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SCHATTEN CLASS TOEPLITZ OPERATORS ON THE PARABOLIC BERGMAN SPACE II

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Abstract

Let $0 < \alpha \le 1$ and let $\boldsymbol{b}_\alpha^2$ be a Hilbert space of all square integrable solutions of a parabolic equation $(\partial_t + (-\Delta)^\alpha)u = 0$ on the upper half space. We study the Toeplitz operators on $\boldsymbol{b}_\alpha^2$, which we characterize to be of Schatten class whose exponent is smaller than 1. For the proof, we use an atomic decomposition theorem of parabolic Bergman functions. Generalizations to Schatten class operators for Orlicz type and Herz type are also discussed.

1. Introduction

Following the previous paper [11], we study the Schatten class Toeplitz operators on parabolic Bergman spaces. Let $0 < \alpha \le 1$ and let V be the $(n+1)$-dimensional Lebesgue measure on $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times (0, \infty)$. We denote by $\boldsymbol{b}_\alpha^2$ the Hilbert space

$$\boldsymbol{b}_\alpha^2 := \{u \in L^2(\boldsymbol{R}_+^{n+1}, V); L^{(\alpha)}\text{-harmonic on } \boldsymbol{R}_+^{n+1}\},$$

where $L^{(\alpha)} := \partial_t + (-\Delta)^\alpha$. The orthogonal projection from $L^2(V) := L^2(\boldsymbol{R}_+^{n+1}, V)$ to $\boldsymbol{b}_\alpha^2$ is represented as an integral operator by a kernel R_α, which is called the α-parabolic Bergman kernel. Let μ be a positive Radon measure on $\boldsymbol{R}_+^{n+1}$ satisfying

$$\int (1 + t + |x|^{2\alpha})^{-\tau} \, d\mu(x, t) < \infty \tag{1}$$

for some $\tau \in \boldsymbol{R}$. The Toeplitz operator with symbol μ is an operator defined by

$$(T_\mu u)(X) := \int R_\alpha(X, Y) u(Y) \, d\mu(Y)$$

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for $u \in \boldsymbol{b}_\alpha^2$. As for the compactness of $T_\mu : \boldsymbol{b}_\alpha^2 \to \boldsymbol{b}_\alpha^2$, it is known that its necessary and sufficient condition is that $\lim_{Y \to \mathscr{A}} \hat{\mu}^{(\alpha)}(Y) = 0$, where $\mathscr{A}$ is the infinite point of the one point compactification of $\boldsymbol{R}_+^{n+1}$. Also it is equivalent to $\lim_{Y \to \mathscr{A}} \tilde{\mu}^{(\alpha)}(Y) = 0$ (see [8]). Here $\hat{\mu}^{(\alpha)}$ and $\tilde{\mu}^{(\alpha)}$ are the averaging function and the Berezin transformation of μ, respectively, which are defined by

$$\hat{\mu}^{(\alpha)}(Y) := \mu(Q^{(\alpha)}(Y))/V(Q^{(\alpha)}(Y))$$

and

$$\tilde{\mu}^{(\alpha)}(Y) := \int R_\alpha(X, Y)^2 \, d\mu(X) \Big/ \int R_\alpha(X, Y)^2 \, dV(X),$$

where $Q^{(\alpha)}(Y)$ is an α-parabolic Carleson box (see §2.2). These functions are very useful for the study of Toeplitz operators (cf. [7], [8], [11] and [12]).

Let $0 < \sigma < \infty$. A compact operator T_μ is said to be of Schatten σ-class if the sequence of all its eigenvalues belongs to the sequence space l^σ. We have already shown in [11] that when $\sigma \geq 1$, T_μ is of Schatten σ-class if and only if the averaging function $\hat{\mu}^{(\alpha)}$ is in $L^\sigma(V^*)$ or equivalently the Berezin transformation $\tilde{\mu}^{(\alpha)}$ is in $L^\sigma(V^*)$, where

$$dV^*(X) := t^{-(n/2\alpha+1)} \, dV(X), \quad (X = (x, t)).$$

In the present paper, we study the remainder case $0 < \sigma < 1$, and we have

Theorem 1. *Let $0 < \sigma < 1$. For a Radon measure $\mu \geq 0$ on $\boldsymbol{R}_+^{n+1}$ satisfying* (1), *the Toeplitz operator T_μ on $\boldsymbol{b}_\alpha^2$ is of Schatten σ-class if and only if $\hat{\mu}^{(\alpha)} \in L^\sigma(V^*)$.*

We remark here that when $\alpha = 1/2$, our Bergman space $\boldsymbol{b}_{1/2}^2$ coincides with the usual harmonic Bergman space (see [6]), and the related result for space of holomorphic or harmonic functions has already studied (e.g. [1], [2], [5] and [13]).

This paper will be organized as follows: We make some preparations in section 2. In §2.1 we recall the fundamental estimates of parabolic Bergman kernels. An atomic decomposition theorem is given in §2.2, which plays an important role in the proof of Theorem 1. In §2.3 we recall some definitions of compact operators of Schatten class. The proof of Theorem 1 is given in section 3. We make some related remarks in section 4. In §4.1, we study the Carleson inclusion of Schatten class. A norm relation between averaging functions and Berezin transformations in the context of L^σ space is given in §4.2. A generarlzation to the Orlicz type class for concave functions is considered in §4.3. Contrary to the convex case, the assertion of Theorem 1 does not hold for concave functions. In fact, we give an exapmle of ψ such that $\hat{\mu}^{(\alpha)} \in L^\psi(V^*)$ but T_μ is not of Schatten ψ-class. In §4.4, we discuss the Herz type class of Toeplitz operators.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

2. Preliminaries

2.1. $L^{(\alpha)}$-harmonic functions and reproducing kernels. Throughout this paper, we denote by $X = (x,t)$, $Y = (y,s)$ and $Z = (z,r)$ points in $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times (0,\infty)$.

Let $0 < \alpha \le 1$. A continuous function u on $\boldsymbol{R}_+^{n+1}$ is said to be $L^{(\alpha)}$-harmonic, if $L^{(\alpha)}u = 0$ in the sense of distribution, i.e., $\iint u \cdot (L^{(\alpha)})^* \varphi \, dV = 0$ for every $\varphi \in C_c^\infty(\boldsymbol{R}_+^{n+1})$, where $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$,

$$(L^{(\alpha)})^* \varphi(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) - c_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y|>\delta} (\varphi(x+y,t) - \varphi(x,t))|y|^{-n-2\alpha} \, dy$$

and

$$c_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0.$$

In this paper, we use a fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ given by

$$W^{(\alpha)}(x,t) = \begin{cases} (2\pi)^{-n} \int_{\boldsymbol{R}^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1} x \cdot \xi) \, d\xi & t > 0 \\ 0 & t \le 0. \end{cases}$$

It has the following homogeneity:

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(s^{1/2\alpha}x, st) = s^{-((n+|\beta|)/2\alpha + k)} (\partial_x^\beta \partial_t^k W^{(\alpha)})(x,t), \tag{2}$$

where $\beta = (\beta_1, \ldots, \beta_n) \in \boldsymbol{N}_0^n$ is a multi-index and $k \ge 0$ is an integer. Here $\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$ denotes the set of all nonnegative integers. The following estimate is fundamental. There exists a constant $C > 0$ such that

$$|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)| \le C(t + |x|^{2\alpha})^{-((n+|\beta|)/2\alpha + k)} \tag{3}$$

for all $(x,t) \in \boldsymbol{R}_+^{n+1}$ (see [6]). For $1 \le p < \infty$, we denote by $\boldsymbol{b}_\alpha^p$ the p-th order parabolic Bergman space, i.e., the set of all p-th integrable $L^{(\alpha)}$-harmonic functions on $\boldsymbol{R}_+^{n+1}$.

Next we list some properties of α-parabolic Bergman kernels R_α and R_α^m for $m \in \boldsymbol{N}_0$. Recall that

$$R_\alpha(x,t;y,s) := -2\partial_t W^{(\alpha)}(x-y, t+s)$$

and

$$R_\alpha^m(x,t;y,s) := \frac{(-2)^m}{m!} s^m \partial_s^m R_\alpha(x,t;y,s) = \frac{(-2)^{m+1}}{m!} s^m \partial_t^{m+1} W^{(\alpha)}(x-y, t+s).$$

They have the following reproducing property: Let $m \ge 0$ and $1 \le p < \infty$. For every $u \in \boldsymbol{b}_\alpha^p$, $u = R_\alpha^m[u]$, i.e.,

$$u(X) = R_\alpha^m[u](X) := \int R_\alpha^m(X,Y) u(Y) \, dV(Y). \tag{4}$$

By (3), there exists a constant $C > 0$ such that

$$|R_\alpha^m(x,t;y,s)| \le C \bar{R}_\alpha^m(x,t;y,s), \tag{5}$$

where

$$\bar{R}^m_\alpha(x,t;y,s) := s^m(t+s+|x-y|^{2\alpha})^{-(n/2\alpha+1)-m}.$$

We consider the corresponding integral operator, which we also denote by $\bar{R}^m_\alpha$. These kernels have the following homogeneity property: For any $Z=(z,r)\in \boldsymbol{R}^{n+1}_+$,

$$R^m_\alpha(\Phi_Z(X),\Phi_Z(Y)) = r^{-(n/2\alpha+1)}R^m_\alpha(X,Y) \tag{6}$$

and

$$\bar{R}^m_\alpha(\Phi_Z(X),\Phi_Z(Y)) = r^{-(n/2\alpha+1)}\bar{R}^m_\alpha(X,Y), \tag{7}$$

where

$$\Phi_Z(x,t) := (r^{1/2\alpha}x+z, rt)$$

is the α-parabolic similarity on $\boldsymbol{R}^{n+1}_+$ with respect to $Z=(z,r)$ (see [10] and [11]). Thus by [11, Proposition 2], we have the following boundedness.

LEMMA 1. *Let* $1\le p<\infty$ *and* $m\ge 1$. *Then the integral operator* $\bar{R}^m_\alpha : L^p(V)\to L^p(V)$ *is bounded.*

We shall frequently use the following integrable estimate.

LEMMA 2. *Let* $\lambda,\tau\in\boldsymbol{R}$. *If* $-1<\lambda<\tau-\left(\frac{n}{2\alpha}+1\right)$, *then*

$$\int t^\lambda(1+t+|x|^{2\alpha})^{-\tau}\,dV(x,t)<\infty \tag{8}$$

and

$$\int t^\lambda(s+t+|x-y|^{2\alpha})^{-\tau}\,dV(x,t) = Cs^{\lambda-\tau+(n/2\alpha+1)} \tag{9}$$

with some constant $C>0$.

This lemma and the homogeneity (2) show that, if $m>\left(\frac{n}{2\alpha}+1\right)\left(\frac{1}{p}-1\right)$,

$$\|R^m_\alpha(\cdot,Y)\|_{L^p(V)} = Cs^{(n/2\alpha+1)(1/p-1)} \tag{10}$$

with some constant $C>0$ independent of $Y=(y,s)\in\boldsymbol{R}^{n+1}_+$.

