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TORUS INVARIANT SPECIAL LAGRANGIAN SUBMANIFOLDS IN THE CANONICAL BUNDLE OF TORIC POSITIVE KÄHLER EINSTEIN MANIFOLDS

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Abstract

In this paper we construct torus invariant special Lagrangian submanifolds in the canonical bundle K_M of the toric positive Kähler Einstein manifold M. We construct a Ricci-flat metric on K_M using the Calabi ansatz to show that K_M is a Calabi-Yau manifold. Then, using moment map techniques developed in [6], we construct special Lagrangian submanifolds in K_M.

1. Introduction

In 1982, Harvey and Lawson [4] introduced the notion of special Lagrangian submanifolds in the study of minimal submanifolds. In general, it is known by the Wirtinger inequality that the complex submanifold in a Kähler manifold minimize its volume in its homology class. Generalizing this property, Special Lagrangian submanifolds are defined in Calabi-Yau manifolds.

The study of special Lagrangian submanifolds is important in relation to mirror symmetry in physics due to the SYZ conjecture. This conjecture was presented by Strominger, Yau and Zaslow [12] in 1996 and it explains mirror symmetry of compact Calabi-Yau 3-folds in terms of dual fibrations by special Lagrangian 3-tori, including singular fibers. They also propose the constructing way of the mirror of a compact Calabi-Yau manifold by an appropriate compactification of the dual of the special Lagrangian torus fibration.

So to understand mirror symmetry more deeply, examples of special Lagrangian submanifolds are constructed using various techniques. In the beginning of the study, examples are mainly constructed on $\mathbf{C}^m$. Joyce [8], [9], [10] constructed special Lagrangian submanifolds in $\mathbf{C}^m$ using the method of ruled submanifolds, integrable systems and evolution of quadrics, and Haskins [5] gave examples of special Lagrangian cones in $\mathbf{C}^3$, etc.

Recently, some examples have also been constructed in non-flat Calabi-Yau manifolds. Anciaux [1] constructed $SO(n)$-invariant examples in the cotangent

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bundle of the n-dimensional sphere with the Ricci-flat Stenzel metric. Ionel and Min-Oo [6] constructed T^2-invariant and $SO(3)$-invariant special Lagrangian submaniolds in the deformed conifold and the resolved conifold using moment map techniques. The case of the deformed conifold was extended to the higher dimensional case by Kanemitsu [11].

In this paper, using the Calabi ansatz and moment map techniques, we construct special Lagrangian submanifolds in the canonical bundle K_M of the toric positive Kähler Einstein manifold M. The Calabi ansatz is the method searching for Kähler forms ω_{K_M} of the form

$$\omega_{K_M} = \pi^* \omega_M + dd^c F(t)$$

where $\pi : K_M \to M$ is the projection, the form ω_M is the Kähler form of the positive Kähler Einstein metric on M, the function t is the logarithm of the norm function and F is a function of one variable. Special Lagrangian submanifolds are defined in Calabi-Yau manifolds and Calabi-Yau manifolds are defined to have the Ricci-flat Kähler form ω and the holomorphic volume form Ω that satisfy the equation (2.1).

First, we will see that K_M is a Calabi-Yau manifold. We construct a Ricci-flat metric on K_M using the Calabi ansatz. This construction is inspired by [3]. We also define the holomorphic volume form $\Omega := d\alpha$ on K_M for some concrete form α on K_M. Using those, we can see that K_M is a Calabi-Yau manifold.

Then we will construct torus invariant special Lagrangian submanifolds by the moment map techniques developed in [6]. Namely, in general, in Hamiltionian G-space, connected G-invariant Lagrangian submanifolds must be in the level set of the moment map. We search for the submanifolds in the level set of the moment map with the additional condition for special Lagrangian that $\operatorname{Im} \Omega$ vanishes on the submanifolds. At this point, the form α plays an important role. In the case of K_M, there exists a torus action preserving the Calabi-Yau structure. For this torus action, we apply the above construction.

These are summarized in Theorem 3.2. (our main theorem) and the essentials are proved in Propositon 3.1.

We now give a brief description of the contents of this paper. In section 2, we review the basic definitions such as Calabi-Yau manifolds, and explain the moment map techniques developed in [6]. In section 3, we prove that K_M is a Calabi-Yau manifold and describe the moment map. Then we construct special Lagrangian submaniolds on K_M using moment map techniques. In section 4, we give examples of special Lagrangian submanifolds by applying our method.

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2. Preliminaries

2.1. Basic definitions

Definition 2.1. Let (M, J, ω) be an m-dimensional Kähler manifold where M is a complex manifold, J is the complex structure on M, and ω is the Kähler form on M.

A Kähler manifold (M, J, ω) is a **positive Kähler Einstein manifold** if M is the Kähler Einstein manifold with the positive Einstein constant. A Kähler manifold (M, J, ω) is **toric** if m-dimensional torus $T^m = (S^1)^m$ acts on (M, J, ω) effectively as holomorphic isometries.

Definition 2.2. The pair (M, J, ω, Ω) is an m-dimensional **Calabi-Yau manifold** if the following conditions are satisfied:

- (M, J, ω) is a Kähler manifold.
- Ω is a nonzero holomorphic section of the canonical bundle K_M on M.
- $$\frac{\omega^m}{m!} = (-1)^{m(m-1)/2}\left(\frac{\sqrt{-1}}{2}\right)^m \Omega \wedge \bar{\Omega}. \tag{2.1}$$

Remark 2.3. If a Kähler form ω and a holomorphic m-form Ω satisfies (2.1), the corresponding Riemannian metric g of ω is Ricci-flat and Ω is parallel with respect to the Levi-Civita connection of g.

