T. BAO AND T. ADACHI
KODAI MATH. J.
34 (2011), 485–504

BEHAVIORS OF CIRCULAR TRAJECTORIES ON HYPERSURFACES OF TYPE (A_1) IN A COMPLEX HYPERBOLIC SPACE

TUYA BAO AND TOSHIAKI ADACHI

Abstract

We study circular trajectories for Sasakian magnetic fields on geodesic spheres, horospheres and tubes around totally geodesic complex hypersurfaces in a complex hyperbolic space. Investigating their extrinsic shapes in the ambient complex hyperbolic space, we give conditions for them to be bounded and to be closed. By use of information on lengths of circles in complex space forms, we give expressions of lengths of circular trajectories on those real hypersurfaces and show that their length spectrum is a discrete subset of a real line.

1. Introduction

As a generalization of static magnetic fields on a Euclidean 3-space, a closed 2-form $\mathbf{B}$ on a Riemannian manifold N is said to be a magnetic field. We define a skew symmetric operator $\Omega_{\mathbf{B}} : TN \to TN$ on the tangent bundle TN by $\langle v, \Omega_{\mathbf{B}}(w)\rangle = \mathbf{B}(v, w)$ for all $v, w \in T_pN$ at an arbitrary point $p \in N$ with Riemannian metric $\langle\,,\rangle$. A magnetic field is said to be uniform if $\Omega_{\mathbf{B}}$ is parallel. On a Kähler manifold, constant multiples of its Kähler form are uniform magnetic fields. In [1] and its sequels the second author has studied some of their properties. Since Kähler manifolds are real even dimensional, we are interested in some objects corresponding to Kähler magnetic fields on odd-dimensional Riemannian manifolds. On a real hypersurface M in a Kähler manifold $\tilde{M}$ with complex structure J, we can consider a canonical closed 2-form $\mathbf{F}_\phi$ given by $\mathbf{F}_\phi(v, w) = \langle v, \phi w\rangle$. Here, $\phi : TM \to TM$ denotes the characteristic tensor induced by J which is defined by $\phi w = Jw + \langle w, J\mathcal{N}\rangle\mathcal{N}$ with a unit normal $\mathcal{N}$

2000 *Mathematics Subject Classification*. Primary 53C40, Secondary 53C22.

Key words and phrases. geodesic spehres, horospheres, tubes around totally geodesic complex hypersurfaces, Sasakian magnetic fields, circular trajectories, extrinsic shapes, length spectrum, complex hyperbolic spaces.

The first author is partially supported by NGK Foundation for International Students.

The second author is partially supported by Grant-in-Aid for Scientific Research (C) (No. 20540071), Japan Society for the Promotion of Sciences.

Received March 1, 2011; revised March 29, 2011.

on M in $\tilde{M}$. We call constant multiples of this form *Sasakian magnetic fields* (see [2, 7]).

For a magnetic field $\mathbf{B}$ we call a smooth curve γ parameterized by its arclength *trajectory* if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \Omega_{\mathbf{B}}(\dot\gamma)$. For a Sasakian magnetic field $\mathbf{F}_\kappa = \kappa\mathbf{F}_\phi$ $(\kappa \in \mathbf{R})$, its trajectory is hence a curve γ which is parameterized by its arclength and satisfies $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$. Unfortunately, being different from Kähler magnetic fields, Sasakian magnetic fields are not uniform. Therefore their trajectories are not "elementary" as Frenet curves, in general. Since every trajectory for a Kähler magnetic field is a circle, we restrict ourselves on circular trajectories, which are trajectories and are also circles of positive geodesic curvatures, for Sasakian magnetic fields.

In this paper we study circular trajectories on geodesic spheres, on horospheres and on tubes around complex hypersurfaces $\mathbf{C}H^{n-1}$ in a complex hyperbolic space $\mathbf{C}H^n$. These submanifolds are typical "nice" examples of homogeneous Riemannian manifolds, because their geodesics are homogeneous curves, that is, each of them is an orbit of a one-parameter subgroup of the isometry group of the base manifold. In [6] we studied geodesics on these real hypersurfaces and showed conditions for them to be bounded and to be closed. Since trajectories are considered as perturbed objects of geodesics, we are interested in their properties. As was mentioned in [5], trajectories for Sasakian magnetic fields on these real hypersurfaces are also homogeneous. It is hence natural to consider that they have a resemblance to geodesics.

In the preceding paper [7], we showed there exist infinitely many circular trajectories which are not congruent to each other on geodesic spheres in a complex projective space $\mathbf{C}P^n$ and horospheres, geodesic spheres and tubes around complex hypersurfaces congruent to $\mathbf{C}H^{n-1}$ in $\mathbf{C}H^n$. As we studied properties of circular trajectories on geodesic spheres in a complex projective space in [8], we here treat circular trajectories on other hypersurfaces in complex hyperbolic spaces. On horospheres and on tubes around complex hypersurfaces, because they are noncompact, we first study whether circular trajectories are bounded or not (Theorems 1 and 4). For trajectories on geodesic spheres and for bounded trajectories on tubes around complex hypersurfaces, we also study conditions for them to be closed and give expressions of lengths of them (Theorems 2 and 5). Just like geodesics on these real hypersurfaces, we find there are infinitely many closed circular trajectories and infinitely many bounded open circular trajectories. But being different from geodesics, the expressions of lengths are a bit complicated. Giving rough estimates of lengths of closed circular trajectories, we investigate how they are distributed on a real line. We show that the set of lengths are discrete and that the number of congruence classes of closed circular trajectories which are shorter than a given arbitrary constant is finite (Theorems 3 and 6).

The authors are grateful to the referee who gives an important advice from the viewpoint of submanifold-theory.

2. Circular condition on trajectories

In a complex hyperbolic space $\mathbf{C}H^n(-c)$ of constant holomorphic sectional curvature $-c$ (< 0), we consider in this paper a horosphere HS, a geodesic sphere $G(r)$ of radius r and a tube $T(r)$ of radius r around totally geodesic $\mathbf{C}H^{n-1}$. These real hypersurfaces have common properties on shape operators:

1) The characteristic vector field ξ defined by $\xi = -J\mathcal{N}$ is a principal vector field;
2) The number of principal curvatures are two, and they are constant on each real hypersurface;
3) The shape operator A and the characteristic tensor field ϕ are simultaneously diagonalizable (i.e. $A\phi = \phi A$).