2.2. Representation of parabolic Bergman functions. We shall show an atomic decomposition theorem for parabolic Bergman spaces $\boldsymbol{b}^p_\alpha$ for $1\le p<\infty$. We use this result for $p=2$ in the proof of our main theorem.

For $\delta>0$ and $Y=(y,s)\in\boldsymbol{R}^{n+1}_+$, we put

$$Q^{(\alpha)}_\delta(Y) := \left\{\Phi_Y(x_1,\dots,x_n,t);\, 1\le t\le 1+\delta, |x_j|\le\frac{\delta^{1/2\alpha}}{2}\ (j=1,\dots,n)\right\}.$$

Note that when $\delta = 1$, $Q_1^{(\alpha)}(Y) = Q^{(\alpha)}(Y)$, where

$$Q^{(\alpha)}(Y) := \left\{(x_1, \dots, x_n, t); s \le t \le 2s, |x_j - y_j| \le \frac{s^{1/2\alpha}}{2},\ \ j = 1, \dots, n\right\}$$

is the α-parabolic Carleson box which was used in our previous papers.

Now for $\kappa = (m_1, \dots, m_n, k) \in \boldsymbol{Z}^{n+1}$, we put

$$X_\kappa = (x_\kappa, t_\kappa) := (m_1\delta^{1/2\alpha}(1+\delta)^{k/2\alpha}, \dots, m_n\delta^{1/2\alpha}(1+\delta)^{k/2\alpha}, (1+\delta)^k)$$

and

$$Q_\kappa := Q_\delta^{(\alpha)}(X_\kappa).$$

We call $\{X_\kappa\}$ and $\{Q_\kappa\}$ the standard δ-lattice on $\boldsymbol{R}_+^{n+1}$ and the standard δ-decomposition of $\boldsymbol{R}_+^{n+1}$, respectively. Then $Q_\kappa = \Phi_{X_\kappa} Q_0$, where $Q_0 = Q_{(0,\dots,0,0)}$ and $V(Q_\kappa) = (\delta t_\kappa)^{n/2\alpha+1}$.

Let $m \ge 1$ and $1 \le p < \infty$. Let $l^p = l^p(\boldsymbol{Z}^{n+1})$. Consider the mappings $B_{p,\delta} : \boldsymbol{b}_\alpha^p \to l^p$ and $U_{p,\delta}^m : l^p \to \boldsymbol{b}_\alpha^p$ defined by

$$B_{p,\delta}[u] := (t_\kappa^{(n/2\alpha+1)(1/p)} u(X_\kappa))_\kappa$$

for $u \in \boldsymbol{b}_\alpha^p$, and

$$U_{p,\delta}^m[(\lambda_\kappa)_\kappa](X) := \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \lambda_\kappa R_\alpha^m(X, X_\kappa) t_\kappa^{(n/2\alpha+1)(1-1/p)}$$

for $(\lambda_\kappa)_\kappa \in l^p$, respectively. It is known that both operators are bounded ([9, Theorem 1 and Lemma 5]). Our atomic decomposition theorem is to assure that $U_{p,\delta}^m$ has a bounded right-inverse. Then $U_{p,\delta}^m$ would be surjective, and hence every element $u \in \boldsymbol{b}_\alpha^p$ can be represented by an (infinite) linear combination of atoms $\{R_\alpha^m(\cdot, X_\kappa)\}_\kappa$.

For our purpose, we consider the composition of two operators. Let

$$A_\delta^m[u](X) := \sum_{\kappa \in \boldsymbol{Z}^{n+1}} V(Q_\kappa) R_\alpha^m(X, X_\kappa) u(X_\kappa)$$

for $u \in \boldsymbol{b}_\alpha^p$. Then

$$A_\delta^m = \delta^{(n/2\alpha+1)} U_{p,\delta}^m B_{p,\delta},$$

so that $A_\delta^m : \boldsymbol{b}_\alpha^p \to \boldsymbol{b}_\alpha^p$ is bounded. Further we obtain

Proposition 1. *Let $1 \le p < \infty$ and $m \ge 1$. Then there exists $\delta_0 > 0$ such that A_δ^m is invertible for any $0 < \delta < \delta_0$.*

Proof. First we recall Lipschitz estimates of parabolic Bergman kernels (cf. [12, Proposition 3.2]). There exist constants $\delta_1 > 0$ and $C > 0$ such that for every $0 < \delta < \delta_1$ and for all $X, Y, Z \in \boldsymbol{R}_+^{n+1}$ with $Y \in Q_\delta^{(\alpha)}(Z)$, we have

$$|R_\alpha^m(X, Y) - R_\alpha^m(X, Z)| \le C(\delta + \delta^{1/2\alpha}) \bar{R}_\alpha^m(X, Y) \tag{11}$$

and

(12) $$|R_\alpha^m(Y,X) - R_\alpha^m(Z,X)| \le C(\delta + \delta^{1/2\alpha})\bar{R}_\alpha^m(Y,X).$$

These inequalities bring us

(13) $$|R_\alpha^m(X,X_\kappa)| \le C\bar{R}_\alpha^m(X,Y) \quad \text{and} \quad |R_\alpha^m(X_\kappa,X)| \le C\bar{R}_\alpha^m(Y,X)$$

for every $X \in \boldsymbol{R}_+^{n+1}$ and $Y \in Q_\kappa$, where $(X_\kappa)_\kappa$ is the standard δ-lattice and $(Q_\kappa)_\kappa$ is the standard δ-decomposition of $\boldsymbol{R}_+^{n+1}$, respectively.

Now we shall estimate the operator norm of $I - A_\delta^m$ for $0 < \delta < \delta_1$, where I is the identity on $\boldsymbol{b}_\alpha^p$. By the reproducing property of $u \in \boldsymbol{b}_\alpha^p$, we have

$$u(X) = \int R_\alpha^m(X,Y)u(Y)\,dV(Y) = \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \int_{Q_\kappa} R_\alpha^m(X,Y)u(Y)\,dV(Y),$$

so that (11) gives

$$\begin{aligned}
&\left| u(X) - \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \int_{Q_\kappa} R_\alpha^m(X,X_\kappa)u(Y)\,dV(Y) \right| \\
&\quad \le \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \int_{Q_\kappa} |R_\alpha^m(X,Y) - R_\alpha^m(X,X_\kappa)|\,|u(Y)|\,dV(Y) \\
&\quad \le C(\delta + \delta^{1/2\alpha}) \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \int_{Q_\kappa} \bar{R}_\alpha^m(X,Y)|u(Y)|\,dV(Y) \\
&\quad = C(\delta + \delta^{1/2\alpha})\bar{R}_\alpha^m[|u|](X).
\end{aligned}$$

Similarly, for any $Y \in Q_\kappa$, (12) gives

$$\begin{aligned}
|u(Y) - u(X_\kappa)| &= \left| \int (R_\alpha^m(Y,Z) - R_\alpha^m(X_\kappa,Z))u(Z)\,dV(Z) \right| \\
&\le C(\delta + \delta^{1/2\alpha}) \int \bar{R}_\alpha^m(Y,Z)|u(Z)|\,dV(Z) \\
&= C(\delta + \delta^{1/2\alpha})\bar{R}_\alpha^m[|u|](Y),
\end{aligned}$$

and hence by (13) we get

$$\begin{aligned}
&\left| \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \int_{Q_\kappa} R_\alpha^m(X,X_\kappa)u(Y)\,dV(Y) - A_\delta^m u(X) \right| \\
&\quad \le C \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \int_{Q_\kappa} \bar{R}_\alpha^m(X,X_\kappa)|u(Y) - u(X_\kappa)|\,dV(Y) \\
&\quad \le C(\delta + \delta^{1/2\alpha}) \int \bar{R}_\alpha^m(X,Y)\bar{R}_\alpha^m[|u|](Y)\,dV(Y) \\
&\quad = C(\delta + \delta^{1/2\alpha})\bar{R}_\alpha^m[\bar{R}_\alpha^m[|u|]](X).
\end{aligned}$$

Therefore

$$|u(X) - A_\delta^m u(X)| \le C(\delta + \delta^{1/2\alpha})\{\bar{R}_\alpha^m[|u|](X) + \bar{R}_\alpha^m[\bar{R}_\alpha^m[|u|]](X)\}.$$

By the L^p boundedness of $\bar{R}_\alpha^m$ in Lemma 1, we obtain

$$\|u - A_\delta^m u\|_{\boldsymbol{b}_\alpha^p} \le C(\delta + \delta^{1/2\alpha})\|u\|_{\boldsymbol{b}_\alpha^p}.$$

Hence $\|I - A_\delta^m\| < 1$ for sufficiently small $\delta > 0$, which implies that A_δ^m is invertible. □

Our atomic decomposition theorem is stated as follows.

THEOREM 2. *Let $1 \le p < \infty$ and $m \ge 1$. Then there exists $\delta_0 > 0$ such that $U_{p,\delta}^m$ has a bounded right-inverse for every $0 < \delta < \delta_0$. In particular $U_{p,\delta}^m : l^p \to \boldsymbol{b}_\alpha^p$ is bounded and surjective.*

Proof. Let $\delta_0 > 0$ be taken in Proposition 1. Since $A_\delta^m = \delta^{(n/2\alpha+1)} U_{p,\delta}^m B_{p,\delta} : \boldsymbol{b}_\alpha^p \to \boldsymbol{b}_\alpha^p$ is invertible for $0 < \delta < \delta_0$, $U_{p,\delta}^m(\delta^{(n/2\alpha+1)} B_{p,\delta}(A_\delta^m)^{-1})$ is the identity on $\boldsymbol{b}_\alpha^p$. □

Remark 1. (1) Let $g_\alpha := t^{-1/\alpha}|dx|^2 + t^{-2}\,dt^2$ be an invariant Riemannian metric under α-parabolic similarities $\{\Phi_X; X \in \boldsymbol{R}_+^{n+1}\}$. We denote by d_α the distance induced by g_α and by $B_\rho^{(\alpha)}(X)$ the geodesic ball with center $X \in \boldsymbol{R}_+^{n+1}$ and redius $\rho > 0$. We can obtain the following generalization of Theorem 2. Let $1 \le p < \infty$ and $m \ge 1$. For a sequence $(X_j)_j = ((x_j, t_j))_j$ in $\boldsymbol{R}_+^{n+1}$, we set

$$U_p^m[(\lambda_j)_j](X) := \sum_j \lambda_j R_\alpha^m(X, X_j) t_j^{(n/2\alpha+1)(1-1/p)} \quad ((\lambda_j)_j \in l^p).$$

If there exists a constant $\rho_0 > 0$ such that

$$\sup_j(\#\{k; X_k \in B_{\rho_0}^{(\alpha)}(X_j)\}) < \infty,$$

where $\#A$ denotes the number of elements of a set A, then the operator $U_p^m : l^p \to \boldsymbol{b}_\alpha^p$ is bounded. Furthermore, if ρ_0 is sufficiently small and

$$\bigcup_j B_{\rho_0}^{(\alpha)}(X_j) = \boldsymbol{R}_+^{n+1},$$

then U_p^m has a bounded right-inverse.

(2) In the above argument, we can take $m = 0$ when $1 < p < \infty$.