Definition 2.4. Let (M, J, ω, Ω) be a Calabi-Yau m-fold and $L \subset M$ a real oriented m-dimensional submanifold of M. Then L is called a **special Lagrangian submanifold** of M if $\omega|_L \equiv 0$, $\operatorname{Im} \Omega|_L \equiv 0$.

Remark 2.5. The condition $\omega|_L \equiv 0$ says that L is a Lagrangian submanifold. Therefore, special Lagrangian submanifolds are Lagrangian with the extra condition $\operatorname{Im} \Omega|_L \equiv 0$.

2.2. Moment map techniques

To construct special Lagrangian submanifolds, we use the moment map techniques developed in [6].

In the moment map techniques, we search for G-invariant special Lagrangian submanifolds for some Lie group G. Though only concrete examples are discussed in [6], they are essentially stated as follows.

Let G be a Lie group, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{g}^*$ the dual of $\mathfrak{g}$, and $Ad^{\#}$ the coadjoint action of G on $\mathfrak{g}^*$. We define the center $Z(\mathfrak{g}^*)$ of $\mathfrak{g}^*$ as

$$Z(\mathfrak{g}^*) := \{\xi \in \mathfrak{g}^* \mid Ad^{\#}(g)\xi = \xi \ \ (\forall g \in G)\}.$$

Fact 2.6. *Let (M, ω, G, μ) be a Hamiltonian G-space. Namely, (M, ω) is a symplectic manifold and a connected Lie group G acts on M preserving ω with the moment map $\mu : M \to \mathfrak{g}^*$.*

Let $\mathcal{O} \subset M$ be any G-orbit. Then the G-orbit $\mathcal{O}$ is isotropic (*i.e.* $\omega|_{\mathcal{O}} = 0$) *if and only if $\mathcal{O} \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

We also see that if $L \subset M$ is the connected G-invariant Lagrangian submanifold,

$$L \subset \mu^{-1}(c)$$

for some $c \in Z(\mathfrak{g}^*)$.

Using this fact, we will construct G-invariant special Lagrangian submanifolds for some Lie group G as follows.

Proposition 2.7. *Assume the following four conditions*:

1. *Let (M, J, ω, Ω) be a Calabi-Yau manifold of complex dimension m.*
2. *A compact connected Lie group G of real dimension $m-1$ acts on M preserving Calabi-Yau structure. Namely, G-action preserves J, ω, Ω. Its generic orbits in M are of real dimension $m-1$.*
3. *There exists a moment map $\mu : M \to \mathfrak{g}^*$ for G-action.*
4. *There exists a G-invariant $(m-1)$-form α such that for any $v_1, \dots, v_{m-1} \in \mathfrak{g}$,*

$$\operatorname{Im}\Omega(\cdot, v_1^*, \dots, v_{m-1}^*) = d(\alpha(v_1^*, \dots, v_{m-1}^*))$$

on M where v_i^* is the real vector field on M generated by v_i.

Then for any $c \in Z(\mathfrak{g}^*)$, $c' \in \mathbf{R}$ and any basis $\{X_1, \dots, X_{m-1}\} \subset \mathfrak{g}$, the set

$$L_{c,c'} := \mu^{-1}(c) \cap (\alpha(X_1^*, \dots, X_{m-1}^*))^{-1}(c')$$

is a G-invariant special Lagrangian submanifold of M. The set $L_{c,c'}$ is singular where the isotropy group is not discrete.

We sketch the proof of this proposition.

If there exists a G-invariant special Lagrangian submanifold L, we see from Fact 2.6. that $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.

Next, since L is G-invariant, for any $X \in \mathfrak{g}$, $X_p^* \in T_pL$ at any point $p \in L$. So if we take the any basis $\{X_1, \dots, X_{m-1}\} \subset \mathfrak{g}$, $X_1, \dots, X_{m-1}$ must satisfy $\operatorname{Im}\Omega(\cdot, X_1^*, \dots, X_{m-1}^*)|_L = 0$ for $\operatorname{Im}\Omega|_L \equiv 0$. From the assumption 4, this condition can be described as $\alpha(X_1^*, \dots, X_{m-1}^*)|_L = c'$ for some $c' \in \mathbf{R}$.

So we see $L \subset \mu^{-1}(c) \cap (\alpha(X_1^*, \dots, X_{m-1}^*))^{-1}(c')$.

But the right hand is G-invariant and already satisfies the special Lagrangian condition. Its real dimension is generically m because of the assumption 2. So we can construct special Lagrangian submanifolds as Proposition 2.7.

Concerning the sigularities, it is clear that if the isotropy group at $p \in L_{c,c'}$ is not discrete, the set $L_{c,c'} = \{\langle \mu, X_i \rangle - \langle c, X_i \rangle = 0 \ (1 \le i \le m-1), \alpha(X_1^*, \dots, X_{m-1}^*) - c' = 0\}$ is singular at p. We can also show that if the isotropy group at $p \in L_{c,c'}$ is discrete, namely the tangent vectors $\{(X_1^*)_p, \dots, (X_{m-1}^*)_p\}$ are linearly independent, the set $L_{c,c'}$ is smooth at p. Calculate the Jacobian matrix about the linearly independent set $\{(X^*_{1,hol})_p, \dots, (X^*_{m-1,hol})_p, v\}$. Here, $X^*_{i,hol}$ is the holomorphic vector field generated by $X_i \in \mathfrak{g}$ such that $X^*_i = X^*_{i,hol} + \overline{X^*_{i,hol}}$ and $v \in T_pM - \ker(\operatorname{Im}\Omega(\cdot, (X^*_1)_p, \dots, (X^*_{m-1})_p))$. Since $d\langle\mu, X_i\rangle = -\omega(X^*_i, \cdot)$ and the assumption 4, the Jacobian matrix about this set is as follows:

$$\begin{pmatrix} (-\omega(\overline{X^*_{i,hol}}, X^*_{j,hol}))_{1\le i,j\le m-1} & * \\ 0 & \gamma \end{pmatrix}$$

where $\gamma = \operatorname{Im}\Omega(v, X^*_1, \dots, X^*_{m-1})$. This shows that the set $L_{c,c'}$ is smooth at p.

