S. ZENG AND I. LAHIRI
KODAI MATH. J.
35 (2012), 105–114

A NORMALITY CRITERION FOR MEROMORPHIC FUNCTIONS

Shanpeng Zeng and Indrajit Lahiri

Abstract

In the paper we prove a normality criterion for a family of meromorphic functions which involves sharing of a non-zero finite value by certain differential polynomials generated by the members of the family.

1. Introduction and results

Let $\mathfrak{D}$ be a domain in the open complex plane $\mathbf{C}$ and $\mathfrak{F}$ be a family of meromorphic functions defined in $\mathfrak{D}$. The family $\mathfrak{F}$ is said to be normal in $\mathfrak{D}$, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathfrak{F}$, there exists a subsequece $\{f_{n_j}\}$ converging spherically locally uniformly to a meromorphic function or ∞.

Let f and g be two meromorphic functions and $a \in \mathbf{C}$. If f and g have the same set of a-points, then we say that f and g share the value a IM (ignoring multiplicities).

In 1998 Y. F. Wang and M. L. Fang [9] proved the following result.

Theorem A [9]. *Let $k, n(\geq k+1)$ be positive integers and f be a transcendental meromorphic function. Then $(f^n)^{(k)}$ assumes every finite non-zero value infinitely often.*

Following normality criterion corresponds to Theorem A.

Theorem B [8]. *Let $\mathfrak{F}$ be a family of meromorphic functions defined in a domain $\mathfrak{D}$ and $k, n(\geq k+3)$ be positive integers. If $(f^n)^{(k)} \neq 1$ for every $f \in \mathfrak{F}$, then $\mathfrak{F}$ is normal.*

In 2009 Y. T. Li and Y. X. Gu [4] improved Theorem B in the following manner.

2010 *Mathematics Subject Classification*. 30D35.
Key words and phrases. Meromorphic function, Normal family, Value sharing.
Received February 23, 2011; revised May 18, 2011.

THEOREM C [4]. *Let* $\mathfrak{F}$ *be a family of meromorphic functions in a domain* $\mathfrak{D}$, $k, n(\geq k+2)$ *be positive integers and* $a \in \mathbf{C}\backslash\{0\}$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share the value a IM in* $\mathfrak{D}$ *for each pair of functions* $f, g \in \mathfrak{F}$, *then* $\mathfrak{F}$ *is normal.*

In [4] it is shown that Theorem C does not hold for $n = k+1$. So it is an interesting problem to investigate the situation under which the condition $n = k+1$ can be accommodated. In this direction we prove the following theorem.

THEOREM 1.1. *Let* $\mathfrak{F}$ *be a family of meromorphic functions defined in a domain* $\mathfrak{D}$, $a \in \mathbf{C}\backslash\{0\}$ *and* k, n *be positive integers such that* $n \geq 1$ *if* $k = 1$ *and* $n \geq 2$ *if* $k \geq 2$. *If* $f^n(f^{k+1})^{(k)}$ *and* $g^n(g^{k+1})^{(k)}$ *share the value a IM in* $\mathfrak{D}$ *for each pair of functions* $f, g \in \mathfrak{F}$, *then* $\mathfrak{F}$ *is normal.*

Following corollary immediately follows from Theorem 1.1.

COROLLARY 1.1. *Let* $\mathfrak{F}$ *be a family of meromorphic functions defined in a domain* $\mathfrak{D}$, $a \in \mathbf{C}\backslash\{0\}$ *and* k, n *be positive integers such that* $n \geq 1$ *if* $k = 1$ *and* $n \geq 2$ *if* $k \geq 2$. *If* $f^n(f^{k+1})^{(k)} \neq a$ *for every* $f \in \mathfrak{F}$, *then* $\mathfrak{F}$ *is normal.*

Remark 1.1. *If the members of* $\mathfrak{F}$ *have no simple zero, then Theorem* 1.1 *and Corollary* 1.1 *hold for* $n = 1$ *and* $k \geq 2$.

Remark 1.2. *Considering the family* $\mathfrak{F} = \{e^{mz} : m = 1, 2, 3, \ldots\}$ *and the domain* $\mathfrak{D} = \{z : |z| < 1\}$ *we can verify that* $a \neq 0$ *is essential for Theorem* 1.1 *and Corollary* 1.1.