2.3. Schatten class. We recall some necessary properties of compact operators of Schatten class. Let T be a compact operator from a Hilbert space $\mathscr{H}_1$ to $\mathscr{H}_2$. Then there exists a nonincreasing sequence of nonnegative numbers $(\lambda_j)_j$

tending to 0 and there exist orthonormal systems $(h_j)_j$ in $\mathscr{H}_1$ and $(f_j)_j$ in $\mathscr{H}_2$ such that T can be represented

$$T = \sum_j \lambda_j \langle \cdot, h_j \rangle f_j \tag{14}$$

as a singular decomposition. Here λ_j is the j-th singular value of T, which is defined to be the j-th eigenvalue of the positive operator $|T| := \sqrt{T^*T}$ on $\mathscr{H}_1$. Note that our Toeplitz operator T_μ on $\boldsymbol{b}_\alpha^2$ is positive, so that $|T_\mu| = T_\mu$ holds.

Let $\psi : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism. For a compact operator T represented as above, T is said to be of Schatten ψ-class, if $(\lambda_j)_j \in l^\psi$, i.e., there exits a constant $\tau > 0$ such that

$$\sum_j \psi(\lambda_j/\tau) < \infty.$$

Then we denote $T \in \mathscr{S}^\psi = \mathscr{S}^\psi(\mathscr{H}_1, \mathscr{H}_2)$. When $\psi(t) = t^\sigma$, we write $\mathscr{S}^\sigma$ simply.

For every $r > 0$, we define

$$\|T\|_{r,\mathscr{S}^\psi} := \|(\lambda_j)_j\|_{r,l^\psi},$$

where

$$\|(\lambda_j)_j\|_{r,l^\psi} := \inf\left\{\tau > 0; \sum_j \psi(\lambda_j/\tau) \le r\right\}.$$

In particular, when $r = 1$, we write $\|T\|_{\mathscr{S}^\psi} = \|T\|_{1,\mathscr{S}^\psi}$ and $\|(\lambda_j)_j\|_{l^\psi} = \|(\lambda_j)_j\|_{1,l^\psi}$. Although $\|T\|_{r,\mathscr{S}^\psi}$ is not a norm in general (see Remark 2 (1) below), we sometimes call it Schatten norm (with respect to ψ and $r > 0$) for convenience' sake.

Note that by definition, $T \in \mathscr{S}^\psi$ if and only if $\|T\|_{r,\mathscr{S}^\psi} < \infty$ for some $r > 0$. However, since

$$\lim_{\tau \to \infty} \sum_j \psi(\lambda_j/\tau) = 0,$$

$\|T\|_{r,\mathscr{S}^\psi} < \infty$ holds for every $r > 0$. We now make two remarks.

Remark 2. Let $\psi : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism. Then there exist two positive constants a and b such that

$$\psi(t + s) \le a(\psi(bt) + \psi(bs))$$

holds. In fact, we may take $a = 1$ and $b = 2$ in general. Moreover if ψ is convex, we can take $a = 1/2$, $b = 2$, and if ψ is concave, we can take $a = b = 1$ (see Lemma 3 below). Let $\mathscr{S}^\psi = \mathscr{S}^\psi(\mathscr{H}_1, \mathscr{H}_2)$. By the same argument as in [11, Appendix] (see also [3]), we have:

(1) For $r > 0$ and $T_1, T_2 \in \mathscr{S}^\psi$, we have

$$\|T_1 + T_2\|_{2ar,\mathscr{S}^\psi} \le b(\|T_1\|_{r,\mathscr{S}^\psi} + \|T_2\|_{r,\mathscr{S}^\psi}),$$

so that $\mathscr{S}^\psi$ is a vector space.

(2) $\mathscr{S}^{\psi}$ is complete in the following sense: If $(T_k)_k$ is a sequence in $\mathscr{S}^{\psi}$ such that

$$\lim_{k,\ell\to\infty} \|T_k - T_\ell\|_{r,\mathscr{S}^{\psi}} = 0 \tag{15}$$

for every $r > 0$, then there exists $T \in \mathscr{S}^{\psi}$ such that $\lim_{k\to\infty} \|T_k - T\|_{r,\mathscr{S}^{\psi}} = 0$ for any $r > 0$.

(3) $\mathscr{S}^{\psi}$ is a both side ideal in the space of all bounded linear operators, i.e., if $T \in \mathscr{S}^{\psi}(\mathscr{H}_1, \mathscr{H}_2)$, $A \in \mathscr{L}(\mathscr{H}_0, \mathscr{H}_1)$ and $B \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_3)$, then $BTA \in \mathscr{S}^{\psi}(\mathscr{H}_0, \mathscr{H}_3)$ and $\|BTA\|_{r,\mathscr{S}^{\psi}} \le \|B\| \cdot \|T\|_{r,\mathscr{S}^{\psi}} \cdot \|A\|$ holds for every $r > 0$, where $\mathscr{L}$ denotes the space of all bounded linear operators.

(4) Let $T \in \mathscr{S}^{\psi}$. The min-max principle shows that the operator norm of T is equal to λ_0 (see [11, Appendix]), and hence $\|T\| \le \psi^{-1}(r)\|T\|_{r,\mathscr{S}^{\psi}}$ holds.

Remark 3. In general, Schatten norms $\|\cdot\|_{r,\mathscr{S}^{\psi}}$ are not comparable with respect to $r > 0$. However, when $\psi(t) = t^{\sigma}$ $(\sigma > 0)$ we have

$$\|\cdot\|_{r,\mathscr{S}^{\sigma}} = \frac{1}{r^{1/\sigma}} \|\cdot\|_{\mathscr{S}^{\sigma}}$$

for $r > 0$. Hence in the case $\psi(t) = t^{\sigma}$, we only consider the simple one $\|\cdot\|_{\mathscr{S}^{\sigma}}$.

Let Φ be the set of all concave and strictly increasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. In this paper, we mainly treat the Schatten class operators for $\psi \in \Phi$. A typical example is $\psi(t) = t^{\sigma}$ with $0 < \sigma \le 1$. We recall a fundamental property of functions in Φ.

LEMMA 3. *Let $\psi \in \Phi$. Then for any sequence $(a_j)_j$ of positive numbers, we have*

$$\psi\left(\sum_j a_j\right) \le \sum_j \psi(a_j). \tag{16}$$

Moreover we have

$$\psi(s_0 t) \le s_0 \psi(t) \tag{17}$$

for every $s_0 \ge 1$ and $t > 0$.

Proof. It is sufficient to show that $\psi(t+s) \le \psi(t) + \psi(s)$ for $t, s > 0$. Since ψ is concave and $\psi(0) = 0$, we see

$$\frac{t}{t+s}\psi(t+s) \le \psi(t) \quad \text{and} \quad \frac{s}{t+s}\psi(t+s) \le \psi(s),$$

from which the claim $\psi(t+s) \le \psi(t) + \psi(s)$ follows. □

Lemma 4. *Let T be a linear operator from $\mathscr{H}_1$ to $\mathscr{H}_2$, $(e_j^k)_j$ a complete orthonormal system on $\mathscr{H}_k$ where $k = 1, 2$ and $\psi(t) = \phi(t^2)$ for some $\phi \in \Phi$. If $(\langle Te_j^1, e_k^2 \rangle)_{jk} \in l^\psi$, then $T \in \mathscr{S}^\psi$ and*

$$\|T\|_{r,\mathscr{S}^\psi} \leq \|(\langle Te_j^1, e_k^2 \rangle)_{jk}\|_{r,l^\psi}$$

for every $r > 0$. Moreover, the infimum of the right hand side over all complete orthonormal systems is equal to the left hand side.

Proof. Let $r > 0$ be fixed and take $\tau > 0$ arbitrary such that

$$\sum_{j,k} \psi\left(\frac{|\langle Te_j^1, e_k^2 \rangle|}{\tau}\right) \leq r.$$

Since $\phi(\sum_{j,k} |\langle Te_j^1, e_k^2 \rangle|^2/\tau^2) \leq \sum_{j,k} \phi(|\langle Te_j^1, e_k^2 \rangle|^2/\tau^2) \leq r$, we have

$$\sum_{j,k} |\langle Te_j^1, e_k^2 \rangle|^2 = \tau^2 \phi^{-1}(r) < \infty.$$

This means that T is a Hilbert-Schmidt operator, so it is compact. Using the singular decomposition

$$T = \sum_j \lambda_j \langle \cdot, h_j \rangle f_j,$$

we have

$$\sum_k \left(\frac{|\langle Te_j^1, e_k^2 \rangle|}{\tau}\right)^2 = \left\|\frac{Te_j^1}{\tau}\right\|^2 = \sum_m \left|\frac{\lambda_m}{\tau} \langle e_j^1, h_m \rangle\right|^2.$$

Since $\sum_m |\langle e_j^1, h_m \rangle|^2 \leq 1$ and ϕ is concave,

$$\phi\left(\left\|\frac{Te_j^1}{\tau}\right\|^2\right) \geq \sum_m \phi\left(\left|\frac{\lambda_m}{\tau}\right|^2\right) |\langle e_j^1, h_m \rangle|^2,$$

which implies

$$r \geq \sum_j \sum_m \phi\left(\left|\frac{\lambda_m}{\tau}\right|^2\right) |\langle e_j^1, h_m \rangle|^2 = \sum_m \phi\left(\left|\frac{\lambda_m}{\tau}\right|^2\right) = \sum_m \psi\left(\frac{\lambda_m}{\tau}\right).$$

Thus we find that $T \in \mathscr{S}^\psi$ and $\|T\|_{r,\mathscr{S}^\psi} \leq \tau$. For the last claim, we have only to choose complete orthonormal systems which contain $(h_j)_j$ and $(f_j)_j$. □

When $\psi(t) = t^\sigma$, we have the following.

Corollary 1. *Let $0 < \sigma \leq 2$. Let T be a linear operator on a Hilbert space $\mathscr{H}$ and $(e_j)_j$ be a complete orthonormal system on $\mathscr{H}$. If $\sum_{j,k} |\langle Te_j, e_k \rangle|^\sigma < \infty$, then $T \in \mathscr{S}^\sigma$ and*

$$\|T\|_{\mathscr{S}^\sigma} \leq \left(\sum_{j,k} |\langle Te_j, e_k \rangle|^\sigma\right)^{1/\sigma}.$$

Lemma 5. *Let $\psi \in \Phi$ and T be a positive operator on a Hilbert space. If $(\langle Te_j, e_j \rangle)_j \in l^{\psi}$ for some complete orthonormal system $(e_j)_j$, then $T \in \mathscr{S}^{\psi}$ and*

$$\|T\|_{r,\mathscr{S}^{\psi}} \leq \|(\langle Te_j, e_j \rangle)_j\|_{r,l^{\psi}}$$

for every $r > 0$. Moreover, the infimum of the right hand side over all complete orthonormal systems is equal to the left hand side.