Remark 2.8. From the construction, these are "maximal" G-invariant special Lagrangian submanifolds. Namely, for any connected G-invariant special Lagrangian submanifold L, there exists $c \in Z(\mathfrak{g}^*)$ and $c' \in \mathbf{R}$ such that $L \subset L_{c,c'}$.

3. Constructing special Lagrangian submanifolds in the canonical bundle

Let M be an m-dimensional toric positive Kähler Einstein manifold. For simplicity, we suppose that M is connected. It is easy to extend to the non-connected case.

We lift the T^m-action on M and consider T^m acts on the canonical bundle K_M. Since T^m acts effectively, the generic orbit of T^m is of real dimension m.

Using moment map techniques developed in [6], we construct T^m-invariant special Lagrangian submanifolds in the canonical bundle K_M of M.

Proposition 3.1. *Consider the condition above. Multiplying some constant to the Kähler form, we may assume the Kähler form ω_M and its Ricci form ρ_M satisfy*

$$\rho_M = 2\omega_M.$$

Then we have

1. *The canonical bundle K_M admits the Calabi-Yau structure. Its complex structure J_{K_M} is the canonical one, the holomorphic volume form Ω is $d\alpha$ for a concrete T^m-invariant m-form α on K_M, and the Kähler form ω_{K_M} is given by*

$$\omega_{K_M} = \pi^*\omega_M + dd^cF(t)$$

where $\pi : K_M \to M$ is the canonical projection and $F \in C^\infty(\mathbf{R})$ with

$$F'(t) = ((m+1)e^{2t} + 1)^{1/(m+1)} - 1.$$

Here, $t = \log r \in C^\infty(K_M - \{0\text{-section}\})$, and r is the distance function form the 0-section measured by the induced fiber metric from ω_M. The Kähler form ω_{K_M} extends to the 0-section smoothly.
2. *The T^m-action preserves the Calabi-Yau structure.*
3. *There exists a moment map $\Phi : K_M \to (\mathfrak{t}^m)^*$ for the T^m-action.*

4. *For any* $v_1, \dots, v_m \in \mathfrak{t}^m$,

$$\operatorname{Im} \Omega(\cdot, \tilde{v}_1^*, \dots, \tilde{v}_m^*) = d(\operatorname{Im} \alpha(\tilde{v}_1^*, \dots, \tilde{v}_m^*))$$

where $\tilde{v}_i^*$ *is the real vector field on* K_M *generated by* v_i.

From this proposition, we can apply Proposition 2.7. to K_M to construct T^m-invariant special Lagrangian submanifolds in K_M. Remark $Z((\mathfrak{t}^m)^*) = (\mathfrak{t}^m)^*$.

Theorem 3.2. *Let* M *be a connected* m*-dimensional toric positive Kähler Einstein manifold. Multiplying some constant to the Kähler form, we may assume the Kähler form* ω_M *and its Ricci form* ρ_M *satisfy*

$$\rho_M = 2\omega_M.$$

From Proposition 3.1., *the canonical bundle* K_M *is a Calabi-Yau manifold with the canonical complex structure* J_{K_M}, *the Kähler form* ω_{K_M}, *and the holomorphic volume form* $\Omega = d\alpha$ *for some concrete* m*-form* α *on* K_M. *The* m*-dimensional torus* T^m *acts on* K_M *preserving the Calabi-Yau structure, and there exists a moment map* $\Phi : K_M \to (\mathfrak{t}^m)^*$ *for the* T^m*-action.*

For any $X \in \mathfrak{t}^m$, *let* $\tilde{X}^*$ *and* X^* *be the real vector field on* K_M *and* M *generated by* X, *respectively. Denote* ∇^M *the Levi-Civita connection of* ω_M *and* J_M *the complex structure on* M.

Then T^m*-invariant special Lagrangian submanifolds in* $(K_M, J_{K_M}, \omega_{K_M}, \Omega)$ *are given by the equations*:

$$\begin{cases} \langle \Phi, X_i \rangle = A_i \quad (1 \le i \le m) \\ \operatorname{Im}(\alpha(\tilde{X}_1^*, \dots, \tilde{X}_m^*)) = A_{m+1} \end{cases}$$

$$\Leftrightarrow \begin{cases} ((m+1)r^2 + 1)^{1/(m+1)}\ tr(\nabla^M(J_M X_i^*)) = A_i \quad (1 \le i \le m) \\ \operatorname{Im}(\alpha(\tilde{X}_1^*, \dots, \tilde{X}_m^*)) = A_{m+1} \end{cases}$$

where $\{X_1, \dots, X_m\}$ *is the any basis of* $\mathfrak{t}^m$ *and* $A_1, \dots, A_{m+1}$ *are any real constants.*

Proof of Proposition 3.1.:

Proof of 1:

- Let the complex structure J_{K_M} be the canonical one on K_M.
- We construct the holomorphic volume form Ω on K_M as follows.