2. Lemmas

In this section we present some necessary lemmas.

LEMMA 2.1 {p. 101 [7], [6]}. *Let* $\mathfrak{F}$ *be a family of meromorphic functions in a domain* $\mathfrak{D} \subset \mathbf{C}$. *If* $\mathfrak{F}$ *is not normal in* $\mathfrak{D}$, *then there exist*
(i) *a number r with* $0 < r < 1$,
(ii) *points* z_j *satisfying* $|z_j| < r$,
(iii) *functions* $f_j \in \mathfrak{F}$,
(iv) *positive numbers* $\rho_j \to 0$ *as* $j \to \infty$,
such that $f_j(z_j + \rho_j\zeta) \to g(\zeta)$ *as* $j \to \infty$ *locally spherically uniformly, where g is a non-constant meromorphic function in* $\mathbf{C}$ *with* $g^{\#}(\zeta) \leq g^{\#}(0) = 1$. *In particular, g has order at most* 2.

A differential polynomial P of a meromorphic function f is defined by $P(z) = \sum_{i=1}^{n} \phi_i(z)$, where $\phi_i(z) = \alpha_i(z)\prod_{j=0}^{p}(f^{(j)}(z))^{S_{ij}}$, where $\alpha_i(z) \not\equiv 0$ are small functions of f and S_{ij}'s are non-negative integers. The numbers $\bar{d}(P) = \max_{1\leq i\leq n}\sum_{j=0}^{p} S_{ij}$ and $\underline{d}(P) = \min_{1\leq i\leq n}\sum_{j=0}^{p} S_{ij}$ are respectively called the degree and the lower degree of the differential polynomial P.

LEMMA 2.2 [3]. *Let f be transcendental and meromorphic and P be a non-constant differential polynomial of f such that* $\underline{d}(P) > 1$. *Then*

$$T(r,f) \le \frac{Q+1}{\underline{d}(P)-1}\overline{N}(r,0;f) + \frac{1}{\underline{d}(P)-1}\overline{N}(r,a;P) + S(r,f),$$

where $Q = \max_{1\le i\le n}\sum_{j=1}^{p} jS_{ij}$.

LEMMA 2.3 [2, 5]. *Let f be a transcendental meromorphic function and $a \in \mathbf{C}\setminus\{0\}$. Then $f^n f'$ has infinitely many a-points, where $n(\ge 2)$ is an integer.*

LEMMA 2.4. *Let f be a transcendental meromorphic function and k, n be positive integers such that $n \ge 1$ if $k = 1$ and $n \ge 2$ if $k \ge 2$. Then $f^n(f^{k+1})^{(k)}$ assumes every value $a \in \mathbf{C}\setminus\{0\}$ infinitly often.*

Proof. Without loss of generality we may choose $a = 1$. Let $P = f^n(f^{k+1})^{(k)}$. If $k = 1$, then $P = 2f^{n+1}f'$ assumes the value 1 infinitely often by Lemma 2.3.

Let $k \ge 2$. Then $\underline{d}(P) = n+k+1$ and $Q = k$ in Lemma 2.2. So by Lemma 2.2 we get

$$T(r,f) \le \frac{k+1}{n+k}\overline{N}(r,0;f) + \frac{1}{n+k}\overline{N}(r,1;P) + S(r,f)$$

and so

$$\frac{n-1}{n+k}T(r,f) \le \frac{1}{n+k}\overline{N}(r,1;P) + S(r,f),$$

which shows that P assumes the value 1 infinitely often. This proves the lemma. □

Let $R = \dfrac{A}{B}$ be a rational function. We denote by $(R)_\infty$ the number $\deg(A) - \deg(B)$. Using the Laurent expansion around ∞ we can easily obtain the following lemma (or see the proof of Lemma 6 of [10]).

LEMMA 2.5. *If $(R)_\infty < 0$, then $(R^{(k)})_\infty = (R)_\infty - k$.*

LEMMA 2.6. *Let $R = \dfrac{A}{B}$ be rational and B be non-constant. Then $(R^{(k)})_\infty \le (R)_\infty - k$.*

Proof. We consider the following cases.

CASE 1. Let $(R)_\infty < 0$. Then the lemma follows from Lemma 2.5.