Proof. Let $r > 0$ be fixed and take $\tau > 0$ arbitrary such that

$$\sum_j \psi\left(\frac{\langle Te_j, e_j \rangle}{\tau}\right) \leq r.$$

By using the spectral decomposition of T:

$$T = \int_0^{\infty} \lambda \, dE(\lambda)$$

and the Jensen inequality, we have

$$\psi\left(\frac{\langle Te_j, e_j \rangle}{\tau}\right) \geq \int_0^{\infty} \psi\left(\frac{\lambda}{\tau}\right) d\|E(\lambda)e_j\|^2 = \left\langle \psi\left(\frac{T}{\tau}\right) e_j, e_j \right\rangle,$$

and hence,

$$\operatorname{tr}\left[\psi\left(\frac{T}{\tau}\right)\right] = \sum_j \left\langle \psi\left(\frac{T}{\tau}\right) e_j, e_j \right\rangle \leq \sum_j \psi\left(\frac{\langle Te_j, e_j \rangle}{\tau}\right) \leq r < \infty,$$

where $\operatorname{tr}[\psi(T/\tau))]$ denotes the trace of an operator $\psi(T/\tau)$. This shows that $\psi(T/\tau)$ is compact, so that $T = \tau\psi^{-1}(\psi(T/\tau))$ is also compact. Let $(\lambda_j)_j$ be the sequence of all singular values (= eigenvalues) of T. Then

$$\sum_j \psi\left(\frac{\lambda_j}{\tau}\right) \leq \sum_j \psi\left(\frac{\langle Te_j, e_j \rangle}{\tau}\right).$$

When we consider the case that $(e_j)_j$ is the normalized eigenvectors of $(\lambda_j)_j$, the last claim follows. □

Corollary 2. *Let T and S be positive operators on a Hilbert space with $S \leq T$ and $\psi \in \Phi$. If $T \in \mathscr{S}^{\psi}$, then $S \in \mathscr{S}^{\psi}$ and*

$$\|S\|_{r,\mathscr{S}^{\psi}} \leq \|T\|_{r,\mathscr{S}^{\psi}}$$

for every $r > 0$.

In fact, for any complete orthonormal system $(e_j)_j$, we have $\langle Se_j, e_j \rangle \leq \langle Te_j, e_j \rangle$, so that the assertion follows from Lemma 5 immediately.

3. Proof of Theorem 1

In this section, we shall prove our main theorem. First, we introduce auxiliary functions. Let $\delta > 0$ and let $\mu \geq 0$ be a Radon measure on $\boldsymbol{R}_+^{n+1}$. For $Y \in \boldsymbol{R}_+^{n+1}$, we put

$$\hat{\mu}_\delta^{(\alpha)}(Y) := \frac{\mu(Q_\delta^{(\alpha)}(Y))}{V(Q_\delta^{(\alpha)}(Y))}.$$

This function is closely related to the original averaging function $\hat{\mu}^{(\alpha)} (= \hat{\mu}_1^{(\alpha)})$. In fact, the following assertion holds.

Lemma 6. *Let $\delta > 0$ and $0 < \sigma \leq 1$. For a Radon measure $\mu \geq 0$ on $\boldsymbol{R}_+^{n+1}$, the following three conditions are equivalent*:

$$\hat{\mu}^{(\alpha)} \in L^\sigma(V^*), \quad \hat{\mu}_\delta^{(\alpha)} \in L^\sigma(V^*) \quad \textit{and} \quad (\hat{\mu}_\delta^{(\alpha)}(X_\kappa))_\kappa \in l^\sigma,$$

where $(X_\kappa)_\kappa$ is the standard δ-lattice on $\boldsymbol{R}_+^{n+1}$. To be accurate, the values

$$\int \hat{\mu}^{(\alpha)}(X)^\sigma \, dV^*(X), \quad \int \hat{\mu}_\delta^{(\alpha)}(X)^\sigma \, dV^*(X), \quad \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \hat{\mu}_\delta^{(\alpha)}(X_\kappa)^\sigma$$

are comparable to one another.

Proof. The equivalence $\hat{\mu}^{(\alpha)} \in L^\sigma(V^*) \Leftrightarrow \hat{\mu}_\delta^{(\alpha)} \in L^\sigma(V^*)$ can be shown similarly to [11, Lemma 1].

To show the implication $(\hat{\mu}_\delta^{(\alpha)}(X_\kappa))_\kappa \in l^\sigma \Rightarrow \hat{\mu}_\delta^{(\alpha)} \in L^\sigma(V^*)$, let $(Q_\kappa)_\kappa$ be the standard δ-decomposition of $\boldsymbol{R}_+^{n+1}$. We remark that

$$N_1 := \sup_\kappa \#\{\nu \in \boldsymbol{Z}^{n+1}; \exists X \in Q_\nu \text{ such that } Q_\delta^{(\alpha)}(X) \cap Q_\kappa \neq \emptyset\} < \infty.$$

For each $\nu \in \boldsymbol{Z}^{n+1}$, we put

$$\tilde{Q}_\nu := \bigcup_{X \in Q_\nu} Q_\delta^{(\alpha)}(X)$$

and set $K_\nu := \{\kappa \in \boldsymbol{Z}^{n+1}; \tilde{Q}_\nu \cap Q_\kappa \neq \emptyset\}$. Since for every $X \in Q_\nu$,

$$\mu(Q_\delta^{(\alpha)}(X)) \leq \sum_{\kappa \in K_\nu} \mu(Q_\kappa), \quad \text{i.e.,} \quad \hat{\mu}_\delta^{(\alpha)}(X)^\sigma \leq C \sum_{\kappa \in K_\nu} \hat{\mu}_\delta^{(\alpha)}(X_\kappa)^\sigma,$$

we have

$$\frac{1}{V(Q_\nu)} \int_{Q_\nu} \hat{\mu}_\delta^{(\alpha)}(X)^\sigma \, dV(X) \leq C \sum_{\kappa \in K_\nu} \hat{\mu}_\delta^{(\alpha)}(X_\kappa)^\sigma,$$

which implies

$$\int \hat{\mu}_\delta^{(\alpha)}(X)^\sigma \, dV^*(X) = \sum_{\nu \in \boldsymbol{Z}^{n+1}} \int_{Q_\nu} \hat{\mu}_\delta^{(\alpha)}(X)^\sigma \, dV^*(X) \le C \sum_{\nu \in \boldsymbol{Z}^{n+1}} \sum_{\kappa \in K_\nu} \hat{\mu}_\delta^{(\alpha)}(X_\kappa)^\sigma$$
$$\le CN_1 \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \hat{\mu}_\delta^{(\alpha)}(X_\kappa)^\sigma.$$

To show the opposite implication, let $\delta' > \max\{\delta + \delta(1+\delta), \delta + \delta(1+\delta)^{1/2\alpha}\}$. Then for any κ, there exists ν such that

$$Q_\kappa \subset \bigcap_{X \in Q_\nu} Q_{\delta'}^{(\alpha)}(X),$$

and hence we have

$$\hat{\mu}_\delta^{(\alpha)}(X_\kappa)^\sigma \le \frac{C}{V(Q_\nu)} \int_{Q_\nu} \hat{\mu}_{\delta'}^{(\alpha)}(X)^\sigma \, dV(X) \le C \int_{Q_\nu} \hat{\mu}_{\delta'}^{(\alpha)}(X)^\sigma \, dV^*(X).$$

Since

$$N_2 := \sup_\nu \#\{\kappa \in \boldsymbol{Z}^{n+1}; Q_\nu \cap Q_\kappa\} < \infty,$$

we obtain

$$\sum_{\kappa \in \boldsymbol{Z}^{n+1}} \hat{\mu}_\delta^{(\alpha)}(X_\kappa)^\sigma \le CN_2 \int \hat{\mu}_{\delta'}^{(\alpha)}(X)^\sigma \, dV^*(X),$$

which completes the proof. □

By the similar argument, we have the above assertion for general ψ which is an increasing homeomorphism on $[0, \infty)$. Here we recall the definition of the Orlicz space $L^\psi(V_\eta)$. For $\eta \in \boldsymbol{R}^n$, we set $dV_\eta(X) = t^\eta \, dV(X)$. A Borel measurable function f on $\boldsymbol{R}^{n+1}$ belongs to $L^\psi(V_\eta)$, if there exists $\tau > 0$ such that

$$\int \psi\left(\frac{|f|(X)}{\tau}\right) dV_\eta < \infty.$$

For $f \in L^\psi(V_\eta)$ and $r > 0$, we set

$$\|f\|_{r, L^\psi(V_\eta)} := \inf\left\{\tau > 0; \int \psi\left(\frac{|f|(X)}{\tau}\right) dV_\eta(X) \le r\right\}.$$

Note that $V^* = V_{-(n/2\alpha+1)}$ and if $\psi(t) = t^\sigma$ $(\sigma > 0)$, then $L^{t^\sigma}(V_\eta)$ is nothing but the usual $L^\sigma(V_\eta)$ and

$$\|f\|_{r, L^{t^\sigma}(V_\eta)} = \left(\int |f(X)|^\sigma \, dV_\eta(X)\right)^{1/\sigma}$$

holds.

Remark 4. If $\hat{\mu}^{(\alpha)} \in L^{\psi}(V^*)$, then μ satisfies the growth condition (1) with $\tau > \dfrac{n}{2\alpha} + 1$. Moreover, if $\psi = t^{\sigma}$ $(0 < \sigma \le 1)$, we can choose $\tau = \dfrac{n}{2\alpha} + 1$. In fact, from the above lemma, if $\hat{\mu}^{(\alpha)} \in L^{\sigma}(V^*)$, then $(\hat{\mu}_{\delta}^{(\alpha)}(X_{\kappa}))_{\kappa} \in l^{\sigma}$, i.e.,

$$\sum_{\kappa \in \boldsymbol{Z}^{n+1}} \left(\int_{Q_{\kappa}} t_{\kappa}^{-(n/2\alpha+1)} \, d\mu(X) \right)^{\sigma} < \infty,$$

which implies

$$\left(\int (1 + t + |x|^{2\alpha})^{-\tau} \, d\mu(X) \right)^{\sigma} \le \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \left(\int_{Q_{\kappa}} t^{-(n/2\alpha+1)} \, d\mu(X) \right)^{\sigma} < \infty,$$

if $0 < \sigma \le 1$ and $\tau \ge \dfrac{n}{2\alpha} + 1$. For general ψ, the condition $\hat{\mu}^{(\alpha)} \in L^{\psi}(V^*)$ yields the boundedness of the following sequence

$$\left(\int_{Q_{\kappa}} t^{-(n/2\alpha+1)} d\mu(X) \right)_{\kappa},$$

which gives the inequality

$$\int (1 + t + |x|^{2\alpha})^{-\tau} \, d\mu(x,t) \le C \int (1 + t + |x|^{2\alpha})^{-\tau} \, dV(x,t).$$

The integral of the right hand side is finite if $\tau > \dfrac{n}{2\alpha} + 1$ by Lemma 2.