Let $\pi : K_M \to M$ be the projection. For any $(x, \xi) \in K_M$ where $x \in M$, $\xi \in (K_M)_x$, we can define the pull back map

$$(d\pi)'^*_{(x,\xi)} : T'^*_x M \to T'^*_{(x,\xi)} K_M$$

where $T'^* M$, $T'^* K_M$ is a holomorphic cotangent bundle of M, K_M, respectively. We also use the same notation for the extension:

$$(d\pi)'^*_{(x,\xi)} : \bigwedge^* T'^*_x M \to \bigwedge^* T'^*_{(x,\xi)} K_M$$

Then we define a holomorphic m-form α on K_M and Ω as

$$\alpha_{(x,\xi)} := (d\pi)'^{*}_{(x,\xi)}(\xi)$$
$$\Omega := d\alpha.$$

Using local coordinates, α and Ω can be described as follows.

Let $(z^1, \dots, z^m)$ be the local coordinate of M, and z is a fiber coordinate of K_M with respect to $dz^1 \wedge \dots \wedge dz^m$. We also denote $(z^1, \dots, z^m)$ for the pull-backed local coordinates on K_M. When $\xi = z\, dz^1 \wedge \dots \wedge dz^m$,

$$\alpha = z\, dz^1 \wedge \dots \wedge dz^m$$
$$\Omega = dz \wedge dz^1 \wedge \dots \wedge dz^m.$$

• Next, we construct the Ricci-flat Kähler form ω_{K_M} on K_M.

We search for ω_{K_M} of the form

$$\omega_{K_M} = \pi^* \omega_M + dd^c F(t)$$

where $F \in C^\infty(\mathbf{R})$, $t = \log r$, and r is the distance function form the 0-section measured by the induced fiber metric from ω_M.

For the construction, it is enough to show the following lemma.

Lemma 3.3. *Under the condition above, there exists a function $F \in C^\infty(\mathbf{R})$ satisfying the following conditions*:

• 1A.

$$\frac{\omega_{K_M}^{m+1}}{(m+1)!} = (-1)^{m(m+1)/2} \left(\frac{\sqrt{-1}}{2} \right)^{m+1} \Omega \wedge \bar{\Omega}$$

• 1B. *The form ω_{K_M} extends to the* 0*-section.*

• 1C. *The form ω_{K_M} determines the metric on K_M.*

First, we consider the condition 1A. It is enough to consider locally.

Let $(z, z^1, \dots, z^m)$ be the local coordinate of K_M where $(z^1, \dots, z^m)$ is the pull back of the local coordinate of M and z is the fiber coordinate with respect to $dz^1 \wedge \dots \wedge dz^m$.

Then the Kähler form ω_M on M and the distance function r can be described as follows:

$$\omega_M = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{M,i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

$$r^2 = \det(g_M^{i\bar{j}})|z|^2$$

where $\Sigma g_M^{i\bar{j}} g_{M,k\bar{j}} = \delta_k^i$.

Then

$$\begin{aligned} dd^c F(t) &= d(F'(t)d^c t) \\ &= F''(t)\, dt \wedge d^c t + F'(t)\, dd^c t \\ &= F''(t)\, dt \wedge d^c t + F'(t)\pi^*\rho_M \\ &= F''(t)\, dt \wedge d^c t + F'(t)\pi^*\omega_M \\ 2\partial t &= \partial(\log r^2) \\ &= \partial \log(\det(g_M^{i\bar{j}})) + \frac{dz}{z} \\ dt \wedge d^c t &= \sqrt{-1}\partial t \wedge \bar{\partial} t \\ &= \frac{\sqrt{-1}}{4}(\gamma_0 + \gamma_1 + \gamma_2) \\ \gamma_0 &= \partial \log(\det(g_M^{i\bar{j}})) \wedge \bar{\partial} \log(\det(g_M^{i\bar{j}})) \\ \gamma_1 &= \partial \log(\det(g_M^{i\bar{j}})) \wedge \frac{d\bar{z}}{\bar{z}} + \frac{dz}{z} \wedge \bar{\partial} \log(\det(g_M^{i\bar{j}})) \\ \gamma_2 &= \frac{dz \wedge d\bar{z}}{|z|^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \omega_{K_M} &= \pi^*\omega_M + dd^c F(t) \\ &= (1 + F'(t))\pi^*\omega_M + F''(t)\, dt \wedge d^c t \\ \frac{\omega_{K_M}^{m+1}}{(m+1)!} &= (1 + F'(t))^m F''(t) \frac{(\pi^*\omega_M)^m}{m!} \wedge dt \wedge d^c t \\ &= \frac{(1 + F'(t))^m F''(t)}{2r^2} (-1)^{m(m+1)/2} \left(\frac{\sqrt{-1}}{2}\right)^{m+1} \Omega \wedge \bar{\Omega}. \end{aligned}$$

To satisfy the Calabi-Yau condition, it is enough to solve the following equation:

$$\frac{(1 + F'(t))^m F''(t)}{2e^{2t}} = 1$$

We can solve this equation easily,

$$\frac{d}{dt}((1 + F'(t))^{m+1}) = 2(m+1)e^{2t}.$$

For simplicity, we choose the following solution:

$$F'(t) = ((m+1)e^{2t} + 1)^{1/(m+1)} - 1.$$

Next, we consider the condition 1B. From the construction of ω_{K_M}, to show ω_{K_M} extends to the 0-section, it is enough to prove the following:

$$\lim_{t \to -\infty} \frac{F''(t)}{e^{2t}} < \infty$$

Differentiating $F'(t)$ obtained above,

$$F''(t) = 2e^{2t}((m+1)e^{2t}+1)^{1/(m+1)-1}.$$

From this, we can also see that ω_{K_M} extends to the 0-sexton smoothly.