CASE 2. Let $(R)_\infty = 0$. Then we can write

$$R = c + \frac{p}{B}, \tag{2.1}$$

where c is a non-zero constant and p is a polynomial with $\deg(p) < \deg(B)$.

Since $\deg(A) = \deg(B) > \deg(p)$, we get

$$\left(\frac{p}{B}\right)_\infty < \left(\frac{A}{B}\right)_\infty. \tag{2.2}$$

So from (2.1), (2.2) and Lemma 2.5 we obtain

$$(R^{(k)})_\infty = \left(\left(\frac{p}{B}\right)^{(k)}\right)_\infty = \left(\frac{p}{B}\right)_\infty - k < \left(\frac{A}{B}\right)_\infty - k = (R)_\infty - k.$$

CASE 3. Let $(R)_\infty > 0$. Then we can express R as follows

$$R = a_m z^m + \cdots + a_1 z + a_0 + \frac{p}{B}, \tag{2.3}$$

where $a_0, a_1, \ldots, a_{m-1}, a_m (\neq 0)$ are constants, m is a positive integer and p is a polynomial with $\deg(p) < \deg(B)$.

We now further consider the following subcases.

SUBCASE 3.1. Let $k > m$. Since $\left(\frac{p}{B}\right)_\infty < 0$, by Lemma 2.5 we get from (2.3)

$$(R^{(k)})_\infty = \left(\left(\frac{p}{B}\right)^{(k)}\right)_\infty = \left(\frac{p}{B}\right)_\infty - k < (R)_\infty - k.$$

SUBCASE 3.2. Let $k = m$. Then $(R)_\infty = m = k$. By Lemma 2.5 we get

$$\left(\left(\frac{p}{B}\right)^{(k)}\right)_\infty = \left(\frac{p}{B}\right)_\infty - k < -k < 0. \tag{2.4}$$

We put $\left(\frac{p}{B}\right)^{(k)} = \frac{P}{Q}$, where P, Q are polynomials. From (2.4) we get $\deg(P) < \deg(Q)$ and so $\deg(a_m Q m! + P) = \deg(Q)$. Hence

$$\begin{aligned}(R^{(k)})_\infty &= \left(a_m m! + \left(\frac{p}{B}\right)^{(k)}\right)_\infty = \left(a_m m! + \frac{P}{Q}\right)_\infty = \left(\frac{a_m Q m! + P}{Q}\right)_\infty \\ &= 0 = k - k = (R)_\infty - k.\end{aligned}$$

SUBCASE 3.3. Let $k < m$. Then $(R)_\infty = m$ and by Lemma 2.5 we get

$$\left(\left(\frac{p}{B}\right)^{(k)}\right)_\infty = \left(\frac{p}{B}\right)_\infty - k < -k < 0. \tag{2.5}$$

We put $\left(\frac{p}{B}\right)^{(k)} = \frac{P}{Q}$, where P, Q are polynomials. From (2.5) we see that $\deg(P) < \deg(Q)$ and so

$$\deg\left[\left(\frac{a_m m!}{(m-k)!}z^{m-k}+\cdots+k!\right)Q+P\right]=\deg\left[\left(\frac{a_m m!}{(m-k)!}z^{m-k}+\cdots+k!\right)Q\right].$$

Therefore

$$\begin{aligned}(R^{(k)})_\infty &= \left(\frac{a_m m!}{(m-k)!}z^{m-k}+\cdots+k!+\left(\frac{p}{B}\right)^{(k)}\right)_\infty\\ &= \left(\frac{\left(\frac{a_m m!}{(m-k)!}z^{m-k}+\cdots+k!\right)Q+P}{Q}\right)_\infty\\ &= m-k\\ &= (R)_\infty - k.\end{aligned}$$

This proves the lemma. □

LEMMA 2.7. *Let f be a non-constant rational function, k, n be positive integers and $a \in \mathbf{C}\backslash\{0\}$. Then $f^n(f^{k+1})^{(k)}$ has at least two distinct a-points.*

Proof. We consider the following cases.

CASE 1. Suppose $f^n(f^{k+1})^{(k)}$ has exactly one a-point.