Proof of Theorem 1. Let $0 < \sigma < 1$ and take $m \ge 1$ such that

$$m\sigma + \left(\frac{n}{2\alpha} + 1 \right) \left(\frac{\sigma}{2} - 1 \right) > -1. \tag{18}$$

For $\delta > 0$, let $(Q_{\kappa})_{\kappa}$ be the standard δ-decomposition and $(X_{\kappa})_{\kappa}$ be the standard δ-lattice of $\boldsymbol{R}_{+}^{n+1}$ and put

$$g_{\kappa}(X) := R_{\alpha}^{m}(X, X_{\kappa}) t_{\kappa}^{(1/2)(n/2\alpha+1)}.$$

Now suppose $\hat{\mu}^{(\alpha)} \in L^{\sigma}(V^*)$. First we shall show

$$\sum_{j,k \in \boldsymbol{Z}^{n+1}} |\langle T_{\mu} g_j, g_k \rangle|^{\sigma} \le C \sum_{l \in \boldsymbol{Z}^{n+1}} \hat{\mu}_{\delta}(X_l)^{\sigma}. \tag{19}$$

Note that the right hand side is finite by Lemma 6. The condition (1) guarantees

$$|\langle T_{\mu} g_j, g_k \rangle| = \left| \int g_j g_k \, d\mu \right| \le \int |g_j g_k| \, d\mu.$$

To show (19), it is sufficient to prove

$$\sum_{j,k \in \boldsymbol{Z}^{n+1}} \left(\int |g_j g_k| \, d\mu \right)^{\sigma} \le C \sum_{l \in \boldsymbol{Z}^{n+1}} \hat{\mu}_{\delta}(X_l)^{\sigma}.$$

By (13) we have

$$\begin{aligned}\int |g_j g_k| \, d\mu &= \sum_{l \in \boldsymbol{Z}^{n+1}} \int_{Q_l} |R_{\alpha}^m(X, X_j) R_{\alpha}^m(X, X_k)| t_j^{(1/2)(n/2\alpha+1)} t_k^{(1/2)(n/2\alpha+1)} \, d\mu(X) \\ &\le C \sum_{l \in \boldsymbol{Z}^{n+1}} \bar{R}_{\alpha}^m(X_l, X_j) \bar{R}_{\alpha}^m(X_l, X_k) t_j^{(1/2)(n/2\alpha+1)} t_k^{(1/2)(n/2\alpha+1)} \mu(Q_l).\end{aligned}$$

Hence $\mu(Q_l) = C t_l^{(n/2\alpha+1)} \hat{\mu}_{\delta}(X_l)$ implies

$$\begin{aligned}&\sum_{j,k \in \boldsymbol{Z}^{n+1}} \left(\int |g_j g_k| \, d\mu \right)^{\sigma} \\ &\quad \le C \sum_{j,k \in \boldsymbol{Z}^{n+1}} \sum_{l \in \boldsymbol{Z}^{n+1}} \bar{R}_{\alpha}^m(X_l, X_j)^{\sigma} \bar{R}_{\alpha}^m(X_l, X_k)^{\sigma} t_j^{(\sigma/2)(n/2\alpha+1)} t_k^{(\sigma/2)(n/2\alpha+1)} \mu(Q_l)^{\sigma} \\ &\quad = C \sum_{l \in \boldsymbol{Z}^{n+1}} \left(\sum_{j \in \boldsymbol{Z}^{n+1}} \bar{R}_{\alpha}^m(X_l, X_j)^{\sigma} t_j^{(\sigma/2)(n/2\alpha+1)} \right)^2 \mu(Q_l)^{\sigma} \\ &\quad \le C \sum_{l \in \boldsymbol{Z}^{n+1}} \hat{\mu}_{\delta}(X_l)^{\sigma},\end{aligned}$$

because

$$\begin{aligned}&\sum_{j \in \boldsymbol{Z}^{n+1}} \bar{R}_{\alpha}^m(X_l, X_j)^{\sigma} t_j^{(\sigma/2)(n/2\alpha+1)} \\ &\quad = C \sum_{j \in \boldsymbol{Z}^{n+1}} t_j^{m\sigma + (n/2\alpha+1)(\sigma/2-1)} (t_l + t_j + |x_l - x_j|^{2\alpha})^{-(n/2\alpha+1+m)\sigma} V(Q_j) \\ &\quad \le C \int s^{m\sigma + (n/2\alpha+1)(\sigma/2-1)} (t_l + s + |x_l - y|^{2\alpha})^{-(n/2\alpha+1+m)\sigma} \, dV(y, s) \\ &\quad = t_l^{-(n/2\alpha+1)(\sigma/2)}\end{aligned}$$

by Lemma 2. In fact, by (18), $\lambda := m\sigma + \left(\dfrac{n}{2\alpha} + 1\right)\left(\dfrac{\sigma}{2} - 1\right)$ and $\tau := \left(\dfrac{n}{2\alpha} + 1 + m\right)\sigma$ satisfy the integrability conditions in Lemma 2. Hence (19) is estabilshed.

Let $e_j \in l^2$ be the j-th unit element, where $l^2 = l^2(\boldsymbol{Z}^{n+1})$. Then $(e_j)_j$ is a complete orthonormal system of l^2. Considering the operator $U = U_{2,\delta}^m : l^2 \to \boldsymbol{b}_{\alpha}^2$, we can write

$$\langle T_{\mu} g_j, g_k \rangle = \langle T_{\mu} U[e_j], U[e_k] \rangle = \langle U^* T_{\mu} U[e_j], e_k \rangle,$$

so that an operator $U^*T_\mu U$ on l^2 belongs to Schatten σ-class by Corollary 1 and (19). Hence if $\delta > 0$ is taken small enough, Theorem 2 ensures us that U has a bounded right-inverse, which we denote by $\tilde{B}$. Then we have

$$T_\mu = \tilde{B}^* U^* T_\mu U \tilde{B} \in \mathcal{S}^\sigma.$$

Next, we assume $T_\mu \in \mathcal{S}^\sigma$. For $\delta > 0$, let $(Q_\kappa)_\kappa$ be the standard δ-decomposition and $(X_\kappa)_\kappa$ be the standard δ-lattice in $\boldsymbol{R}_+^{n+1}$ again. For a given $M > 0$, we divide $\boldsymbol{Z}^{n+1} = \{\kappa\}$ into a finite union $\{\kappa\} = \bigcup_\nu \{\kappa_{\nu,j}\}_j$ such that

$$d_\alpha(Q_{\kappa_{\nu,j}}, Q_{\kappa_{\nu,k}}) > M$$

whenever $j \neq k$. Here d_α is the distance we used in Remark 1. We denote by $B_\rho(X_0)$ the geodesic ball with center $X_0 = (0,1)$ and radius $\rho > 0$. Now take ν arbitrarily and fix a positive number ρ. Writing $\mu_\kappa := \chi_{Q_\kappa}\mu$, where χ_{Q_κ} is the characteristic function, we set

$$\mu_\nu := \sum_{j \in F_{\nu,\rho}} \mu_{\kappa_{\nu,j}},$$

where $F_{\nu,\rho} := \{j \in \boldsymbol{Z}^{n+1}; X_{\kappa_{\nu,j}} \in B_\rho(X_0)\}$. Then $T_{\mu_\nu} \in \mathcal{S}^\sigma$ and $\|T_{\mu_\nu}\|_{\mathcal{S}^\sigma} \leq \|T_\mu\|_{\mathcal{S}^\sigma}$ by Corollary 2, because

$$\langle T_{\mu_\nu} u, u\rangle = \int |u|^2 \, d\mu_\nu \leq \int |u|^2 \, d\mu = \langle T_\mu u, u\rangle$$

for every $u \in \boldsymbol{b}_\alpha^2$.

Since the operator $U = U_{2,\delta}^m : l^2 \to \boldsymbol{b}_\alpha^2$ is bounded, the matrix

$$T_\nu := (\langle T_{\mu_\nu} g_j, g_k\rangle)_{jk} = (\langle U^* T_\mu U[e_j], e_k\rangle)_{jk}$$

defines an operator $U^* T_{\mu_\nu} U$ on $\mathcal{S}^\sigma(l^2)$, where $g_\kappa(X) := R_\alpha^m(X, X_\kappa) t_\kappa^{(1/2)(n/2\alpha+1)}$ as above. We divide T_ν into two parts, the diagonal part D_ν of T_ν and the off-diagonal part $E_\nu := T_\nu - D_\nu$.

First we estimate the norm of D_ν from below. If we take $\delta > 0$ small enough, then $|R_\alpha^m(X, X_\kappa)| \geq C R_\alpha^m(X_\kappa, X_\kappa)$ for $X \in Q_\kappa$. We fix such a $\delta > 0$. Then

$$\begin{aligned}
\|D_\nu\|_{\mathcal{S}^\sigma(l^2)}^\sigma &= \sum_{j \in F_{\nu,\rho}} |\langle T_\nu g_{\kappa_{\nu,j}}, g_{\kappa_{\nu,j}}\rangle|^\sigma = \sum_{j \in F_{\nu,\rho}} \left(\int |g_{\kappa_{\nu,j}}|^2 \, d\mu_\nu \right)^\sigma \\
&\geq \sum_{j \in F_{\nu,\rho}} \left(\int_{Q_{\kappa_{\nu,j}}} R_\alpha^m(X, X_{\kappa_{\nu,j}})^2 t_{\kappa_{\nu,j}}^{(n/2\alpha+1)} \, d\mu_\nu \right)^\sigma \\
&\geq C \sum_{j \in F_{\nu,\rho}} (R_\alpha^m(X_{\kappa_{\nu,j}}, X_{\kappa_{\nu,j}})^2 t_{\kappa_{\nu,j}}^{(n/2\alpha+1)} \mu_\nu(Q_{\kappa_{\nu,j}}))^\sigma \\
&= C R_\alpha^m(X_0, X_0)^{2\sigma} \sum_{j \in F_{\nu,\rho}} \hat{\mu}_\delta^{(\alpha)}(X_{\kappa_{\nu,j}})^\sigma.
\end{aligned}$$

Next we estimate the norm of E_ν from above. We have

$$\|E_\nu\|^\sigma_{\mathscr{S}^\sigma(l^2)} = \sum_{k\neq l, k\in F_{\nu,\rho}} |\langle T_{\mu_\nu} g_{\kappa_{\nu,k}}, g_{\kappa_{\nu,l}}\rangle|^\sigma = \sum_{k\neq l, k\in F_{\nu,\rho}} \left|\int g_{\kappa_{\nu,k}} g_{\kappa_{\nu,l}}\, d\mu_\nu\right|^\sigma$$
$$\leq C \sum_{k\neq l, k\in F_{\nu,\rho}} \sum_{j\in F_{\nu,\rho}} [\mu(Q_{\kappa_{\nu,j}}) \overline{R}^m_\alpha(X_{\kappa_{\nu,j}}, X_{\kappa_{\nu,k}})$$
$$\times \overline{R}^m_\alpha(X_{\kappa_{\nu,j}}, X_{\kappa_{\nu,l}}) t^{(1/2)(n/2\alpha+1)}_{\kappa_{\nu,k}} t^{(1/2)(n/2\alpha+1)}_{\kappa_{\nu,l}}]^\sigma$$
$$\leq C \sum_{j\in F_{\nu,\rho}} [\mu(Q_{\kappa_{\nu,j}})^\sigma \times \iint_{d_\alpha(Y,Z)>M} \overline{R}^m_\alpha(X_{\kappa_{\nu,j}}, Y)^\sigma \overline{R}^m_\alpha(X_{\kappa_{\nu,j}}, Z)^\sigma$$
$$\times s^{(\sigma/2-1)(n/2\alpha+1)} r^{(\sigma/2-1)(n/2\alpha+1)}\, dV(Y)\, dV(Z)].$$