We here list their principal curvatures. We denote by ν_M the principal curvature of ξ and λ_M the principal curvature of vectors orthogonal to ξ. As all tangent vectors orthogonal to ξ are principal, these real hypersurfaces are said to be totally η-umbilic.

TABLE 1. principal curvatures of totally η-umbilic hypersurfaces

M	HS in $\mathbf{C}H^n(-c)$	$G(r)$ in $\mathbf{C}H^n(-c)$	$T(r)$ in $\mathbf{C}H^n(-c)$
λ_M	$\sqrt{c}/2$	$(\sqrt{c}/2)\coth(\sqrt{c}r/2)$	$(\sqrt{c}/2)\tanh(\sqrt{c}r/2)$
ν_M	$\sqrt{c}$	$\sqrt{c}\coth(\sqrt{c}r)$	$\sqrt{c}\coth(\sqrt{c}r)$

In these real hypersurfaces, trajectories for Sasakian magnetic fields are classified into congruence classes by their structure torsions. We say two smooth curves γ_1, γ_2 on a Riemannian manifold N parameterized by their arclengths are *congruent* to each other if there exist an isometry φ of N and a constant t_0 with $\gamma_2(t + t_0) = \varphi \circ \gamma_1(t)$ for all t. For a trajectory γ for a Sasakian magnetic field $\mathbf{F}_\kappa$ on a real hypersurface M in $\mathbf{C}H^n$, we define its structure torsion ρ_γ by $\rho_\gamma = \langle \dot\gamma, \xi_\gamma \rangle$. It is known that the structure torsion of γ is not necessarily constant along γ. But on our real hypersurfaces, as their shape operators and their characteristic tensors are simultaneously diagonalizable, each trajectory for a Sasakian magnetic field has constant structure torsion (see [7]). We find that a trajectory γ_1 for $\mathbf{F}_{\kappa_1}$ and a trajectory γ_2 for $\mathbf{F}_{\kappa_2}$ are congruent to each other if and only if one of the following conditions holds (see [2]):

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,
ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ and $|\kappa_1| = |\kappa_2|$,
iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$ and $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

In [7] we studied features of trajectories for Sasakian magnetic fields. A smooth curve σ parameterized by its arclength on a Riemannian manifold N is said to be a *helix of proper order d* if it satisfies the following system of ordinary differential equations $\nabla_{\dot\gamma} Y_j = -\kappa_{j-1} Y_{j-1} + \kappa_j Y_{j+1}$ $(1 \le j \le d)$ with positive constants $\kappa_1, \dots, \kappa_{d-1}$ and an orthonormal system $\{Y_1 = \dot{\gamma}, Y_2, \dots, Y_d\}$ of vector fields along γ. Here, we put $\kappa_0 = \kappa_d = 0$ and choose Y_0, Y_{d+1} to be null vector fields along γ. We call these constants $\kappa_1, \dots, \kappa_{d-1}$ and the frame $\{Y_1, \dots, Y_d\}$ the geodesic curvatures and the Frenet frame of γ, respectively. For trivial (Sasakian) magnetic field $\mathbf{F}_0$, its trajectories are geodesics. For non-trivial Sasakian magnetic fields we have the following.

Proposition 1 ([7]). *Let M be a real hypersurface which is congruent to one of a horosphere HS, a geodesic sphere $G(r)$ of radius r and a tube $T(r)$ of radius r around totally geodesic $\mathbf{C}H^{n-1}$ in $\mathbf{C}H^n(-c)$. A trajectory γ for a non-trivial Sasakian magnetic field $\mathbf{F}_\kappa$ on M satisfies the following*:

(1) *If $\rho_\gamma = \pm 1$, it is a geodesic on M.*
(2) *If it satisfies $\kappa\rho_\gamma = \lambda_M$, it is a circle of geodesic curvature $|\kappa|\sqrt{1-\rho_\gamma^2}$ on M,*
(3) *Otherwise, it is a helix of proper order* 3.

We call a trajectory for a Sasakian magnetic field *circular* if it is also a circle of positive geodesic curvature. On a horosphere, a geodesic sphere and a tube around totally geodesic complex hypersurface in $\mathbf{C}H^n$, a trajectory γ for $\mathbf{F}_\kappa$ is circular if and only if $\kappa\rho_\gamma = \lambda_M$. We hence have that two circular trajectories γ_i for Sasakian magnetic fields $\mathbf{F}_{\kappa_i}$ $(i = 1, 2)$ are congruent to each other if and only if $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_1}| < 1$.

3. Extrinsic shapes of circular trajectories

In order to study curves on real hypersrufaces in a complex hyperbolic space, it is one of basic idea to investigate their extrinsic shapes. For a smooth curve γ on a submanifold M in $\mathbf{C}H^n$, we call the curve $\iota \circ \gamma$ with an immersion $\iota : M \to \mathbf{C}H^n$ its *extrinsic shape*. We call a helix on $\mathbf{C}H^n$ *Killing* if it is an orbit of a one-parameter family of isometries of $\mathbf{C}H^n$. It is known that a helix of proper order d on $\mathbf{C}H^n$ is Killing if and only if all its complex torsions τ_{ij} $(1 \le i < j \le d)$ defined by $\tau_{ij} = \langle Y_i, JY_j \rangle$ with its Frenet frame $\{Y_i\}_{i=1}^d$ are constant functions (see [11] and also [10]).

We here recall the influence of homothetical change of metrics. Let σ be a helix of proper order d with geodesic curvatures $k_1, \dots, k_{d-1}$ on a Riemannian manifold N. If we change the metric $\langle\,,\rangle$ on N homothetically to $\lambda^2\langle\,,\rangle$ with some positive λ, then the curve $\sigma_1(t) = \sigma(t/\lambda)$ is a helix of proper order d with geodesic curvatures $k_1/\lambda, \dots, k_{d-1}/\lambda$. When N is a real hypersurface in a Kähler manifold and σ is a trajectory for a Sasakian magnetic field $\mathbf{F}_\kappa$, under such a homothetic change of metric, we find σ_1 is a trajectory for $\mathbf{F}_{\kappa/\lambda}$. Since the sectional curvatures change λ^{-2}-times of the original sectional curvatures, we may only treat the case $\mathbf{C}H^n(-4)$.