First we suppose that f is a non-constant polynomial. We set $f^n(f^{k+1})^{(k)} - a = A(z-z_0)^l$, where A is a non-zero constant and l is a positive integer satisfying $l \geq n+(k+1-k) = n+1 \geq 2$. Then $[f^n(f^{k+1})^{(k)}]' = Al(z-z_0)^{l-1}$. Since a zero of f is a zero of $f^n(f^{k+1})^{(k)}$ of multiplicity at least 2, it is also a zero of $[f^n(f^{k+1})^{(k)}]'$. Since $[f^n(f^{k+1})^{(k)}]'$ has exactly one zero at z_0 and f is a non-constant polynomial, it follows that z_0 is a zero of f and so is a zero of $f^n(f^{k+1})^{(k)}$, which is a contradiction. Therefore f is a non-polynomial rational function. We set

$$f(z) = A\frac{(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.6}$$

where $A(\neq 0)$ is a constant and $m_1, m_2, \dots, m_s$, $n_1, n_2, \dots, n_t$ are positive integers.
We put

$$M = (k+1)\sum_{j=1}^{s} m_j, \quad M' = n\sum_{j=1}^{s} m_j, \quad N = (k+1)\sum_{i=1}^{t} n_i \quad \text{and} \quad N' = n\sum_{i=1}^{t} n_i.$$

From (2.6) we get

$$f^{k+1}(z) = A^{k+1}\frac{(z-\alpha_1)^{m_1(k+1)}(z-\alpha_2)^{m_2(k+1)}\cdots(z-\alpha_s)^{m_s(k+1)}}{(z-\beta_1)^{n_1(k+1)}(z-\beta_2)^{n_2(k+1)}\cdots(z-\beta_t)^{n_t(k+1)}} \tag{2.7}$$

and so

$$(f^{k+1})^{(k)} = \frac{(z-\alpha_1)^{m_1(k+1)-k}(z-\alpha_2)^{m_2(k+1)-k}\cdots(z-\alpha_s)^{m_s(k+1)-k}}{(z-\beta_1)^{n_1(k+1)+k}(z-\beta_2)^{n_2(k+1)+k}\cdots(z-\beta_t)^{n_t(k+1)+k}}g(z), \tag{2.8}$$

where g is a polynomial.

From (2.6) and (2.8) we get

$$\begin{aligned} f^n(f^{k+1})^{(k)} &= A^n \frac{(z-\alpha_1)^{m_1(n+k+1)-k}(z-\alpha_2)^{m_2(n+k+1)-k}\cdots(z-\alpha_s)^{m_s(n+k+1)-k}}{(z-\beta_1)^{n_1(n+k+1)+k}(z-\beta_2)^{n_2(n+k+1)+k}\cdots(z-\beta_t)^{n_t(n+k+1)+k}}g(z) \\ &= \frac{p_1}{q_1}, \quad \text{say}, \end{aligned} \tag{2.9}$$

where p_1, q_1 are polynomials.

Since $f^n(f^{k+1})^{(k)}$ has exactly one a-point at z_0, say, we get from (2.9)

$$\begin{aligned} f^n(f^{k+1})^{(k)} &= a + \frac{B(z-z_0)^l}{(z-\beta_1)^{n_1(n+k+1)+k}(z-\beta_2)^{n_2(n+k+1)+k}\cdots(z-\beta_t)^{n_t(n+k+1)+k}} \\ &= \frac{p_1}{q_1}, \end{aligned} \tag{2.10}$$

where B is a non-zero constant and l is a positive integer.

From (2.9) and (2.10) we obtain respectively

$$\begin{aligned} [f^n(f^{k+1})^{(k)}]' &= \frac{(z-\alpha_1)^{m_1(n+k+1)-k-1}(z-\alpha_2)^{m_2(n+k+1)-k-1}\cdots(z-\alpha_s)^{m_s(n+k+1)-k-1}}{(z-\beta_1)^{n_1(n+k+1)+k+1}(z-\beta_2)^{n_2(n+k+1)+k+1}\cdots(z-\beta_t)^{n_t(n+k+1)+k+1}} \\ &\quad \times g_1(z) \end{aligned} \tag{2.11}$$

and

$$[f^n(f^{k+1})^{(k)}]' = \frac{(z-z_0)^{l-1}g_2(z)}{(z-\beta_1)^{n_1(n+k+1)+k+1}(z-\beta_2)^{n_2(n+k+1)+k+1}\cdots(z-\beta_t)^{n_t(n+k+1)+k+1}}, \tag{2.12}$$

where g_1, g_2 are polynomials.