Here changing variables $Y = \Phi_{X_{\kappa_{\nu,j}}}(\tilde{Y})$ and $Z = \Phi_{X_{\kappa_{\nu,j}}}(\tilde{Z})$, we have

$$\iint_{d_\alpha(Y,Z)>M} \overline{R}^m_\alpha(X_{\kappa_{\nu,j}}, Y)^\sigma \overline{R}^m_\alpha(X_{\kappa_{\nu,j}}, Z)^\sigma s^{(\sigma/2-1)(n/2\alpha+1)} r^{(\sigma/2-1)(n/2\alpha+1)}\, dV(Y)\, dV(Z)$$
$$= c_M t^{-\sigma(n/2\alpha+1)}_{\kappa_{\nu,j}} = \delta^{\sigma(n/2\alpha+1)} c_M V(Q_{\kappa_{\nu,j}})^{-\sigma},$$

where

$$c_M = \iint_{d_\alpha(\tilde{Y},\tilde{Z})>M} \overline{R}^m_\alpha(X_0, \tilde{Y})^\sigma \overline{R}^m_\alpha(X_0, \tilde{Z})^\sigma$$
$$\times \tilde{s}^{(\sigma/2-1)(n/2\alpha+1)} \tilde{r}^{(\sigma/2-1)(n/2\alpha+1)}\, dV(\tilde{Y})\, dV(\tilde{Z}).$$

By (18) and Lemma 2, we have

$$\iint \overline{R}^m_\alpha(X_0, \tilde{Y})^\sigma \overline{R}^m_\alpha(X_0, \tilde{Z})^\sigma \tilde{s}^{(\sigma/2-1)(n/2\alpha+1)} \tilde{r}^{(\sigma/2-1)(n/2\alpha+1)}\, dV(\tilde{Y})\, dV(\tilde{Z})$$
$$= \left(\int \overline{R}^m_\alpha(X_0, \tilde{Y})^\sigma \tilde{s}^{(\sigma/2-1)(n/2\alpha+1)}\, dV(\tilde{Y})\right)^2 < \infty,$$

so that the constant c_M can be taken arbitrarily small, if we choose $M > 0$ large enough. In this way, we have

$$\|E_\nu\|^\sigma_{\mathscr{S}^\sigma(l^2)} = \sum_{k\neq l, k\in F_{\nu,\rho}} |\langle T_\nu g_{\kappa_{\nu,k}}, g_{\kappa_{\nu,l}}\rangle|^\sigma \leq C\delta^{\sigma(n/2\alpha+1)} c_M \sum_{j\in F_{\nu,\rho}} \hat{\mu}^{(\alpha)}_\delta(X_{\kappa_{\nu,j}})^\sigma < \infty,$$

which shows

$$(C - C\delta^{\sigma(n/2\alpha+1)} c_M) \sum_{j\in F_{\nu,\rho}} \hat{\mu}^{(\alpha)}_\delta(X_{\kappa_{\nu,j}})^\sigma \leq \|D_\nu\|^\sigma_{\mathscr{S}^\sigma(l^2)} - \|E_\nu\|^\sigma_{\mathscr{S}^\sigma(l^2)} \leq \|T_\nu\|^\sigma_{\mathscr{S}^\sigma(l^2)}.$$

Since $\|T_\nu\|_{\mathscr{S}^\sigma(l^2)} = \|U^* T_{\mu_\nu} U\|_{\mathscr{S}^\sigma(l^2)} \le \|U\|^2 \|T_\mu\|_{\mathscr{S}^\sigma}$, $\sum_{j \in F_{\nu,\rho}} \hat{\mu}_\delta^{(\alpha)}(X_{\kappa_{\nu,j}})^\sigma \le C\|T_\mu\|_{\mathscr{S}^\sigma}^\sigma$ holds. Hence letting $\rho \to \infty$ and taking the finite summation over ν, we have

$$\sum_{\kappa \in \boldsymbol{Z}^{n+1}} \hat{\mu}_\delta^{(\alpha)}(X_\kappa)^\sigma \le C\|T_\mu\|_{\mathscr{S}^\sigma}^\sigma.$$

Lemma 6 shows $\hat{\mu}^{(\alpha)} \in L^\sigma(V^*)$, which completes the proof. □

4. Remarks

4.1. Carleson inclusion of Schatten class. We discuss the Carleson inclusions on $\boldsymbol{b}_\alpha^2$. Let μ be a positive Radon maesure on $\boldsymbol{R}_+^{n+1}$ satisfying (1). The inclusion map $i_\mu : \boldsymbol{b}_\alpha^2 \mapsto L^2(\mu)$ is called a Carleson inclusion. It was shown in [8] that i_μ is compact if and only if $T_\mu : \boldsymbol{b}_\alpha^2 \mapsto \boldsymbol{b}_\alpha^2$ is comapct. Moreover, in this case,

$$T_\mu = i_\mu^* \cdot i_\mu$$

holds, where $i_\mu^* : L^2(\mu) \mapsto \boldsymbol{b}_\alpha^2$ is the adjoint of i_μ. On the other hand, by defintion, the Schatten norm of i_μ is that of $\sqrt{i_\mu^* \cdot i_\mu}$. Hence

$$\|i_\mu\|_{\mathscr{S}^\sigma} = \|T_\mu\|_{\mathscr{S}^{\sigma/2}}^{1/2}.$$

This gives us the following consequence.

Corollary 3. *Let $\sigma > 0$. For a Radon measure $\mu \ge 0$ on $\boldsymbol{R}_+^{n+1}$ satisfying* (1), *the Carleson inclusion i_μ on $\boldsymbol{b}_\alpha^2$ into $L^2(\mu)$ is of Schatten σ-class if and only if $\hat{\mu}^{(\alpha)} \in L^{\sigma/2}(V^*)$.*

4.2. Relation between averaging functions and Berezin transformations. As for a relation between averaging functions and Berezin transformations, we shall show the following.

Proposition 2. *Let $\sigma > n/(n+2\alpha)$. Then $\hat{\mu}^{(\alpha)} \in L^\sigma(V^*)$ if and only if $\tilde{\mu}^{(\alpha)} \in L^\sigma(V^*)$.*

We showed this equivalence in [11, Theorem 1] for $\sigma \ge 1$. Proposition 2 will be proved along the almost similar way to the case $\sigma \ge 1$, but some modifications are necessary.

First, we recall weighted averaging functions and Berezin transformations, introduced in [11]. We put

$$A_{K,\lambda}\mu(X) := \int_{\Phi_X(K)} s^\lambda \, d\mu(X) \bigg/ \int_{\Phi_X(K)} s^\lambda \, dV(X)$$

and

$$B_{m,p,\lambda}\mu(X) := \int |R_\alpha^m(Y,X)|^p s^\lambda \, d\mu(Y) \bigg/ \int |R_\alpha^m(Y,X)|^p s^\lambda \, dV(Y).$$

Note that $A_{Q^{(\alpha)}(X_0),0}\mu = \hat{\mu}^{(\alpha)}$ and $B_{0,2,0}\mu = \tilde{\mu}^{(\alpha)}$ and we also define

$$\bar{B}_{m,p,\lambda}\mu(X) := \int |\bar{R}_\alpha^m(Y,X)|^p s^\lambda \, d\mu(Y) \Big/ \int |\bar{R}_\alpha^m(Y,X)|^p s^\lambda \, dV(Y).$$

It is not difficult to verify that

$$B_{m,p,\lambda}\mu(X) \le C\bar{B}_{m,p,\lambda}\mu(X). \tag{20}$$

If we use α-parabolic similarities instead of [11, Proposition 1], the proof of [11, Lemma 3] also gives us the following.

Lemma 7. *Let $0 < \sigma \le \infty$, $m \in \boldsymbol{N}_0$, $0 < p < \infty$ and $\lambda, \tau, \eta \in \boldsymbol{R}$. We assume*

$$-1 < \lambda < \left(\frac{n}{2\alpha} + 1\right)(p-1) + mp \tag{21}$$

For a compact set K of positive Lebesgue measure and a positive Radon measure μ on $\boldsymbol{R}_+^{n+1}$, we have

$$\|A_{K,\tau}\mu\|_{L^\sigma(V_\eta)} \le C\|B_{m,p,\lambda}\mu\|_{L^\sigma(V_\eta)}$$

with some constant $C > 0$ independent of μ. More generally, for any $r_1 > 0$, there exist $r > 0$ and $C > 0$ such that

$$\|A_{K,\tau}\mu\|_{r_1, L^\psi(V_\eta)} \le C\|B_{m,p,\lambda}\mu\|_{r, L^\psi(V_\eta)},$$

where $\psi : [0,\infty) \to [0,\infty)$ is a strictly increasing continuous function with $\psi(0) = 0$ and $\lim_{s\to\infty} \psi(s) = \infty$.

The opposite inequality is much restricted.

Lemma 8 (cf. [11, Lemma 4]). *Let $0 < \sigma \le 1$. Let $m \in \boldsymbol{N}_0$, $0 < p < \infty$ and $\lambda, \tau, \eta \in \boldsymbol{R}$, and put $\kappa := (p-1)\left(\frac{n}{2\alpha} + 1\right) - \lambda$. Take a relatively compact open set $U \ne \emptyset$ in $\boldsymbol{R}_+^{n+1}$. If*

$$-\kappa - pm < \frac{\eta + 1}{\sigma} \quad \text{and} \quad \frac{\eta + \left(\frac{n}{2\alpha} + 1\right)}{\sigma} < \lambda + \left(\frac{n}{2\alpha} + 1\right), \tag{22}$$

then

$$\|\bar{B}_{m,p,\lambda}\mu\|_{L^\sigma(V_\eta)} \le C\|A_{U,\tau}\mu\|_{L^\sigma(V_\eta)}$$

with some constant $C > 0$.

