n\alpha, 0]. \end{cases}$$

Since there are ordering

$$0 \prec (1 - a_{n+1})q_n\alpha \prec (q_n + q_{n-1})\alpha \prec -a_{n+1}q_n\alpha \prec q_{n-1}\alpha,$$

the map $S = R_\alpha^{-q_n}$ sends the interval $[0, (1 - a_{n+1})q_n\alpha]$ onto $[-q_n\alpha, -a_{n+1}q_n\alpha]$. In particular $R_\alpha^{-\nu q_n}[0, -q_n\alpha]$, $0 \le \nu < a_{n+1}$, form a consecutive sequence of intervals contained in $[0, q_{n-1}\alpha]$. Translating into f via the topological conjugacy, we have for any $x \in S^1$

$$\bigcup_{\nu=0}^{a_{n+1}-1} f^{-\nu q_n} L(x) \subset K(x), \tag{3.1}$$

where $L(x) = [x, f^{-q_n}(x)]$ as before and $K(x) = [x, f^{q_{n-1}}(x)]$.

As is well known, easy to show, (2.1) can be extended to:

$$f^j K(x) \text{ are disjoint for } 0 \le j \le q_n - 1. \tag{3.2}$$

So it looks plausible that the total length $l(x)$ of $\bigcup_j f^j L(x)$ is very small, since a large number of its iterates by $f^{-\nu q_n}$ are mutally disjoint (except the end points), by virtue of (3.1) and (3.2). On the other hand the Denjoy distortion lemma actually guarantees that the coefficient e^{-M} in Proposition 2.1 can be replaced by e^{-Ml} where l is the maximum of $l(x)$, and hence if $l(x)$ were small enough, we should be able to prove Theorem 1. However we cannot do this for $L(x)$ itself and instead consider a subinterval $\hat{L}(x)$ defined by

$$\hat{L}(x) = [x, f^{-q_n} \circ f_t^{q_n}(x)],$$

where as before t is the smallest value such that $\rho(t) = p_n/q_n$. We are going to show that the total length of the union of intervals

$$\hat{l}(x) = m\left(\bigcup_{j=1}^{q_n} f^j \hat{L}(x)\right)$$

is small, where m denotes the Lebesgue measure as before. Notice that this is enough for our purpose of applying the Denjoy distortion lemma, that is, Proposition 2.1 can be improved as

$$\frac{\rho(t) - \rho(0)}{t} \ge e^{-M\hat{l}}, \tag{3.3}$$

where $\hat{l} = \max\{\hat{l}(x) \mid x \in S^1\}$. Now we have

$$\hat{L}(x) = \bigcup_{i=1}^{q_n} f^{-i} I_i(x), \tag{3.4}$$

where as before

$$I_i(x) = [f \circ f_t^{i-1}(x), f_t^i(x)] \subset f^i L(x).$$

Put

$$\hat{l}_i(x) = m\left(\bigcup_{j=1}^{q_n} f^{j-i} I_i(x)\right).$$

Then we have by (3.4) and (2.1)

$$\hat{l}(x) = \sum_{i=1}^{q_n} \hat{l}_i(x).$$

Also (3.1) and (3.2) implies that

$$\sum_{v=0}^{a_{n+1}-1} \hat{l}(f^{-vq_n}(x)) \le 1. \tag{3.5}$$

Let us compare $\hat{l}_i(x)$ with $\hat{l}_i(f^{-vq_n}(x))$ for $0 \le v < a_{n+1}$. This is possible since the intervals $I_i(x)$ and $I_i(f^{-vq_n}(x))$ are contained in $f^iK(x)$ and of length t. In fact again by the Denjoy distortion lemma and (3.2), we get

$$m([f^{j-i}I_i(x)])/t \le e^N m([f^{j-i}I_i(f^{-vq_n}(x))])/t \tag{3.6}$$

for any $0 < v < a_{n+1}$ and $1 \le j \le q_n$, where

$$N = \max\{\|(\log f')'\|_{C^0}, \|(\log(f^{-1})')'\|_{C^0}\}.$$

Summing up (3.6) by j, we get

$$\hat{l}_i(x) \le e^N \hat{l}_i(f^{-vq_n}x).$$

Again summing up by i we obtain

$$\hat{l}(x) \le e^N \hat{l}(f^{-vq_n}(x)).$$

Finally we get by (3.5)

$$a_{n+1}\hat{l}(x) \le e^N \sum_{v=0}^{a_{n+1}-1} \hat{l}(f^{-vq_n}(x)) \le e^N.$$

Now N is a constant depending only on f and a_{n+1} can be chosen arbitrarily large. By virtue of (3.3), this completes the proof of Theorem 1 for Liouville $\rho(t)$.

4. Proof of Theorem 1 for non-Liouville $\rho(t)$

Here we shall show that if $\rho(t_0)$ is non Liouville, then ρ is differentiable at t_0 and $\rho'(t_0) \ge 1$.

In the first place we need the following theorem by P. Brunovský ([B]). For the proof see also [H1].