Next, we consider the condition 1C. It is enough to prove the positive definiteness of ω_{K_M} on K_M.

From 1A, the determinant of $\omega_{K_M} \equiv 1 > 0$ on K_M, eigenvalues of ω_{K_M} vary continuously, and K_M is connected since M is connected, it is enough to prove the positive definiteness of ω_{K_M} on K_M at one point of K_M.

If we choose any one point on the 0-section of K_M, using the local coordinate above,

$$\omega_{K_M} = \pi^* \omega_M + \frac{\sqrt{-1}}{2} \det(g_M^{i\bar{j}})\, dz \wedge d\bar{z}.$$

Since this is clearly positive definite, we see ω_{K_M} is the Ricci-flat Kähler form on K_M.

Proof of 2:

We will see the lifted T^m-action preserves the Calabi-Yau structure. Since T^m-action is the holomorphic isometry, ω_{K_M} and J_{K_M} is preserved under the T^m-action. We will see that T^m-action preserves Ω. For that, we will see that T^m-action preserves α.

For $g \in T^m$, let ψ_g be the action of g on M and $\varphi_g := (\psi_g^{-1})^*$ be the lifted action of g on K_M.

$$\begin{array}{ccc} K_M & \xrightarrow{\varphi_g} & K_M \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\psi_g} & M \end{array}$$

Take the local coordinate $(z, z^1, \ldots, z^m)$ as above. From the theory of the Lie group, all the elements of T^m can be described as the finite products of the elements in the neighborhood of the identity element. So it is enough to prove in the case that the image of φ_g is also in the same local coordinate.

If we put

$$\psi_g(z^1, \ldots, z^m) = (\psi_g^1(z^1, \ldots, z^m), \ldots, \psi_g^m(z^1, \ldots, z^m)).$$

Then, from $\varphi_g = (\psi_g^{-1})^*$

$$\varphi_g(z, z^1, \ldots, z^m) = \left(z \cdot \det\left(\frac{\partial \psi_{g^{-1}}^i}{\partial z^j}\right) \circ \psi_g, \psi_g^1, \ldots, \psi_g^m\right).$$

From this, we can see

$$\begin{aligned}\varphi_g^* \alpha &= z \cdot \det\left(\frac{\partial \psi_{g^{-1}}^i}{\partial z^j}\right) \circ \psi_g \, d\psi_g^1 \wedge \cdots \wedge d\psi_g^m \\ &= z \cdot \det\left(\frac{\partial \psi_{g^{-1}}^i}{\partial z^j}\right) \circ \psi_g \cdot \det\left(\frac{\partial \psi_g^i}{\partial z^j}\right) dz^1 \wedge \cdots \wedge dz^m \\ &= \alpha.\end{aligned}$$

Therefore,

$$\varphi_g^* \Omega = \Omega.$$

Proof of 3:

We construct the moment map $\Phi : K_M \to (\mathfrak{t}^m)^*$.

• First, we construct the moment map $\Phi : K_M - \{0\text{-section}\} \to (\mathfrak{t}^m)^*$.

We will see Φ extends to the 0-section later. From the continuity, the extended Φ is also the moment map.

To begin with, ω_{K_M} was the following form:

$$\begin{aligned}\omega_{K_M} &= \omega^T + dd^c F(t) \\ &= d(d^c(t + F(t)))\end{aligned}$$

Since T^m-action preserves r and J, T^m-action also preserves $d^c(t + F(t))$. So for any $X \in \mathfrak{t}^m$,

$$L_{\tilde{X}^*} d^c(t + F(t)) = 0$$

namely,

$$i(\tilde{X}^*)\omega_{K_M} = -d(i(\tilde{X}^*) d^c(t + F(t)))$$

where $i(\cdot)$ is the inner product.

So we can define the moment map $\Phi : K_M - \{0\text{-section}\} \to (\mathfrak{t}^m)^*$ as follows:

$$\begin{aligned}\langle \Phi, X \rangle &= i(\tilde{X}^*) d^c(t + F(t)) \\ &= (1 + F'(t)) d^c t(\tilde{X}^*) \\ &= ((m+1)r^2 + 1)^{1/(m+1)} d^c t(\tilde{X}^*)\end{aligned}$$

• Next, we see Φ extends to the 0-section.

For this, we will compute $\tilde{X}^*$ explicitly in the local coordinate. Let $(z, z^1, \ldots, z^m)$ be the same local coordinate as in the Proof of 2. For $X \in \mathfrak{t}^m$, we put $X^* \in \mathfrak{X}(M)$ as follows:

$$X^* = X^*_{hol} + \overline{X^*_{hol}}$$
$$X^*_{hol} = \Sigma (X^*)^i \frac{\partial}{\partial z^i}$$
$$(X^*)^i = \left.\frac{d\psi^i_{exp(tX)}}{dt}\right|_{t=0}$$

Then from the description of φ_g in local coordinate in the Proof of 2, we see

$$\frac{d}{dt}\det\left(\frac{\partial\psi^i_{exp(-tX)}}{\partial z^j}\right)\circ \psi_{exp(tX)}\bigg|_{t=0} = -\Sigma\frac{\partial(X^*)^i}{\partial z^i}.$$

So

$$\tilde{X}^*_{hol} = X^*_{hol} - \Sigma\frac{\partial(X^*)^i}{\partial z^i} z\frac{\partial}{\partial z}$$
$$\tilde{X}^* = \tilde{X}^*_{hol} + \overline{\tilde{X}^*_{hol}}.$$