Proof. Let $(X_\kappa)_\kappa$ and $(Q_\kappa)_\kappa$ be the standard δ-lattice and δ-decomposition of $\boldsymbol{R}_+^{n+1}$ for $\delta = 1$. Then in a similar manner to the proof of [9, Proposition 2], we have

$$\begin{aligned}\bar{B}_{m,p,\lambda}\mu(Y) &= s^{\kappa}\int (s^m(t+s+|x-y|^{2\alpha})^{-(n/2\alpha+1)-m})^p t^{\lambda}\, d\mu(X)\\ &= s^{\kappa+pm}\sum_{\nu\in\mathbf{Z}^{n+1}}\int_{Q_\nu}(t+s+|x-y|^{2\alpha})^{-p(n/2\alpha+1)-pm}t^{\lambda}\,d\mu(X)\\ &\le Cs^{\kappa+pm}\sum_{\nu\in\mathbf{Z}^{n+1}} t_\nu^{\lambda}\mu(Q_\nu)(t_\nu+s+|x_\nu-y|^{2\alpha})^{-p(n/2\alpha+1)-pm}\\ &= Cs^{\kappa+pm}\sum_{\nu\in\mathbf{Z}^{n+1}} t_\nu^{\lambda+(n/2\alpha+1)}\hat{\mu}^{(\alpha)}(X_\nu)(t_\nu+s+|x_\nu-y|^{2\alpha})^{-p(n/2\alpha+1)-pm}.\end{aligned}$$

Hence Lemma 4 gives us

$$\begin{aligned}&\bar{B}_{m,p,\lambda}\mu(Y)^{\sigma}\\ &\quad\le Cs^{\sigma\kappa+\sigma pm}\sum_{\nu\in\mathbf{Z}^{n+1}} t_\nu^{\sigma\lambda+\sigma(n/2\alpha+1)}\hat{\mu}^{(\alpha)}(X_\nu)^{\sigma}(t_\nu+s+|x_\nu-y|^{2\alpha})^{-\sigma p(n/2\alpha+1)-\sigma pm}\\ &\quad\le Cs^{\sigma\kappa+\sigma pm}\int \hat{\mu}^{(\alpha)}(X)^{\sigma}t^{\sigma\lambda+(\sigma-1)(n/2\alpha+1)}(t+s+|x-y|^{2\alpha})^{-\sigma p(n/2\alpha+1)-\sigma pm}\,dV(X),\end{aligned}$$

and Lemma 2 yields

$$\begin{aligned}&\int \bar{B}_{m,p,\lambda}\mu(Y)^{\sigma}\,dV_\eta(Y)\\ &\qquad\le C\int \hat{\mu}^{(\alpha)}(X)^{\sigma}t^{\sigma\lambda+(\sigma-1)(n/2\alpha+1)}\\ &\qquad\quad\times\int s^{\sigma\kappa+\sigma pm+\eta}(t+s+|x-y|^{2\alpha})^{-\sigma p(n/2\alpha+1)-\sigma pm}\,dV(Y)\,dV(X)\\ &\qquad\le C\int \hat{\mu}^{(\alpha)}(X)^{\sigma}t^{\sigma\lambda+(\sigma-1)(n/2\alpha+1)}t^{\sigma\kappa+\sigma pm+\eta-\sigma p(n/2\alpha+1)-\sigma pm+(n/2\alpha+1)}\,dV(X)\\ &\qquad= C\int \hat{\mu}^{(\alpha)}(X)^{\sigma}\,dV_\eta(X),\end{aligned}$$

because $\kappa=(p-1)\left(\dfrac{n}{2\alpha}+1\right)-\lambda$. Note that Lemma 2 requires the following integrability condition

$$-1<\sigma\kappa+\sigma pm+\eta<\sigma p\left(\frac{n}{2\alpha}+1\right)+\sigma pm-\left(\frac{n}{2\alpha}+1\right),$$

which follows from (21). Thus we get $\|\bar{B}_{m,p,\lambda}\mu\|_{L^{\sigma}(V_\eta)}\le C\|\hat{\mu}^{(\alpha)}\|_{L^{\sigma}(V_\eta)}$. By the α-parabolic similarity, we also see

$$\hat{\mu}^{(\alpha)}(X)=A_{Q^{(\alpha)}(X_0),0}(X)\le CA_{U,\tau}(X),$$

which completes the proof. □

Proof of Proposition 2. Let $p = 2$, $m = 0$, $\lambda = 0$ and $\eta = -\left(\frac{n}{2\alpha} + 1\right)$. Then they satisfy the conditions (21) and (22). Hence if we take $K = U = Q^{(\alpha)}(X_0)$, then two lemmas and (20) show our assertion. □

4.3. A remark on Orlicz type class. Recall that Φ is the set of all concave and strictly increasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. In this section, we remark that the equivalence relation $T_\mu \in \mathscr{S}^\sigma \Leftrightarrow \hat{\mu}^{(\alpha)} \in L^\sigma(V^*)$ does not hold for a general $\psi \in \Phi$. In fact we will give an example of $\psi \in \Phi$ and a Radom measure μ such that $\hat{\mu}^{(\alpha)} \in L^\psi(V^*)$ but $T_\mu \notin \mathscr{S}^\psi$.

We use the following.

Lemma 9. *There exists a positive measure μ on $\boldsymbol{R}_+^{n+1}$ with compact support such that the corresponding Toeplitz operator T_μ on $\boldsymbol{b}_\alpha^2$ is not of finite rank.*

Proof. We shall show that the one dimensional Lebesgue measure μ on the x_1-axis, i.e., $\mu = 1_{[a_1,b_1]}\,dx_1 \otimes \delta(x_2) \otimes \cdots \otimes \delta(x_n) \otimes \delta_c(t)$ is a required measure.

To see this let $R_\alpha^Y := R_\alpha(\cdot, Y)$ for $Y \in \boldsymbol{R}_+^{n+1}$ and put

$$F(X, Y) := T_\mu(R_\alpha^Y)(X) = \int R_\alpha(X, Z) R_\alpha(Y, Z)\, d\mu(Z).$$

By the direct computation, the Fourier transform $\tilde{F}$ in the space variables of F is given by

$$\tilde{F}(\xi, t, \eta, s) := \iint e^{\sqrt{-1}(\xi\cdot x + \eta\cdot y)} F(x, t, y, s)\, dxdy = G(\xi, t) G(\eta, s) f(\xi_1 + \eta_1),$$

where

$$(23) \qquad G(\xi, t) := 2(2\pi)^{-n/2} |\xi|^{2\alpha} e^{-(t+c)|\xi|^{2\alpha}} \quad \text{and} \quad f(w) := \frac{e^{\sqrt{-1}b_1 w} - e^{\sqrt{-1}a_1 w}}{\sqrt{-1}w}.$$

Now we assume that the rank of T_μ is finite. Then the singular decomposition theorem implies

$$T_\mu u = \sum_{j=0}^{k} \lambda_j \langle u, \phi_j \rangle \phi_j$$

with some integer k, where $(\phi_j)_j$ is an orthonormal system of $\boldsymbol{b}_\alpha^2$. Then

$$F(X, Y) = \sum_{j=0}^{k} \lambda_j \langle R_\alpha^Y, \phi_j \rangle \phi_j(X) = \sum_{j=0}^{k} \lambda_j \phi_j(Y) \phi_j(X)$$

and hence

$$\tilde{F}(\xi, t, \eta, s) = \sum_{j=0}^{k} \lambda_j \tilde{\phi}_j(\xi, t) \tilde{\phi}_j(\eta, s).$$

Thus we have

$$f(\xi_1 + \eta_1) = \sum_{j=1}^{k} \lambda_j (\tilde{\phi}_j(\xi, t)/G(\xi, t))(\tilde{\phi}_j(\eta, s)/G(\eta, s)).$$

Considering the right hand side is only the function of ξ_1 and η_1, differentiating in η_1 within $(k+1)$-times and putting $\eta_1 = 0$, we have a non-trivial linear relation between $f, f', \dots, f^{(k)}$, i.e., an ordinary differential equation with constant coefficients. However this solution does not have a form in (23). This contradiction shows that T_μ is of infinite rank. □

Example. Let μ be as in Lemma 9 and let $(\lambda_j)_{j=1}^{\infty}$ be a strictly decreasing subsequence of eigenvalues of T_μ. We now construct a $\psi \in \Phi$ such that $T_\mu \notin \mathscr{S}^\psi$. Take the convex hull K of the set

$$\{(0,0), (\lambda_1, 0)\} \cup \left\{ \left(\lambda_j, \frac{1}{j} \right); j \in \boldsymbol{N} \right\},$$

and define ψ on $[0, \lambda_1]$ by $\psi(\tau) := \sup\{y; (\tau, y) \in K\}$. Then $\lim_{\tau\to 0} \psi(\tau) = 0$ (see Remark 5 below), so that ψ can be prolonged to $[0, \infty)$ as a concave and increasing homeomorphism. Since $\sum_j \psi(\lambda_j) \geq \sum_j 1/j = \infty$ and $\psi(\lambda_j/\tau) \geq \psi(\lambda_j)/\tau$ for $\tau > 1$ by (17), we find that $T_\mu \notin \mathscr{S}^\psi(\boldsymbol{b}_\alpha^2)$. On the other hand, it is obvious that $\hat{\mu}^{(\alpha)} \in L^\psi(V^*)$ because $\hat{\mu}^{(\alpha)}$ is bounded and supported by a compact set.

Remark 5. Let $(\lambda_j)_{j=0}^{\infty}$ and $(a_j)_{j=0}^{\infty}$ be strictly deceasing sequences in $\boldsymbol{R}$ convergent to 0. We define a sequence $(b_j)_{j=0}^{\infty}$ inductively by $b_0 := a_0$ and

$$\frac{b_j - a_j}{\lambda_j} = \frac{b_{j-1} - a_j}{\lambda_{j-1}}$$

for $j \geq 1$ and let K_a and K_b be convex hulls of the sets

$$\{(0,0), (\lambda_0, 0)\} \cup \{(\lambda_j, a_j); j \in \boldsymbol{N}_0\} \quad \text{and} \quad \{(0,0), (\lambda_0, 0)\} \cup \{(\lambda_j, b_j); j \in \boldsymbol{N}_0\},$$

respectively. Then we have (i) $a_j < b_j$ for $j \geq 1$, (ii) $(b_j)_{j=0}^{\infty}$ is strictly decreasing and convergent to 0 and (iii) $K_a \subset K_b$, and (λ_j, b_j) is an extremal point of K_b for every $j \geq 1$. In particular, $\psi(\tau) := \sup\{y; (\tau, y) \in K_a\} \leq b_j$ whenever $0 < \tau \leq \lambda_j$ so that $\lim_{\tau\to 0} \psi(t) = 0$.

As for the converse implication $T_\mu \in \mathscr{S}^\psi \Rightarrow \hat{\mu}^{(\alpha)} \in L^\psi(V^*)$, we do not know whether this is true or not in general. A related result is the following.

Proposition 3. *Let $\psi \in \Phi$ and let $d\mu(X) = f(X)\,dV(V) \geq 0$ be an absolutely continuous Radon measure on $\boldsymbol{R}_+^{n+1}$ satisfying* (1). *Then $T_\mu \in \mathscr{S}^\psi$ implies $f \in L^\psi(V^*)$ and*

$$\|f\|_{r, L^\psi(V^*)} \leq \|T_\mu\|_{rr_0, \mathscr{S}^\psi} \tag{24}$$

holds for any $r > 0$, where $r_0 = R_\alpha(X_0, X_0)$ with $X_0 = (0, 1)$.

Proof. Let $(e_j)_j$ be a complete orthonormal system of $\boldsymbol{b}_\alpha^2$ which consists of eigenvectors of T_μ and put $\lambda_j := \langle e_j, T_\mu e_j \rangle = \int f|e_j|^2\,dV$. Let $r > 0$, and take any $\tau > 0$ with $\sum_j \psi(\lambda_j/\tau) \leq rr_0$. Since ψ^{-1} is convex, by the Jensen inequality we have

$$\psi^{-1}\left(\int \psi\left(\frac{f(X)}{\tau}\right)|e_j(X)|^2\,dV(X)\right) \leq \int \frac{f(X)}{\tau}|e_j(X)|^2\,dV(X) = \frac{\lambda_j}{\tau}.$$

Remarking $\sum_j |e_j(X)|^2\,dV(X) = R_\alpha(X, X)\,dV(X) = R_\alpha(X_0, X_0)\,dV^*(X)$ (see [11, Remark 1]), we have

$$\int \psi\left(\frac{f(X)}{\tau}\right)dV^*(X) \leq \frac{1}{r_0}\sum_j \psi\left(\frac{\lambda_j}{\tau}\right) \leq r,$$

which shows $f \in L^\psi(V^*)$ and (24). □

4.4. Herz type class. In this section, we consider the Herz type Toeplitz operators connected with the Schatten class ones. Herz introduced in [4] a kind of mixed norm spaces. First, we modify the definition of the Lebesgue-Herz spaces.