We denote by ∇ and $\tilde{\nabla}$ the connections of a real hypersurface which is congruent to one of HS, $G(r)$, $T(r)$ and of $\mathbf{C}H^n(-4)$, respectively. They are related with each other by the Gauss formula $\tilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle\mathcal{N}$ and the Weingarten formula $\tilde{\nabla}_X\mathcal{N} = -AX$ for vector fields X, Y tangent to M.

Proposition 2. *The extrinsic shape of a circular* $\mathbf{F}_\kappa$*-trajectory on a real hypersurface which is congruent to one of* HS, $G(r)$ *and* $T(r)$ *in* $\mathbf{C}H^n(-4)$ *is a Killing helix of proper order* 4 *which lies on some totally geodesic* $\mathbf{C}H^2$. *Its geodesic curvatures are*

$$\kappa_1 = \frac{1}{\kappa^2}\sqrt{\kappa^6 + (1+2\kappa^2)\lambda_M^2}, \quad \kappa_2 = \frac{(\kappa^2+1)\lambda_M\sqrt{\kappa^2-\lambda_M^2}}{\kappa^2\sqrt{\kappa^6+(1+2\kappa^2)\lambda_M^2}},$$

$$\kappa_3 = \frac{\kappa^2-\lambda_M^2}{\sqrt{\kappa^6+(1+2\kappa^2)\lambda_M^2}},$$

and its complex torsions satisfy

$$\tau_{12} = \tau_{34} = \frac{-\mathrm{sgn}(\kappa)\cdot(\kappa_1+\kappa_3)}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}}, \quad \tau_{23} = \tau_{14} = \frac{-\mathrm{sgn}(\kappa)\cdot\kappa_2}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}}, \quad \tau_{13} = \tau_{24} = 0.$$

Proof. As we have $A\dot\gamma = \lambda_M\dot\gamma + (\nu_M - \lambda_M)\rho_\gamma\xi = \lambda_M\dot\gamma + \rho_\gamma\lambda_M^{-1}\xi$, by use of the circular condition $\kappa\rho_\gamma = \lambda_M$, we obtain $\tilde{\nabla}_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma + \rho_\gamma(\kappa+\kappa^{-1})\mathcal{N}$. Hence we have

$$\kappa_1 = \sqrt{\kappa^2 + 2\rho_\gamma^2 + \rho_\gamma^2\kappa^{-2}}\ \ (>0), \quad Y_2 = (\kappa J\dot\gamma + \rho_\gamma\kappa^{-1}\mathcal{N})/\kappa_1.$$

By use of Weingarten formula we have

$$\tilde{\nabla}_{\dot\gamma}(\kappa J\dot\gamma + \rho_\gamma\kappa^{-1}\mathcal{N}) = -(\kappa^2+\rho_\gamma^2)\dot\gamma - \rho_\gamma(1+\kappa^{-2})\xi,$$

thus we see

$$\kappa_2 = \kappa_1^{-1}(1+\kappa^{-2})|\rho_\gamma|\sqrt{1-\rho_\gamma^2}\ \ (>0), \quad Y_3 = \mathrm{sgn}(\rho_\gamma)\cdot(\rho_\gamma\dot\gamma - \xi)/\sqrt{1-\rho_\gamma^2}.$$

Continuing calculations we have $\tilde{\nabla}_{\dot\gamma}(\rho_\gamma\dot\gamma - \xi) = \kappa^{-1}(\rho_\gamma^2 - 1)\mathcal{N}$. Hence we see

$$\kappa_3 = \kappa_1^{-1}(1-\rho_\gamma^2)\ \ (>0), \quad Y_4 = \frac{\mathrm{sgn}(\rho_\gamma)}{\kappa_1\sqrt{1-\rho_\gamma^2}}\{\rho_\gamma(\kappa+\kappa^{-1})\phi\dot\gamma - \kappa(1-\rho_\gamma^2)\mathcal{N}\}.$$

As we have $\tilde{\nabla}_{\dot\gamma}\{\rho_\gamma(\kappa+\kappa^{-1})\phi\dot\gamma - \kappa(1-\rho_\gamma^2)\mathcal{N}\} = -(1-\rho_\gamma^2)(\rho_\gamma\dot\gamma - \xi)$, we find the extrinsic shape of γ is a helix of proper order 4. If we compute its complex torsions, then we get

$$\tau_{12} = \frac{1}{\kappa_1}\langle \dot\gamma, -\kappa\dot\gamma - \rho_\gamma\kappa^{-1}\xi\rangle = -\frac{\kappa^2+\rho_\gamma^2}{\kappa\kappa_1} = \frac{-\mathrm{sgn}(\kappa)\cdot(\kappa_1+\kappa_3)}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}} = \tau_{34},$$

$$\begin{aligned}\tau_{23} &= \frac{\mathrm{sgn}(\rho_\gamma)}{\kappa_1\sqrt{1-\rho_\gamma^2}}\langle \kappa J\dot\gamma + \rho_\gamma\kappa^{-1}\mathcal{N}, \rho_\gamma J\dot\gamma - \mathcal{N}\rangle \\ &= -\frac{|\rho_\gamma|\sqrt{1-\rho_\gamma^2}}{\kappa\kappa_1} = \frac{-\mathrm{sgn}(\kappa)\cdot\kappa_2}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}} = \tau_{14}.\end{aligned}$$

Thus we obtain our conclusion. □

Let $\varpi : H_1^{2n+1} \to \mathbf{C}H^n(-4)$ be a canonical fibration of an anti-de Sitter space H_1^{2n+1} $(\subset \mathbf{C}^{n+1})$. This connects the geometry of complex hyperbolic space with the geometry of complex Euclidean space. For a trajectory γ on our real hypersurface, we consider its extrinsic shape $\imath\circ\gamma$ and take its horizontal lift $\hat\gamma$ with respect to ϖ. We regard $\hat\gamma$ as a curve on $\mathbf{C}^{n+1}$. The connections $\overline{\nabla}$ on $\mathbf{C}^{n+1}$ associated with the Hermitian form $\langle\,,\rangle$ and $\tilde\nabla$ on $\mathbf{C}H^n(-4)$ are related as

$$\overline{\nabla}_X Y = \tilde\nabla_X Y + \langle X, Y\rangle\hat{\mathcal{N}} - \langle X, JY\rangle J\hat{\mathcal{N}}$$

for arbitrary vector fields X, Y on $\mathbf{C}H^n(-4)$ with a normal $\hat{\mathcal{N}}$ of H_1^{2n+1} in $\mathbf{C}^{n+1}$ satisfying $\langle\mathcal{N},\mathcal{N}\rangle = -1$ and with the complex structure J on $\mathbf{C}^{n+1}$. We here regard X, Y as horizontal vector fields on H_1^{2n+1}.