Then

$$\begin{aligned}
d^c t(\tilde{X}^*_{hol}) &= -\frac{dr^2}{4r^2}(J\tilde{X}^*_{hol})\\
&= -\frac{\sqrt{-1}}{4}\frac{dr^2}{r^2}\left(X^*_{hol} - \sum_i \frac{\partial(X^*)^i}{\partial z^i} z\frac{\partial}{\partial z}\right)\\
&= -\frac{\sqrt{-1}}{4}\left\{\frac{X^*_{hol}(\det(g^{i\bar{j}}_M))}{\det(g^{i\bar{j}}_M)} - \sum_i \frac{\partial(X^*)^i}{\partial z^i}\right\}\\
&= -\frac{\sqrt{-1}}{4}\left\{\sum_{i,k,l}(X^*)^i\frac{\partial g^{k\bar{l}}_M}{\partial z^i}(g_M)_{k\bar{l}} - \sum_i \frac{\partial(X^*)^i}{\partial z^i}\right\}\\
&= \frac{\sqrt{-1}}{4}\sum_i dz^i(\nabla_{\partial/\partial z^i}X^*_{hol})\}\\
&= \frac{\sqrt{-1}}{4}\; tr(\nabla^M X^*_{hol})\\
\langle\Phi, X\rangle &= \frac{\sqrt{-1}}{4}((m+1)r^2+1)^{1/(m+1)}\{tr(\nabla^M X^*_{hol}) - tr(\nabla^M\overline{X^*_{hol}})\}\\
&= \frac{1}{4}((m+1)r^2+1)^{1/(m+1)}\; tr(\nabla^M(J_M X^*))
\end{aligned}$$

where ∇^M is the Levi-Civita connection of ω_M and J_M is the complex structure on M. From this description, we can see Φ extends to the 0-section.

Proof of 4:

Recall $\Omega = d\alpha$. We will show that α (up to the sign) satisfies the condition. It is shown that α is T^m-invariant in the Proof of 2. So for $X \in \mathfrak{t}^m$,

$$L_{\tilde{X}^*}\alpha = 0$$
$$i(\tilde{X}^*)\Omega = -d(i(\tilde{X}^*)\alpha).$$

Moreover, for $Y \in \mathfrak{t}^m$, since $\mathfrak{t}^m$ is commutative Lie algebra,

$$L_{\tilde{Y}^*}(i(\tilde{X}^*)\alpha) = 0.$$

Therefore,

$$i(\tilde{Y}^*)i(\tilde{X}^*)\Omega = d(i(\tilde{Y}^*)i(\tilde{X}^*)\alpha).$$

Iterating this, we have for any $v_1, \dots, v_m \in \mathfrak{t}^m$,

$$\mathrm{Im}\,\Omega(\cdot, \tilde{v}_1, \dots, \tilde{v}_m) = \pm d(\mathrm{Im}(\alpha(\tilde{v}_1, \dots, \tilde{v}_m))).$$

This completes the proof.

4. Examples

Applying the method of section 4, we construct special Lagrangian submanifolds in the case of $M = \mathbf{C}P^m$.

• First, we will see that $K_{\mathbf{C}P^m}$ is a Calabi-Yau manifold.

Let $U_i := \{[z^1 : \dots : z^{m+1}] \in M \mid z_i \neq 0\} \subset \mathbf{C}P^m$ $(1 \leq i \leq m+1)$ and $\pi : K_M \to M$ be the projection. Let the complex structure J_{K_M}, J_M on K_M, M be the canonical one, respectively. We define the Kähler form ω_M as

$$\omega_M := \frac{m+1}{2}\sqrt{-1}\partial\bar{\partial}\log\left(\sum_{j=1}^{m+1}\left|\frac{z^j}{z^i}\right|^2\right)$$

on U_i. This metric is the Fubini-Study metric (multiplied some constant). We can easily see $\rho_M = 2\omega_M$.

We also define the action of torus T^m on M as follows:

$$(g_1, \dots, g_m) \cdot [z^1 : \dots : z^{m+1}] := [g_1 z^1 : \dots : g_m z^m : z^{m+1}]$$

where $(g_1, \dots, g_m) \in T^m$, $[z^1 : \dots : z^{m+1}] \in M$. We consider $g_i \in S^1 \subset \mathbf{C}$.

From this, (M, ω_M) is toric and from Theorem 3.2 we have the Ricci-flat metric on K_M

$$\omega_{K_M} = \pi^*\omega_M + dd^c F(t)$$
$$F'(t) = ((m+1)e^{2t} + 1)^{1/(m+1)} - 1.$$

This metric is the same (up to constant factors) as the one in [7] (Calabi's metric on $\mathbf{C}P^m$ in [2]). For more details, see Appendix.

If we take $\Omega = d\alpha$ as the former section, $(K_M, J_{K_M}, \omega_{K_M}, \Omega)$ becomes Calabi-Yau manifold.

• Next, we apply the construction in the former section.

We want to describe the moment map $\Phi : K_M \to (\mathfrak{t}^m)^*$ and $\alpha(\tilde{X}_1, \dots, \tilde{X}_m)$ explicitly. For that, we will describe r and $\tilde{X}^* \in \mathfrak{X}(K_M)$ generated by $X \in \mathfrak{t}^m$.