Let $\delta > 0$, $0 < \sigma < \infty$ and $0 < p < \infty$. For the standard δ-decomposition $(Q_\kappa)_\kappa$ of $\boldsymbol{R}_+^{n+1}$, we set

$$L_\delta^{\sigma,p} := \left\{ f; \|f\|_{L_\delta^{\sigma,p}} := \left(\sum_{\kappa \in \boldsymbol{Z}^{n+1}}\left(\int_{Q_\kappa}|f|^\sigma\,dV^*\right)^{p/\sigma}\right)^{1/p} < \infty \right\}.$$

Note that $L_\delta^{\sigma,p}$ does not depend on $\delta > 0$, because

$$\begin{cases} \sup_\kappa \#\{\tau \in \boldsymbol{Z}^{n+1}; Q_\tau \cap Q'_\kappa \neq \emptyset\} < \infty \\ \sup_\tau \#\{\kappa \in \boldsymbol{Z}^{n+1}; Q_\tau \cap Q'_\kappa \neq \emptyset\} < \infty, \end{cases} \tag{25}$$

where $(Q_\kappa)_\kappa$ and $(Q'_\tau)_\tau$ are δ- and δ'-decompositions of $\boldsymbol{R}_+^{n+1}$, respectively.

Now we give the definition of the Schatten-Herz class operators.

Definition 1. Let $\delta > 0$ and take the standard δ-lattice $(X_\kappa)_\kappa$ and δ-decomposition $(Q_\kappa)_\kappa$ of $\boldsymbol{R}_+^{n+1}$. For a Radon measure $\mu \geq 0$ on $\boldsymbol{R}_+^{n+1}$, T_μ is said to be of Schatten-Herz class if

$$\|T_\mu\|_{\mathscr{S}_\delta^{\sigma,p}} := \left(\sum_{\kappa \in \boldsymbol{Z}^{n+1}} \|T_{\mu|_{Q_\kappa}}\|_{\mathscr{S}^\sigma}^p \right)^{1/p} < \infty.$$

Then we denote by $T_\mu \in \mathscr{S}_\delta^{\sigma,p}$.

The purpose of this section is to characterize for the Toeplitz operator to be of Schatten-Herz class by using the averaging function of the given symbol Radon measure. We see the following.

THEOREM 3. *Let $\delta > 0$, $0 < \sigma < \infty$ and $0 < p < \infty$. For a Radon measure $\mu \geq 0$ on $\boldsymbol{R}_+^{n+1}$, $T_\mu \in \mathscr{S}_\delta^{\sigma,p}$ if and only if $\hat{\mu}^{(\alpha)} \in L_\delta^{\sigma,p}$. Moreover, both norms $\|T_\mu\|_{\mathscr{S}_\delta^{\sigma,p}}$ and $\|\hat{\mu}^{(\alpha)}\|_{L_\delta^{\sigma,p}}$ are comparable.*

Proof. First, we assume that $\hat{\mu}^{(\alpha)} \in L_\delta^{\sigma,p}$. For each $\kappa \in \boldsymbol{Z}^{n+1}$, $\widehat{\mu|_{Q_\kappa}}^{(\alpha)} \leq \hat{\mu}^{(\alpha)}$ and the number of sets $Q_{\kappa'}$ which intersect the support of $\widehat{\mu|_{Q_\kappa}}^{(\alpha)}$ is bounded by a constant independent of κ so that there exists a constant $C > 0$ such that

$$\|\widehat{\mu|_{Q_\kappa}}^{(\alpha)}\|_{L^\sigma(V^*)}^p \leq C \sum_{\kappa'} \|\hat{\mu}^{(\alpha)}|_{Q_{\kappa'}}\|_{L^\sigma(V^*)}^p.$$

Here taking the summation over κ, we have

$$\sum_{\kappa \in \boldsymbol{Z}^{n+1}} \|\widehat{\mu|_{Q_\kappa}}^{(\alpha)}\|_{L^\sigma(V^*)}^p \leq C \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \sum_{\kappa'} \|\hat{\mu}^{(\alpha)}|_{Q_{\kappa'}}\|_{L^\sigma(V^*)}^p \leq C \sum_{\kappa' \in \boldsymbol{Z}^{n+1}} \|\hat{\mu}^{(\alpha)}|_{Q_{\kappa'}}\|_{L^\sigma(V^*)}^p$$

because for κ', the number of indices κ such that the support of $\widehat{\mu|_{Q_\kappa}}^{(\alpha)}$ intersects $Q_{\kappa'}$ is also bounded by a constant independent of κ'. Since $\|\widehat{\mu|_{Q_\kappa}}^{(\alpha)}\|_{L^\sigma(V^*)}$ and $\|T_{\mu|_{Q_\kappa}}\|_{\mathscr{S}^\sigma}$ are comparable by Theorem 1 for $0 < \sigma < 1$ and [11, Theorem 2] for $\sigma \geq 1$, we have $\|T_\mu\|_{\mathscr{S}_\delta^{\sigma,p}} \leq C\|\hat{\mu}^{(\alpha)}\|_{L_\delta^{\sigma,p}}$.

Next, we assume that $T_\mu \in \mathscr{S}_\delta^{\sigma,p}$. Since $\hat{\mu}^{(\alpha)} = \sum_{\kappa'} \widehat{\mu|_{Q_{\kappa'}}}^{(\alpha)}$, by the similar argument to the above, we have

$$\|\hat{\mu}^{(\alpha)}|_{Q_\kappa}\|_{L^\sigma(V^*)}^p \leq C \sum_{\kappa'} \|\widehat{\mu|_{Q_{\kappa'}}}^{(\alpha)}\|_{L^\sigma(V^*)}^p$$

and

$$\sum_{\kappa \in \boldsymbol{Z}^{n+1}} \|\hat{\mu}^{(\alpha)}|_{Q_\kappa}\|_{L^\sigma(V^*)}^p \leq C \sum_{\kappa \in \boldsymbol{Z}^{n+1}} \sum_{\kappa'} \|\widehat{\mu|_{Q_{\kappa'}}}^{(\alpha)}\|_{L^\sigma(V^*)}^p \leq C \sum_{\kappa' \in \boldsymbol{Z}^{n+1}} \|\widehat{\mu|_{Q_{\kappa'}}}^{(\alpha)}\|_{L^\sigma(V^*)}^p,$$

which shows the theorem. □

Remark 6. The above space $\mathscr{S}_\delta^{\sigma,p}$ does not depend on δ and also σ. In fact, by (25), it is independent of δ. To show the independence of σ, let

$0 < \sigma_1 < \sigma_2 < \infty$ be arbitrarily and $\mu \geq 0$ be a measure on $\boldsymbol{R}_+^{n+1}$. Then since $\sum_j (\lambda/\tau)^{\sigma_1} \leq 1$ implies $\sum_j (\lambda/\tau)^{\sigma_2} \leq 1$, which shows $\|T_{\mu|_{Q_\kappa}}\|_{\mathscr{S}^{\sigma_2}} \leq \|T_{\mu|_{Q_\kappa}}\|_{\mathscr{S}^{\sigma_1}}$ for any Q_κ. On the other hand, since $V^*(Q_\kappa)$ is independent of κ, the Hölder inequality gives us

$$\|\hat{\mu}^{(\alpha)}|_{Q_\kappa}\|_{L_1^\sigma(V^*)} \leq C\|\hat{\mu}^{(\alpha)}|_{Q_\kappa}\|_{L_2^\sigma(V^*)}$$

with some constant $C > 0$ independent of κ. As a consequence, $\|T_{\mu|_{Q_\kappa}}\|_{\mathscr{S}^{\sigma_2}}$ and $\|T_{\mu|_{Q_\kappa}}\|_{\mathscr{S}^{\sigma_1}}$ are comparable, which shows independence of σ.

The Toeplitz operator whose symbol has compact support is always compact (see [8]). Hence Remark 2 (4) implies the following.

Proposition 4. *Let $\mu \geq 0$ be a Radon measure on $\boldsymbol{R}_+^{n+1}$. If $T_\mu \in \mathscr{S}_\delta^{\sigma,p}$, then T_μ is compact.*

Schatten-Herz class Toeplitz operators for the harmonic Bergman space are discussed in [2], where the cutting process is different from ours. We mention that ours is more natural, because they cut the upper half space into (non compact) strips, but we cut it into compact sets. The above assertion is an advantage of our cutting.

References

[1] B. R. Choe, H. Koo and H. Yi, Positive Toeplitz operators between the harmonic Bergman spaces, Potential Analysis **17** (2002), 307–335.

[2] B. R. Choe, H. Koo and Y. J. Lee, Positive Schatten(-Herz) class Toeplitz operators on the half space, Potential Analysis **27** (2007), 73–100.

[3] N. Dunford and J. Schwartz, Linear operators, Part II, Interscience, 1963.

[4] C. S. Herz, Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms, J. Math. Mech. **18** (1968/69), 283–323.

[5] D. H. Lueking, Trace ideal criteria for Toeplitz operators, J. Funct. Anal. **73** (1987), 345–368.

[6] M. Nishio, K. Shimomura and N. Suzuki, α-parabolic Bergman spaces, Osaka J. Math. **42** (2005), 133–162.

[7] M. Nishio, N. Suzuki and M. Yamada, Toeplitz operators and Carleson measures on parabolic Bergman spaces, Hokkaido Math. J. 36 **3** (2007), 563–583.

[8] M. Nishio, N. Suzuki and M. Yamada, Compact Toeplitz operators on parabolic Bergman spaces, Hiroshima Math. J. **38** (2008), 177–192.

[9] M. Nishio, N. Suzuki and M. Yamada, Interpolating sequences of parabolic Bergman spaces, Potential Analysis **28** (2008), 353–378.

[10] M. Nishio, N. Suzuki and M. Yamada, Parabolic dilations with application to Toeplitz operators on parabolic Bergman spaces, Proceedings of the 15th ICFIDCAA, 307–312, OCAMI Studies Vol. 2, Osaka Municipal Universities Press, 2008.

[11] M. Nishio, N. Suzuki and M. Yamada, Weighted Berezin transformations with application to the Schatten class Toeplitz operators on parabolic Bergman spaces, Kodai Math. J. **32** (2009), 501–520.

[12] M. Nishio, N. Suzuki and M. Yamada, Carleson inequalities on parabolic Bergman spaces, Tohoku Math. J. **62** (2010), 269–286.

[13] K. Zhu, Operator theory in function spaces, second edition, American Mathematical Society, 2007.

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