Lemma 1. *Let γ be a circular trajectory for $\mathbf{F}_\kappa$ $(\kappa \neq 0)$ on a real hypersurface M in $\mathbf{C}H^n(-4)$ which is congruent to one of HS, $G(r)$ and $T(r)$. A horizontal lift $\hat\gamma$ of its extrinsic shape satisfies the following differential equation if we regard it as a curve in $\mathbf{C}^{n+1}$:*

$$\hat\gamma''' - \sqrt{-1}(\kappa+\kappa^{-1})\hat\gamma'' - (2-\rho_\gamma^2)\hat\gamma' + \sqrt{-1}(1-\rho_\gamma^2)\kappa^{-1}\hat\gamma = 0. \tag{3.1}$$

Proof. The extrinsic shape of γ, which is also denoted by γ, is a helix of proper order 4 lying on some totally geodesic $\mathbf{C}H^2$. Therefore we find by Proposition 2 that it is determined by the differential equations

$$\begin{cases}\tilde\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma + \rho_\gamma\kappa^{-1}\mathcal{N} = \kappa_1 Y_2,\\ \tilde\nabla_{\dot\gamma}Y_2 = \kappa_1^{-1}\{-(\kappa^2+\rho_\gamma^2)\dot\gamma + \rho_\gamma(1+\kappa^{-2})J\mathcal{N}\}\\ \qquad\quad = -\kappa_1\dot\gamma + \{(\kappa_1+\kappa_3)\dot\gamma + \mathrm{sgn}(\kappa)\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}JY_2\}\end{cases}$$

(c.f. [3]). We hence have

$$\begin{cases}\overline{\nabla}_{\dot{\hat\gamma}}\dot{\hat\gamma} = \kappa_1 Y_2 + \hat{\mathcal{N}},\\ \overline{\nabla}_{\dot{\hat\gamma}}Y_2 = -\kappa_3\dot{\hat\gamma} + \mathrm{sgn}(\kappa)\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}JY_2 + \dfrac{\mathrm{sgn}(\kappa)\cdot(\kappa_1+\kappa_3)}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}}J\hat{\mathcal{N}}.\end{cases}$$

Hence we obtain

$$\begin{aligned}\overline{\nabla}_{\dot{\hat{\gamma}}}\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} &= \kappa_1\overline{\nabla}_{\dot{\hat{\gamma}}}Y_2 + \dot{\hat{\gamma}} \\ &= (1-\kappa_1\kappa_3)\dot{\hat{\gamma}} + \mathrm{sgn}(\kappa)\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}J\overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} \\ &\quad - \frac{\mathrm{sgn}(\kappa)\cdot\{\kappa_2^2+\kappa_3(\kappa_1+\kappa_3)\}}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}}J\mathcal{N},\end{aligned}$$

and get the conclusion. □

4. Circular trajectories on horospheres

We now study trajectories on our real hypersurfaces individually. A smooth curve σ parameterized by its arc-length on $\mathbf{C}H^n$ is said to be unbounded in both directions if both of the sets $\sigma([0,\infty)), \sigma((-\infty,0])$ are unbounded. Since $\mathbf{C}H^n$ is a typical example of Hadamard manifolds, which are simply connected complete Riemannian manifolds of nonpositive curvature, we can consider its ideal boundary $\partial\mathbf{C}H^n$ and its compactification $\overline{\mathbf{C}H^n} = \mathbf{C}H^n \cup \partial\mathbf{C}H^n$ with the cone topology (see [9]). If we represent $\mathbf{C}H^n$ as a ball model $D^n = \{w = (w_1,\dots,w_n) \in \mathbf{C}^n \mid |w_1|^2+\cdots+|w_n|^2 < 1\}$, its ideal boundary coincides with its topological boundary. When we study asymptotic behaviors of curves, the identification of a point $\varpi(z) \in \mathbf{C}H^n$ given by $z = (z_0,\dots,z_n) \in H_1^{2n+1}$ with a point $(z_1/z_0,\dots,z_n/z_0) \in D^n$ is useful. For a smooth curve σ which is unbounded in both directions, we set $\sigma(\infty) = \lim_{t\to\infty}\sigma(t)$, $\sigma(-\infty) = \lim_{t\to-\infty}\sigma(t) \in \partial\mathbf{C}H^n$ if they exist and call them points at infinity. For curves on a real hypersurface in $\mathbf{C}H^n$, regarding them as curves on $\mathbf{C}H^n$ we employ these terminologies.

THEOREM 1. *Every circular trajectory γ for a Sasakian magnetic field on a horosphere HS in $\mathbf{C}H^n$ is unbounded in both directions. In particular, it has a single point at infinity; $\gamma(\infty) = \gamma(-\infty)$.*

Proof. We are enough to consider a horosphere in $\mathbf{C}H^n(-4)$. Since we have $\kappa\rho_\gamma = 1$, the characteristic equation

$$\Lambda^3 - \sqrt{-1}(\kappa+\kappa^{-1})\Lambda^2 - (2-\kappa^{-2})\Lambda + \sqrt{-1}(\kappa^{-1}-\kappa^{-3}) = 0$$

of (3.1) has a pure imaginary double solution $\sqrt{-1}/\kappa$ and a pure imaginary solution $\sqrt{-1}(\kappa-\kappa^{-1})$. We therefore find that $\hat{\gamma}(t) = e^{\sqrt{-1}t/\kappa}(A+Bt) + Ce^{\sqrt{-1}(\kappa-\kappa^{-1})t}$ with some $A,B,C \in \mathbf{C}^{n+1}$. We hence obtain that γ is unbounded in both directions and has a single point at infinity. □

If we make mention of trajectories on HS with structure torsion ± 1, as they are geodesics on HS, each of them is unbounded in both directions and has a single point at infinity.