We discuss on $\pi^{-1}(U_{m+1})$. We take the local coordinate $(w, w^1, \dots, w^m)$ on $\pi^{-1}(U_{m+1})$ as $w^i = \dfrac{z^i}{z^{m+1}}$, w is a fiber coordinate of K_M with respect to $dw^1 \wedge \cdots \wedge dw^m$. If we put

$$\omega_M|_{U_{m+1}} = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{M,i\bar{j}}\, dw^i \wedge d\overline{w}^j.$$

Then

$$\begin{aligned} r^2 &= \det(g_M^{i\bar{j}})|w|^2 \\ &= \left(\frac{1}{m+1}\right)^{m+1} (1 + \Sigma|w^j|^2)^{m+1}|w|^2. \end{aligned}$$

The action of T^m is:

$$(g_1, \dots, g_m) \cdot (w, w^1, \dots, w^m) = (g_1^{-1} \cdots g_m^{-1} w, g_1 w^1, \dots, g_m w^m).$$

Therefore, the real vector field $\tilde{X}^* \in \mathfrak{X}(K_M)$ generated by $X = (X^1, \dots, X^m) \in (\sqrt{-1}\mathbf{R})^m = \mathfrak{t}^m$ is

$$\begin{aligned} \tilde{X}^* &= \tilde{X}^*_{hol} + \overline{\tilde{X}^*_{hol}} \\ \tilde{X}^*_{hol} &= \sum_{i=1}^m X^i w^i \frac{\partial}{\partial w^i} - (X^1 + \cdots + X^m) w \frac{\partial}{\partial w}. \end{aligned}$$

So

$$\begin{aligned} d^c t(\tilde{X}^*_{hol}) &= -\frac{\sqrt{-1}}{4} \left\{ \Sigma X^i w^i \frac{\frac{\partial}{\partial w^i} \det(g_M^{i\bar{j}})}{\det(g_M^{i\bar{j}})} - \Sigma X^i \right\} \\ &= -\frac{\sqrt{-1}}{4} \Sigma X^i \left\{ \frac{(m+1)|w^i|^2}{1 + \Sigma|w^j|^2} - 1 \right\}. \end{aligned}$$

So we can describe describe the moment map Φ as

$$\langle \Phi, X \rangle = -\frac{\sqrt{-1}}{2} ((m+1)r^2 + 1)^{1/(m+1)} \sum_{i=1}^m X^i \left\{ \frac{(m+1)|z^i|^2}{\sum_{j=1}^{m+1} |z^j|^2} - 1 \right\}.$$

Next, we describe will $\alpha(\tilde{X}_1, \dots, \tilde{X}_m)$ explicitly.

If we put $X_i := (0, \dots, \sqrt{-1}, \dots, 0) \in (\sqrt{-1}\mathbf{R})^m \cong \mathfrak{t}^m$ ($\sqrt{-1}$ is the i-th entry),

$$\tilde{X}^*_{i,hol} = \sqrt{-1}\left(w^i \frac{\partial}{\partial w^i} - w \frac{\partial}{\partial w}\right).$$

Since $\alpha = w\, dw^1 \wedge \cdots \wedge dw^m$,

$$\alpha(\tilde{X}_1, \dots, \tilde{X}_m) = (\sqrt{-1})^m (ww^1 \cdots w^m).$$

On other coordinates $\pi^{-1}(U_i)$ $(1 \le i \le m)$, we can also describe $\alpha(\tilde{X}_1, \dots, \tilde{X}_m)$ in the same way.

THEOREM 4.1. *Let $K_{\mathbf{C}P^m}$ be the canonical bundle of $\mathbf{C}P^m$ and $\pi : K_{\mathbf{C}P^m} \to \mathbf{C}P^m$ the projection. We consider $K_{\mathbf{C}P^m} = \{([z^1 : \dots : z^{m+1}], \xi) \mid [z^1 : \dots : z^{m+1}] \in \mathbf{C}P^m, \xi \in (K_{\mathbf{C}P^m})_{[z^1:\dots:z^{m+1}]}\}$ and r is the distance function between ξ and the 0-section measured by the fiber metric of $K_{\mathbf{C}P^m}$ induced by the Fubini-Study metric on $\mathbf{C}P^m$ of Einstein constant 2.*

Then T^m-invariant special Lagrangian submanifolds in $K_{\mathbf{C}P^m}$ are given by the equations:

$$((m+1)r^2 + 1)^{1/(m+1)} \left\{ \frac{(m+1)|z^i|^2}{\sum_{j=1}^{m+1} |z^j|^2} - 1 \right\} = A_i \quad (1 \le i \le m)$$
$$\mathrm{Im}(\alpha(\tilde{X}_1, \dots, \tilde{X}_m)) = A_{m+1}$$

where $A_1, \dots, A_{m+1}$ are any real constants and $\alpha(\tilde{X}_1, \dots, \tilde{X}_m)$ is a complex valued function on K_M.

On $\pi^{-1}(U_i)$ $(U_i := \{[z^1 : \dots : z^{m+1}] \in \mathbf{C}P^m \mid z_i \ne 0\}$ $(1 \le i \le m+1))$, if we take the local coordinate $(w_{(i)}, w^1_{(i)}, \dots, w^{i-1}_{(i)}, w^{i+1}_{(i)}, \dots, w^{m+1}_{(i)})$ on $\pi^{-1}(U_i)$ as $w^j_{(i)} = \dfrac{z^j}{z^i}$ and $w_{(i)}$ is a fiber coordinate of K_M with respect to $dw^1_{(i)} \wedge \cdots \wedge dw^{i-1}_{(i)} \wedge dw^{i+1}_{(i)} \wedge \cdots \wedge dw^{m+1}_{(i)}$, then

$$\alpha(\tilde{X}_1, \dots, \tilde{X}_m) = (\sqrt{-1})^m (-1)^{m-i+1} (w_{(i)} w^1_{(i)} \cdots w^{i-1}_{(i)} w^{i+1}_{(i)} \cdots w^{m+1}_{(i)}).$$

Appendix: The Metric on $K_{\mathbf{C}P^m}$

On [7] (Example 8.2.5), the metric on $K_{\mathbf{C}P^m}$ is given as follows.