From (2.7) and (2.8) we get

$$(f^{k+1})_\infty = M - N \quad \text{and} \quad ((f^{k+1})^{(k)})_\infty = M - N - (s+t)k + \deg(g).$$

Since by Lemms 2.6 $((f^{k+1})^{(k)})_\infty \le (f^{k+1})_\infty - k$, we get

$$\deg(g) \le k(s+t-1). \tag{2.13}$$

From (2.9) and (2.11) we obtain

$$(f^n(f^{k+1})^{(k)})_\infty = M + M' - ks + \deg(g) - (N + N' + kt) \tag{2.14}$$

and

$$(f^n(f^{k+1})^{(k)})'_\infty = M + M' - (k+1)s + \deg(g_1) - \{N + N' + (k+1)t\}. \tag{2.15}$$

By Lemma 2.6 we see that

$$(f^n(f^{k+1})^{(k)})'_\infty \le (f^n(f^{k+1})^{(k)})_\infty - 1. \tag{2.16}$$

Hence from (2.13)–(2.16) we get

$$\begin{aligned}\deg(g_1) &\le \deg(g) + t + s - 1 \le k(s + t - 1) + s + t - 1 \\ &= (k+1)(s + t - 1).\end{aligned} \tag{2.17}$$

Now we consider the following sub-cases.

SUBCASE 1.1. Let $l < N + N' + kt$. From (2.10) we see that $\deg(p_1) = \deg(q_1)$. From (2.9) and (2.13) we get

$$\begin{aligned}\deg(q_1) = N + N' + kt = \deg(p_1) &= M + M' - ks + \deg(g) \\ &\le M + M' - ks + k(s + t - 1) = M + M' + kt - k.\end{aligned}$$

Hence $(M + M') - (N + N') \ge k$ and so $(n + k + 1)[(m_1 + m_2 + \cdots + m_s) - (n_1 + n_2 + \cdots + n_t)] \ge k$. This implies $(m_1 + m_2 + \cdots + m_s) - (n_1 + n_2 + \cdots + n_t) \ge 1$. So $(f)_\infty \ge 1$ and hence $(f^{k+1})_\infty \ge k + 1$. Therefore we can express f^{k+1} as follows

$$f^{k+1} = a_m z^m + \cdots + a_1 z + a_0 + \frac{p}{B},$$

where $a_0, a_1, \dots, a_{m-1}, a_m (\ne 0)$ are constants, $m (\ge k + 1)$ is an integer, p and B are polynomials with $\deg(p) < \deg(B)$. Since $m > k$, by Subcase 3.3 of the proof of Lemma 2.6 we get

$$((f^{k+1})^{(k)})_\infty = (f^{k+1})_\infty - k \ge k + 1 - k = 1. \tag{2.18}$$

Since $(f)_\infty \ge 1$, from (2.18) we see that $(f^n(f^{k+1})^{(k)})_\infty \ge n + 1$, which by (2.9) contradicts the fact that $\deg(p_1) = \deg(q_1)$.

SUBCASE 1.2. Let $l > N + N' + kt$. Then from (2.10) we see that $(f^n(f^{k+1})^{(k)})_\infty > 0$. We now verify that $m_1 + m_2 + \cdots + m_s > n_1 + n_2 + \cdots + n_t$ and so

$$M > N \quad \text{and} \quad M' > N'. \tag{2.19}$$

If $m_1 + m_2 + \cdots + m_s \le n_1 + n_2 + \cdots + n_t$, then $(f)_\infty \le 0$, $(f^n)_\infty \le 0$ and $(f^{k+1})_\infty \le 0$.

Hence by Lemma 2.6 we get

$$(f^n(f^{k+1})^{(k)})_\infty = (f^n)_\infty + ((f^{k+1})^{(k)})_\infty \le 0 + (f^{k+1})_\infty - k = -k < 0,$$

a contradiction.

From (2.10) and (2.12) we respectively get

$$(f^n(f^{k+1})^{(k)})_\infty = l - (N + N' + kt) \quad \text{and}$$
$$(f^n(f^{k+1})^{(k)})'_\infty = l - 1 + \deg(g_2) - (N + N' + kt) - t.$$

So by Lemma 2.6 we obtain $l - 1 + \deg(g_2) - (N + N' + kt) - t \le l - (N + N' + kt) - 1$ and so $\deg(g_2) \le t$.