5. Lengths of circular trajectories on geodesic spheres

Next we study properties of circular trajectories on geodesic spheres. A smooth curve γ parameterized by its arclength is said to be closed if there is a positive constant t_c satisfying $\gamma(t+t_c)=\gamma(t)$ for all t. The minimum positive t_c with this property is called the length of γ and is denoted by length(γ). For a smooth curve which is not closed we say it is open and set length$(\gamma)=\infty$.

On a geodesic sphere $G(r)$ in $\mathbf{C}H^n(-4)$, a circular trajectory γ satisfies $\kappa\rho_\gamma=\coth r$, hence a horizontal lift $\hat{\gamma}$ of its extrinsic shape satisfies the equation

$$\hat{\gamma}'''-\sqrt{-1}(\kappa+\kappa^{-1})\hat{\gamma}''-(2-\kappa^{-2}\coth^2 r)\hat{\gamma}'+\sqrt{-1}(\kappa^{-1}-\kappa^{-3}\coth^2 r)\hat{\gamma}=0.$$

As $G(r)$ is compact, it is clear that γ is bounded, hence the characteristic equation

$$\Lambda^3-\sqrt{-1}(\kappa+\kappa^{-1})\Lambda^2-(2-\kappa^{-2}\coth^2 r)\Lambda+\sqrt{-1}(\kappa^{-1}-\kappa^{-3}\coth^2 r)=0 \tag{5.1}$$

of this differential equation should have three distinct pure imaginary solutions $\sqrt{-1}a_\kappa, \sqrt{-1}b_\kappa, \sqrt{-1}d_\kappa$ $(a_\kappa<b_\kappa<d_\kappa)$. Thus $\hat{\gamma}$ is of the form

$$\hat{\gamma}(t)=Ae^{\sqrt{-1}a_\kappa t}+Be^{\sqrt{-1}b_\kappa t}+De^{\sqrt{-1}d_\kappa t}$$

with some linearly independent $A,B,D\in\mathbf{C}^{n+1}$. Thus we find that γ is closed if and only if $(b_\kappa-a_\kappa)/(d_\kappa-a_\kappa)$ is rational and that in that case its length is given as $2\pi\times\text{L.C.M.}\{(b_\kappa-a_\kappa)^{-1},(d_\kappa-a_\kappa)^{-1}\}$. Here, L.C.M.$(\alpha,\beta)$ for positive numbers α, β denotes the minimum of the set $\{j\alpha \mid j=1,2,\dots\}\cap\{j\beta \mid j=1,2,\dots\}$. We hence study the cubic equation (5.1). If we put $\theta=-\{3\sqrt{-1}\Lambda+(\kappa+\kappa^{-1})\}/\sqrt{2\{\kappa^2-4+(3\coth^2 r+1)\kappa^{-2}\}}$, it turns to

$$\theta^3-(3/2)\theta+\tau_G(\kappa;r)/\sqrt{2}=0, \tag{5.2}$$

where

$$\tau_G(\kappa;r)=-\mathrm{sgn}(\kappa)\frac{(\kappa^2-2)(2\kappa^4-8\kappa^2+9\coth^2 r-1)}{2(\kappa^4-4\kappa^2+3\coth^2 r+1)^{3/2}}.$$

This cubic equation coincides with the characteristic equation for circles on $\mathbf{C}P^n(4)$ of geodesic curvature $1/\sqrt{2}$ and complex torsion $\tau_{12}=\tau_G(\kappa;r)$ (see (5.1) in [8]). By use of Proposition 4 in [8] (see also [4]) we get the following.

Theorem 2. *Let γ be a circular $\mathbf{F}_\kappa$-trajectory on a geodesic sphere $G(r)$ of radius r in $\mathbf{C}H^n(-4)$.*

(1) *When $r>\log(\sqrt{2}+1)$ and $\kappa=\pm\sqrt{2}$, it is closed and its length is $2\sqrt{2}\pi\sinh r$.*

(2) *Otherwise, it is closed if and only if*

$$\frac{|\kappa^2-2|(2\kappa^4-8\kappa^2+9\coth^2 r-1)}{2(\kappa^4-4\kappa^2+3\coth^2 r+1)^{3/2}}=\frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q satisfying $p>q$. In this case its length is given as $\pi\delta(p,q)|\kappa|\cdot$

$\sqrt{(3p^2+q^2)/(\kappa^4-4\kappa^2+3\coth^2 r+1)}$, *where* $\delta(p,q)=1$ *when* pq *is odd and* $\delta(p,q)=2$ *when* pq *is even.*

For the sake of readers' convenience, we rewrite the above theorem to the case of geodesic spheres in a complex hyperbolic space of constant holomorphic sectional curvature $-c$. As we pointed out in section 3, we make use of homothetical changes of metrics.

Proposition 3. *Let γ be a circular $\mathbf{F}_\kappa$-trajectory on a geodesic sphere $G(r)$ of radius r in $\mathbf{C}H^n(-c)$.*

(1) *When* $r > (2/\sqrt{c})\log(\sqrt{2}+1)$ *and* $\kappa = \pm\sqrt{c/2}$, *it is closed and its length is* $4\sqrt{2/c}\pi\sinh r$.
(2) *Otherwise, it is closed if and only if*

$$\frac{|2\kappa^2-c|\{32\kappa^4-32c\kappa^2+c^2(9\coth^2(\sqrt{c}r/2)-1)\}}{\{16\kappa^4-16c\kappa^2+c^2(3\coth^2(\sqrt{c}r/2)+1)\}^{3/2}}=\frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q satisfying $p>q$. In this case its length is given as

$$4\pi\delta(p,q)|\kappa|\sqrt{(3p^2+q^2)/\{16\kappa^4-16c\kappa^2+c^2(3\coth^2(\sqrt{c}r/2)+1)\}},$$

where $\delta(p,q)=1$ *when* pq *is odd and* $\delta(p,q)=2$ *when* pq *is even.*