Let $\mathbf{C}^{m+1}$ have complex coordinates $(z^1, \dots, z^{m+1})$, let $\zeta = e^{2\pi\sqrt{-1}/(m+1)}$, and let ζ act on $\mathbf{C}^{m+1}$ by $\zeta : (z^1, \dots, z^{m+1}) \mapsto (\zeta z^1, \dots, \zeta z^{m+1})$. Then the group generated by ζ is isomorphic to $\mathbf{Z}_{m+1}$ because $\zeta^{m+1} = 1$, and $\mathbf{Z}_{m+1}$ acts freely on $\mathbf{C}^{m+1} - \{0\}$. Thus the quotient $\mathbf{C}^{m+1}/\mathbf{Z}_{m+1}$ has an isolated singular point at 0. Let (X, ϖ) be the blow-up of $\mathbf{C}^{m+1}/\mathbf{Z}_{m+1}$ at 0. X is biholomorphic to $K_{\mathbf{C}P^m}$. Let $\tilde{r} = (\Sigma|z^i|^2)^{1/2}$ be the radius function on $\mathbf{C}^{m+1}/\mathbf{Z}_{m+1}$.

Define $f : \mathbf{C}^{m+1}/\mathbf{Z}_{m+1} - \{0\} \to \mathbf{R}$ by

$$f = (\tilde{r}^{2m+2} + 1)^{1/(m+1)} + \frac{1}{m+1} \sum_{j=0}^{m} \zeta^j \log((\tilde{r}^{2m+2} + 1)^{1/(m+1)} - \zeta^j).$$

Then, $\omega := dd^c \varpi^*(f)$ defines the Kähler form on $X - \varpi^{-1}(0)$ and extends to all of X. This metric is given by Calabi [2] and in the case $m = 1$, this is Eguchi-Hanson metric.

We will show that this metric is the same (up to constant factors) as the one in Examples of $M = \mathbf{C}P^m$.

• First, we will describe the metric in Examples of $M = \mathbf{C}P^m$ more explicitly. The Kähler form ω_{K_M} is given by

$$\omega_{K_M} = \pi^* \omega_M + dd^c F(t)$$

where $F \in C^\infty(\mathbf{R})$ with

$$F'(t) = ((m+1)e^{2t} + 1)^{1/(m+1)} - 1.$$

We will integrate $F'(t)$. For that, we will change a variable from t to $r = e^t$.

If we describe $G(r) := F(\log r)$, then

$$\frac{dG}{dr}(r) = \frac{((m+1)r^2 + 1)^{1/(m+1)} - 1}{r}.$$

We can easily see that

$$\begin{aligned} G(r) = \frac{m+1}{2} \{ &((m+1)r^2 + 1)^{1/(m+1)} \\ &+ \frac{1}{m+1} \sum_{j=0}^{m} \zeta^j \log(((m+1)r^2 + 1)^{1/(m+1)} - \zeta^j) \} - \log r + const. \end{aligned}$$

Remark for $k^{m+1} \neq 1$,

$$\sum_{j=0}^{m} \frac{1}{k\zeta^j - 1} = \frac{m+1}{k^{m+1} - 1}.$$

• Next, we define $\Psi : \mathbf{C}^{m+1}/\mathbf{Z}_{m+1} - \{0\} \to K_{\mathbf{C}P^m} - \{0\text{-section}\}$ as

$$\begin{aligned} \Psi(z^1, \dots, z^{m+1}) &= (-1)^{i-1} (z^i)^{m+1} \, d\left(\frac{z^1}{z^i}\right) \wedge \cdots \wedge d\left(\frac{z^{i-1}}{z^i}\right) \\ &\quad \wedge d\left(\frac{z^{i+1}}{z^i}\right) \wedge \cdots \wedge d\left(\frac{z^{m+1}}{z^i}\right) \end{aligned}$$

$$\pi \circ \Psi(z^1, \dots, z^{m+1}) = [z^1 : \dots : z^{m+1}]$$

on $\{z^i \neq 0\} \subset \mathbf{C}^{m+1}/\mathbf{Z}_{m+1} - \{0\}$.

Ψ is well-defined and biholomorphic.

Then

$$\begin{aligned}
r^2 \circ \Psi(z^1, \ldots, z^{m+1}) &= \left(\frac{1}{m+1}\right)^{m+1} |z^i|^{2m+2} \left(1 + \sum_{j \neq i} \left|\frac{z^j}{z^i}\right|^2\right)^{m+1} \\
&= \left(\frac{1}{m+1}\right)^{m+1} \left(\sum_j |z^j|^2\right)^{m+1} \\
&= \left(\frac{1}{m+1}\right)^{m+1} \tilde{r}^{2m+2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Psi^* \omega_{K_M} &= \Psi^*(\pi^* \omega_M + dd^c G(r)) \\
&= dd^c \left\{ \frac{m+1}{2} \log\left(1 + \sum_{j \neq i} \left|\frac{z^j}{z^i}\right|^2\right) + G\left(\left(\frac{1}{m+1}\right)^{(m+1)/2} \tilde{r}^{m+1}\right) \right\} \\
&= dd^c \left\{ \frac{m+1}{2} \log(\tilde{r}^2) + G\left(\left(\frac{1}{m+1}\right)^{(m+1)/2} \tilde{r}^{m+1}\right) \right\}.
\end{aligned}$$

From the description of G, we see that metrics are the same up to constant factors.

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