THEOREM 4.1. *Let g_t be a C^1-path of C^1-diffeomorphisms such that g_{t_0} is an irrational rotation for some t_0. Then we have*

$$\frac{d}{dt}\,\mathrm{rot}(g_t)|_{t=t_0} = \int_{S^1} \frac{\partial g_t}{\partial t}(x)|_{t=t_0}\,dx,$$

where $\mathrm{rot}(g_t)$ *denotes the rotation number of a lift of* g_t, *chosen continuously on* t.

Since $\rho(t_0)$ is non Liouville, we have $f_{t_0} = h \circ R_{\rho(t_0)} \circ h^{-1}$ for some C^∞ diffeomorphism h ([Y]). Applying the Brunovský theorem to the family $h^{-1} \circ f_t \circ h$, we get

$$\rho'(t_0) = \int_{S^1} \frac{\partial(h^{-1} \circ f_t \circ h)}{\partial t}(x)\bigg|_{t=t_0} dx = \int_{S^1} (h^{-1})' \circ f_{t_0} \circ h(x) \cdot \frac{\partial f_t}{\partial t}\bigg|_{t=t_0} \circ h(x)\, dx$$
$$= \int_{S^1} (h^{-1})' \circ f_{t_0} \circ h(x)\, dx.$$

Since $f_{t_0} \circ h = h \circ R_{\rho(t_0)}$ and the Lebesgue measure is invariant by the rotation we have

$$\rho'(t_0) = \int_{S^1} (h^{-1})' \circ h(x)\, dx = \int_{S^1} h'(x)^{-1}\, dx.$$

Now the Schwarz inequality concludes the proof of Theorem 1 for non-Liouville $\rho(t)$.

5. Proof of Theorem 2

By the assumption of Theorem 2, for any $p/q \in \mathbf{Q}$, the set $\rho^{-1}(p/q)$ is a nondegenerate interval and C is a Cantor set. It is no loss of generality to assume that $0 \in J$ is the supremum of $\rho^{-1}(p/q)$ and to show

$$\lim_{t \downarrow 0} \frac{\rho(t) - (p/q)}{t} = \infty.$$

The real analyticity of f implies that the periodic points of $f_0 = f$ are finite in number, say x_ν, $1 \le \nu \le ql$. Now since 0 is the supremum of $\rho^{-1}(p/q)$, the graph of $\tilde{f}^q - p$ is above the diagonal and tangent to it just at the points $\pi^{-1}(x_\nu)$, where $\pi : \mathbf{R} \to S^1$ is the universal covering map. For $\varepsilon > 0$, let $I_\nu = [a_\nu, b_\nu]$ be the ε-neighbourhood of x_ν. Choosing ε small enough, one can assume that the intervals I_ν are disjoint. Put $J_\nu = [b_\nu, a_{\nu+1}]$. Choose $N > 0$ big enough so that

$$f^{qj(\nu)}(b_\nu) \notin J_\nu, \;\; 1 \le \exists j(\nu) \le N+1.$$

This means that any orbit by f^q stays consecutively in J_ν for at most N times. Then since $\tilde{f}_t^q > \tilde{f}^q$ for $t > 0$ and the speed for f_t^q is bigger than that for f^q, any orbit by f_t^q stays consecutively in J_ν for at most N times.

On the other hand straightforward computation shows that $\partial f_t^q / \partial t \ge 1$, and therefore $\tilde{f}_t^q - p \ge t$. This shows that any orbit by f_t^q stays consecutively in I_ν for at most $2\varepsilon/t$ times, 2ε being the length of I_ν.

Let us estimates the times of iterations of f_t^q needed for some point to go around S^1 once. The above observation shows that the times needed for a round trip does not exceed $(N + 2\varepsilon/t)ql$. Translated into the language of rotation number we have

$$q\rho(t) - p \ge ((N + 2\varepsilon/t)ql)^{-1}.$$

Therefore if $t < \varepsilon/N$, we have

$$\frac{\rho(t) - (p/q)}{t} \geq (3\varepsilon q^2 l)^{-1},$$

completing the proof of Theorem 2.

Finally let us remark that taking lim sup instead of lim inf is necessary for Theorem 1.

PROPOSITION 5.1. *Assume that f is real analytic and f' is not constantly equal to* 1. *There is a residual subset R in C such that for any $t \in R$*

$$\liminf_{t' \to t} \frac{\rho(t') - \rho(t)}{t' - t} = 0, \quad \text{and} \quad \limsup_{t' \to t} \frac{\rho(t') - \rho(t)}{t' - t} = \infty.$$

Proof. Let $\rho(J) \cap \mathbf{Q} = \{\alpha_n \mid n \in \mathbf{N}\}$ and set $[a_n, b_n] = \rho^{-1}(\alpha_n)$. Then there is $c_n > b_n$ very near b_n such that if $t \in (b_n, c_n)$

$$\frac{\rho(t) - \alpha_n}{t - a_n} < \frac{1}{n} \quad \text{and} \quad \frac{\rho(t) - \alpha_n}{t - b_n} > n.$$

We used Theorem 2 for the second inequality. Now the set

$$R = \bigcap_p \bigcup_{n > p} (b_n, c_n) \cap C$$

is residual in C and satisfies the condition of the theorem. □

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