The above result shows when a circular trajectory is closed. But as the expression of its length is a bit complicated, we are interested in how lengths of closed circular trajectories are distributed. We denote by $\mathscr{T}_\phi(M)$ the set of all congruence classes of circular trajectories on M. Set theoretically it is bijective to the set $\{\kappa \mid \kappa > \lambda_M\}$ when M is one of HS, $G(r)$ and $T(r)$ in $\mathbf{C}H^n$. We define the *length spectrum* $\mathscr{L}:\mathscr{T}_\phi(M)\to\mathbf{R}\cup\{\infty\}$ of circular trajectories on M by $\mathscr{L}([\gamma])=\mathrm{length}(\gamma)$. Here $[\gamma]$ denotes the congruence class containing γ. We put $\mathrm{LSpec}_\phi(M)=\mathscr{L}(\mathscr{T}_\phi(M))\cap\mathbf{R}$ and call it also the length spectrum of circular trajectories on M. We denote by $\lambda_0(M)$ the infimum of the set $\mathrm{LSpec}_\phi(M)$ and call it the bottom of the length spectrum. For $\lambda\in\mathrm{LSpec}_\phi(M)$ we call the cardinality of the set $\mathscr{L}^{-1}(\lambda)$ the *multiplicity* of $\mathscr{L}$ at λ.

Theorem 3. *The length spectrum* $\mathrm{LSpec}_\phi(G(r))$ *of circular trajectories on a geodesic sphere $G(r)$ in $\mathbf{C}H^n(-c)$ has the following properties.*

(1) *It is an unbounded discrete set.*
(2) *For every positive T, the set* $\{[\gamma]\in\mathscr{T}_\phi(G(r)) \mid \mathscr{L}([\gamma])\le T\}$ *is a finite set. In particular, the multiplicity of $\mathscr{L}$ is finite at each point.*

We are enough to consider the case $c=4$. In order to show this theorem, we define two functions $f,g:(\coth^2 r,\infty)\to\mathbf{R}$ by

$$f(s)=\frac{(s-2)(2s^2-8s+9\coth^2 r-1)}{2(s^2-4s+3\coth^2 r+1)^{3/2}},\quad g(s)=\frac{s}{s^2-4s+3\coth^2 r+1}.$$

Theorem 2 shows that if $f(\kappa^2) = \pm q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$ with some relatively prime positive integer p, q satisfying $p > q$ then a circular trajectory for $\mathbf{F}_\kappa$ is closed and is of length $\pi\delta(p.q)\sqrt{(3p^2 + q^2)g(\kappa^2)}$. These functions satisfy the following properties:

i) The function f is monotone increasing, hence satisfies $1 > f(s) > f(\coth^2 r) > -1$;

ii) When $\coth^2 r \geq (\sqrt{13} + 3)/2$, the function g is monotone decreasing and $g(s) < \coth^2 r/(\cot^4 r - \cot^2 r + 1)$;

iii) When $\coth^2 r < (\sqrt{13} + 3)/2$, the function g is monotone increasing in the interval $(\coth^2 r, \sqrt{3\coth^2 r + 1})$ and is monotone decreasing in other part of its domain, hence $g(s) \leq 1/(2\sqrt{3\coth^2 r + 1} - 4)$.

For a pair (p, q) of relatively prime positive integers p, q with $p > q$, we put $\mu(p, q) = q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$. The above properties on g show the following rough estimate.

Lemma 2. *The length of a circular trajectory γ for $\mathbf{F}_\kappa$ on a geodesic sphere $G(r)$ in $\mathbf{C}H^n(-4)$ satisfying either $f(\kappa^2) = \mu(p, q)$ or $f(\kappa^2) = -\mu(p, q)$ is roughly estimated from above as*

$$\text{length}(\gamma) < \pi\delta(p.q)\coth r\sqrt{(3p^2 + q^2)/(\cot^4 r - \cot^2 r + 1)},$$
$$\textit{if } \coth^2 r \geq \frac{1}{2}(\sqrt{13} + 3),$$
$$\text{length}(\gamma) < \pi\delta(p.q)\sqrt{(3p^2 + q^2)/(2\sqrt{3\coth^2 r + 1} - 4)},$$
$$\textit{if } \coth^2 r < \frac{1}{2}(\sqrt{13} + 3).$$

Next we give estimates of lengths of closed circular trajectories from below. For a number τ with $0 < |\tau| < 1$ we denote by s_τ the solution of the equation $f(s) = \tau$ if it exists. Since $f(s) > f(\coth^2 r)$, for negative τ the solution s_τ exists if and only if $\coth^2 r < 2$, which is equivalent to $r > \log(\sqrt{2} + 1)$.

Lemma 3. *Let γ be a circular trajectory for $\mathbf{F}_\kappa$ on a geodesic sphere $G(r)$ with $r > \log(\sqrt{2} + 1)$ in $\mathbf{C}H^n(-4)$ satisfying $f(\kappa^2) = -\mu(p, q)$. Its length is roughly estimated from below as*

$$\text{length}(\gamma) > \pi\delta(p, q)\coth r\sqrt{(3p^2 + q^2)/(\coth^4 r - \coth^2 r + 1)}.$$

Proof. Since $f(s) > 0$ for $s > 2$, we see $s_\tau < 2$ for $\tau = -\mu(p, q)$ if it exists. As we have $\coth^2 r < 2$, the function g is monotone increasing in the interval $(\coth^2 r, 2)$. Therefore we have $g(s_\tau) > g(\coth^2 r)$, and get the conclusion. □

For a pair (p,q) of relatively prime positive integers p, q with $p>q$, we put

$$\begin{aligned}\varepsilon_1(p,q) &= (3p^2+q^2)\{1-\mu(p,q)^2\}/27 = p^2(p^2-q^2)^2(3p^2+q^2)^{-2},\\ \varepsilon_2(p,q) &= (3p^2+q^2)\mu(p,q)\sqrt{1-\mu(p,q)^2}\\ &= 3\sqrt{3}pq(p^2-q^2)(9p^2-q^2)(3p^2+q^2)^{-2}.\end{aligned}$$