Since $\alpha_i \ne z_0$ for $i = 1, 2, \dots, s$, from (2.11) and (2.12) we see that

$$(z - \alpha_1)^{m_1(n+k+1)-k-1}(z - \alpha_2)^{m_2(n+k+1)-k-1} \cdots (z - \alpha_s)^{m_s(n+k+1)-k-1}$$

is a factor of g_2. Therefore

$$M + M' - (k+1)s \le \deg(g_2) \le t. \tag{2.20}$$

From (2.19) and (2.20) we get

$$\begin{aligned} M + M' &\le t + (k+1)s \\ &\le (n_1 + n_2 + \cdots + n_t) + (k+1)(m_1 + m_2 + \cdots + m_s) \\ &= \frac{N'}{n} + M \\ &< M + \frac{M'}{n} \\ &\le M + M', \end{aligned}$$

a contradiction.

Subcase 1.3. Let $l = N + N' + kt$. Then from (2.10) we see that $(f^n(f^{k+1})^{(k)})_\infty \le 0$. We now show that $m_1 + m_2 + \cdots + m_s \le n_1 + n_2 + \cdots + n_t$. If $m_1 + m_2 + \cdots + m_s > n_1 + n_2 + \cdots + n_t$, then $(f^n)_\infty = M' - N' \ge n$ and $(f^{k+1})_\infty = M - N \ge k + 1$. So following the reasoning of Subcase 1.1 and using the proof of Subcase 3.3 of Lemma 2.6 we get $((f^{k+1})^{(k)})_\infty = (f^{k+1})_\infty - k \ge k + 1 - k = 1$ and so $(f^n(f^{k+1})^{(k)})_\infty \ge n + 1$, which is a contradiction.

Since $\alpha_j \ne z_0$ for $j = 1, 2, \dots, s$, from (2.11) and (2.12) we see that $(z - z_0)^{l-1}$ is a factor of g_1. So by (2.17) we get $l - 1 \le \deg(g_1) \le (k+1)(s + t - 1)$. Now

$$\begin{aligned} N + N' &= l - kt \\ &\le (k+1)(s + t - 1) + 1 - kt \\ &= (k+1)s + t - k \\ &\le (k+1)(m_1 + m_2 + \cdots + m_s) + (n_1 + n_2 + \cdots + n_t) - k \\ &= M + \frac{N'}{n} - k \\ &\le N + N' - k, \end{aligned}$$

which is a contradiction.

CASE 2. Suppose $f^n(f^{k+1})^{(k)}$ has no a-point. Then f cannot be a polynomial because in this case $f^n(f^{k+1})^{(k)}$ becomes a polynomial of degree at least $n+1$. Hence f is a non-polynomial rational function. Now putting $l=0$ in (2.10) and proceeding as Subcase 1.1 we arrive at a contradiction. This proves the lemma. □

LEMMA 2.8 [1]. *Let f be an entire function. If the spherical derivative $f^{\#}$ is bounded in* $\mathbf{C}$, *then the order of f is at most* 1.

3. Proof of Theorem 1.1

Proof. We suppose that $\mathfrak{F}$ is not normal in $\mathfrak{D}$. Then by Lemma 2.1 there exist

(i) a number r with $0<r<1$,
(ii) points z_j satisfying $|z_j|<r$,
(iii) functions $f_j \in \mathfrak{F}$,
(iv) positive numbers $\rho_j \to 0$ as $j \to \infty$,

such that $f_j(z_j+\rho_j\zeta) \to g(\zeta)$ as $j\to\infty$ locally spherically uniformly, where g is a non-constant meromorphic function in $\mathbf{C}$ with $g^{\#}(\zeta) \le g^{\#}(0) = 1$. In particular, g has order at most 2.

We put $g_j(\zeta) = f_j(z_j+\rho_j\zeta)$. Then $g_j^n(\zeta)(g_j^{k+1}(\zeta))^{(k)} \to g^n(\zeta)(g^{k+1}(\zeta))^{(k)}$ as $j\to\infty$ locally spherically uniformly.