Lemma 4. *Let γ be a circular trajectory for $\mathbf{F}_\kappa$ on a geodesic sphere $G(r)$ in $\mathbf{C}H^n(-4)$ satisfying $f(\kappa^2)=\mu(p,q)$. Its length is roughly estimated from below in the following manner: If* $\coth^2 r \geq (3+\sqrt{13})/2$,

$$\text{length}(\gamma) > \pi\delta(p,q)\sqrt{18\varepsilon_1(p,q)\sinh^2 r+\varepsilon_2(p,q)\sinh r},$$

and if $\coth^2 r < (3+\sqrt{13})/2$,

$$\begin{aligned}&\text{length}(\gamma)\\ &\quad > \pi\delta(p,q)\min\{\sqrt{18\varepsilon_1(p,q)\sinh^2 r+\varepsilon_2(p,q)\sinh r}, \sqrt{2(3p^2+q^2)/3}\sinh r\}.\end{aligned}$$

In particular, it is estimated from below as

$$\text{length}(\gamma) > 3\sqrt{2}\pi\sinh r\delta(p,q)p(p^2-q^2)(3p^2+q^2)^{-1}.$$

Proof. For $\tau=\mu(p,q)$ we have $s_\tau>2$. In the domain $\{s \mid s> \max(2,\coth^2 r)\}$, we see

$$f(s) > (s-2)(s^2-4s+3\coth^2 r+1)^{-1/2}.$$

Hence, if we set $u_\tau = 2+3\tau\sqrt{(\coth^2-1)/(1-\tau^2)}$, which is the solution of the equation $(s-2)(s^2-4s+3\coth^2 r+1)^{-1/2}=\tau$ with $u_\tau>2$, we find $s_\tau<u_\tau$.

In the case $\coth^2 r\geq(3+\sqrt{13})/2$, as g is monotone decreasing, we have

$$\begin{aligned}g(s_\tau) > g(u_\tau) &= \frac{2(1-\tau^2)+3\tau\sqrt{(\coth^2 r-1)(1-\tau^2)}}{3(\coth^2 r-1)}\\ &= \frac{2}{3}(1-\tau^2)\sinh^2 r+\tau\sqrt{1-\tau^2}\sinh r\\ &= \frac{1}{3p^2+q^2}\{18\varepsilon_1(p,q)\sinh^2 r+\varepsilon_2(p,q)\sinh r\}.\end{aligned}$$

In the case $\coth^2 r<(3+\sqrt{13})/2$, we have $g(s_\tau)>\min\{g(u_\tau),g(2)\}$. As $g(2)= 2\sinh^2 r/3$, we obtain the conclusion. $\square$

Proof of Theorem 3. We find that $\lim_{q\to\infty}\varepsilon_2(q+2,q)=\infty$ and $\lim_{q\to\infty}\mu(q+2,q)=1$. Thus Lemma 4 guarantees that $\text{LSpec}_\phi(G(r))$ is unbounded.

For a pair (p,q) of relatively prime positive integers p, q with $p > q$, the number of the solutions of κ for the equation $f(\kappa^2) = \mu(p.q)$ is at most 6, so is the number of the solutions for the equation $f(\kappa^2) = -\mu(p.q)$. In order to show the second assertion, we use Lemmas 3 and 4. For arbitrary positive T_1, it is clear that the number of pairs (p,q) of positive integers with $3p^2 + q^2 \leq T_1$ is finite. Next we consider the situation that $\varepsilon_2(p,q) \leq T_2$ for a given T_2. Given a positive integer q, we can easily check that $\varepsilon_2(p,q)$ is monotone increasing with respect to p $(> q)$. Thus, under the condition that $\varepsilon_2(p,q) \leq T_2$ we have $\varepsilon_2(q+1,q) \leq T_2$. Since we can see $\varepsilon_2(q+1,q)$ is monotone increasing, the number of q satisfying $\varepsilon_2(q+1,q) \leq T_2$ is finite. As we have $\varepsilon_2(q+[T_2]+1,q) > T_2$, where $[T_2]$ denotes the integer part of T_2, we obtain that the number of pairs (p,q) of positive integers satisfying $\varepsilon_2(p,q) \leq T_2$ and $p > q$ is finite. By use of Lemmas 3 and 4, we get the set $\{[\gamma] \in \mathscr{T}_\phi(G(r)) \mid \mathscr{L}([\gamma]) \leq T\}$ is finite for each positive T. This shows that $\mathrm{LSpec}_\phi(G(r))$ is discrete. □

We here give estimates on the bottom of the length spectrum of circular trajectories.

Proposition 4. *The bottom $\lambda_0(G(r))$ of the length spectrum of circular trajectories on a geodesic sphere $G(r)$ in $\mathbf{C}H^n(-4)$ is roughly estimated from below as follows*:

$$\begin{cases} \lambda_0(G(r)) > \frac{6}{7}\pi\sqrt{2\sinh r(9\sinh r + 5\sqrt{3})}, & \text{if } \coth^2 \geq (3+\sqrt{13})/2, \\ \lambda_0(G(r)) > 2\sqrt{14/3}\pi\sinh r, & \text{if } 2 \leq \coth^2 r < (3+\sqrt{13})/2, \\ \lambda_0(G(r)) = 2\sqrt{2}\pi\sinh r, & \text{if } \varsigma_* < \coth^2 r < 2, \\ \lambda_0(G(r)) > 2\pi\coth r\sqrt{7/(\coth^4 r - \coth^2 r + 1)}, & \text{if } \coth^2 r \leq \varsigma_*, \end{cases}$$

where ς_* is the solution of the cubic equation $7x^3 - 15x^2 + 8x - 1 = 0$ with $\frac{88}{63} < \varsigma_* < \frac{7}{5}$.