Let

$$g^n(\zeta)(g^{k+1}(\zeta))^{(k)} \equiv a. \tag{3.1}$$

Then g is entire having no zero. So in view of Lemma 2.8 we put $g(\zeta) = \exp(c\zeta+d)$, where $c(\neq 0)$ and d are constants. Therefore from (3.1) we get

$$(k+1)^k c^k \exp\{(n+k+1)c\zeta + (n+k+1)d\} \equiv a,$$

which is impossible unless $(n+k+1)c = 0$, a contradiction. Hence $g^n(\zeta)(g^{k+1}(\zeta))^{(k)} \not\equiv a$.

So by Lemma 2.4 and Lemma 2.7 the function $g^n(\zeta)(g^{k+1}(\zeta))^{(k)}$ has at least two distinct a-points ζ_0 and ζ_0^*, say. We now choose two circular neighbourhoods D_1 and D_2 of ζ_0 and ζ_0^* respectively such that $D_1\cap D_2 = \emptyset$ and $D_1 \cup D_2$ does not contain any a-point of $g^n(\zeta)(g^{k+1}(\zeta))^{(k)}$ other than ζ_0 and ζ_0^*.

Now by Hurwitz's theorem there exist two sequences of points $\{\zeta_j\}\subset D_1$ and $\{\zeta_j^*\}\subset D_2$ converging to ζ_0 and ζ_0^* respectively such that $g_j^n(\zeta_j)(g_j^{k+1}(\zeta_j))^{(k)} = a$ and $g_j^n(\zeta_j^*)(g_j^{k+1}(\zeta_j^*))^{(k)} = a$.

By the given condition for any integer m and for all j we get $g_m^n(\zeta_j)(g_m^{k+1}(\zeta_j))^{(k)} = a$ and $g_m^n(\zeta_j^*)(g_m^{k+1}(\zeta_j^*))^{(k)} = a$. By (ii) and (iv), if necessary considering a subsequence, we see that there exists a point ξ, $|\xi|\le r$, such that $z_j+\rho_j\zeta_j \to \xi$ and $z_j+\rho_j\zeta_j^* \to \xi$ as $j\to\infty$. So $f_m^n(\xi)(f_m^{k+1}(\xi))^{(k)} = a$ and since a-points are isolated, for sufficiently large j we get $z_j+\rho_j\zeta_j = \xi$ and $z_j+\rho_j\zeta_j^* = \xi$.

Hence $\zeta_j = \dfrac{\xi - z_j}{\rho_j} = \zeta_j^*$, which is impossible as $D_1 \cap D_2 = \emptyset$. This proves the theorem. □

References

[1] J. Clunie and W. K. Hayman, The spherical derivative of integral and meromorphic functions, Comment. Math. Helv. **40** (1966), 117–148.

[2] W. K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. of Math. **70** (1959), 9–42.

[3] J. D. Hinchliffe, On a result of Chuang related to Hayman's alternative, Computational Methods and Function Theory **2** (2002), 293–297.

[4] Y. T. Li and Y. X. Gu, On normal families of meromorphic functions, J. Math. Anal. Appl. **354** (2009), 421–425.

[5] E. Mues, Über ein problem von Hayman, Math. Z. **164** (1979), 239–259.

[6] X. C. Pang and L. Zalcman, Normal families and shared values, Bull. London Math. Soc. **32** (2000), 325–331.

[7] J. Schiff, Normal families, Springer-Verlag, Berlin, 1993.

[8] W. Schwick, Normality criteria for normal families of meromorphic function, J. Anal. Math. **52** (1989), 241–289.

[9] Y. F. Wang and M. L. Fang, Picard values and normal families of meromorphic functions with multiple zeros, Acta Math. Sinica (Chin. Ser.) **41** (1998), 743–748.

[10] Y. Xu, Normal families and exceptional functions, J. Math. Anal. Appl. **329** (2007), 1343–1354.

Shanpeng Zeng
Department of Mathematics Science
Nanjing Normal University
Nanjing, Jiangsu 210046
P. R. China
E-mail: zengshanpeng@163.com

Current Address:
Department of Mathematics
Hangzhou Electronic Information Vocational School
(Dingqiao Campus)
Hangzhou, Zheiang 310021
P. R. China

Indrajit Lahiri
Department of Mathematics
University of Kalyani
West Bengal 741235
India
E-mail: indr9431@dataone.in
ilahiri@hotmail.com