Proof. One can easily check that for $i = 1, 2$
i) $\varepsilon_i(p+1,q) > \varepsilon_i(p,q)$ for arbitrary (p,q),
ii) $\varepsilon_i(q+1,q)$ is monotone increasing,
iii) $4\varepsilon_1(2,1) = 144/169 > 36/49 = \varepsilon_1(3,1)$,
iv) $4\varepsilon_2(2,1) = 2520\sqrt{3}/169 > 360\sqrt{3}/49 = \varepsilon_2(3,1)$.
When $\coth^2 r \geq (3+\sqrt{13})/2$, we find by Lemma 4 that

$$\lambda_0(G(r)) > \pi\sqrt{18\varepsilon_1(3,1)\sinh^2 r + \varepsilon_2(3,1)\sinh r}.$$

Here we note that $\mu(3,1)$ might be smaller than $f(\coth^2 r)$, hence that there are no circular trajectories with $f(\kappa^2) = \mu(3,1)$. But this estimate can work. When $2 \leq \coth^2 r < (3+\sqrt{13})/2$, by Lemma 4 we need to compare

$$\pi\sqrt{18\varepsilon_1(3,1)\sinh^2 r + \varepsilon_2(3,1)\sinh r} \quad \text{and} \quad 2\sqrt{14/3}\pi\sinh r.$$

Under the condition on the radius r, we have $(\sqrt{13}-1)/6 < \sinh^2 r \le 1$, hence we find the latter is smaller. When $\coth^2 r < 2$, by Lemmas 3, 4 and Theorem 2, we need to compare

$$\pi\sqrt{18\varepsilon_1(3,1)\sinh^2 r + \varepsilon_2(3,1)\sinh r},\, 2\sqrt{2}\pi \sinh r,\, 2\cosh r\sqrt{\frac{7}{\cosh^4 r - \cosh^2 r + 1}}.$$

Clearly the first is larger than the second. Comparing the second and the third, we can get the conclusion. □

For about trajectories for Sasakian magnetic fields with structure torsion ± 1, they are geodesics and are closed of length $\pi \sinh 2r$. Thus we find closed circular trajectories are longer than these geodesics when $\coth^2 r \ge 2$.

6. Behavior of circular trajectories on tubes

In this section we study asymptotic behaviors of unbounded circular trajectories and lengths of closed circular trajectories on tubes around totally geodesic complex hypersurfaces in a complex hyperbolic space. Again we consider the characteristic equation of the differential equation for a horizontal lift of the extrinsic shape $\hat{\gamma}$ of a circular trajectory γ on $T(r)$ in $\mathbf{C}H^n(-4)$:

$$\begin{aligned}\Lambda^3 - \sqrt{-1}(\kappa+\kappa^{-1})\Lambda^2 - (2-\kappa^{-2}\tanh^2 r)\Lambda \\ + \sqrt{-1}(\kappa^{-1} - \kappa^{-3}\tanh^2 r) = 0.\end{aligned} \tag{6.1}$$

If we put $\Omega = -\sqrt{-1}\Lambda - (\kappa+\kappa^{-1})/3$, we find this cubic equation turns to

$$\begin{aligned}\Omega^3 - \frac{1}{3}\{\kappa^2 - 4 + (1+3\tanh^2 r)\kappa^{-2}\}\Omega \\ - \frac{1}{27}\{2\kappa^3 - 12\kappa + 3(5+3\tanh^2 r)\kappa^{-1} + 2(1-9\tanh^2 r)\kappa^{-3}\} = 0.\end{aligned} \tag{6.2}$$

We set $\zeta(\kappa; r) = \kappa^2 - 4 + (1+3\tanh^2 r)\kappa^{-2}$. If $\zeta(\kappa; r) \le 0$, as the cubic equation (6.2) is not $\Omega^3 = 0$ even when $\zeta(\kappa; r) = 0$, we find the cubic equation (6.2) has only one real solution. When $\zeta(\kappa; r) > 0$, by putting $\theta = 3\Omega/\sqrt{2\zeta(\kappa; r)}$ we see (6.2) turns to

$$\theta^3 - (3/2)\theta + \tau_T(\kappa; r)/\sqrt{2} = 0, \tag{6.3}$$

where

$$\tau_T(\kappa; r) = -\mathrm{sgn}(\kappa)\frac{(\kappa^2-2)(2\kappa^4 - 8\kappa^2 + 9\tanh^2 r - 1)}{2(\kappa^4 - 4\kappa^2 + 3\tanh^2 r + 1)^{3/2}}.$$

We first study conditions for circular trajectories to be bounded.

Theorem 4. *On a tube $T(r)$ of radius r around $\mathbf{C}H^{n-1}(-4)$ in $\mathbf{C}H^n(-4)$ the behavior of a circular trajectory γ for $\mathbf{F}_\kappa$ is as follows;*

(1) *If κ satisfies $2\{1-(\cosh r)^{-1}\} < \kappa^2 < 2\{1+(\cosh r)^{-1}\}$, it is unbounded in both directions and has two distinct points at infinity.*

(2) *When $\kappa^2 = 2\{1 \pm (\cosh r)^{-1}\}$, it is also unbounded in both directions but has a single point at infinity.*

(3) *If κ satisfies either $\tanh^2 r < \kappa^2 < 2\{1-(\cosh r)^{-1}\}$ or $\kappa^2 > 2\{1+(\cosh r)^{-1}\}$, then it is bounded.*

Proof. We first consider the case $\zeta(\kappa, r) \le 0$ for κ with $|\kappa| > \tanh r$. Such case occurs when $2 - \sqrt{3}(\cosh r)^{-1} \le \kappa^2 \le 2 + \sqrt{3}(\cosh r)^{-1}$. In this case, the left hand side of (6.2) is monotone increasing with respect to Ω. We hence find that (6.1) has one pure imaginary solution and two distinct solutions which are not pure imaginary. Thus we obtain that γ is unbounded in both directions. More precisely, the solutions of (6.2) are of the form $-2\alpha_\kappa$, $\alpha_\kappa \pm \sqrt{-1}\beta_\kappa$ with real numbers α_κ, β_κ satisfying $3(3\alpha_\kappa^2 - \beta_\kappa^2) = \zeta(\kappa; r)$, $\alpha_\kappa \ne 0$ and $\beta_\kappa \ne 0$. We hence find that a horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ is of the form

$$\hat{\gamma}(t) = Ae^{-\sqrt{-1}\{2\alpha_\kappa - (\kappa+\kappa^{-1})/3\}t} + (Be^{\beta_\kappa t} + Ce^{-\beta_\kappa t})e^{\sqrt{-1}\{\alpha_\kappa + (\kappa+\kappa^{-1})/3\}t}$$

with $\mathbf{C}$-linearly independent $A, B, C \in \mathbf{C}^{n+1}$. Hence, rewriting this expression on the ball model D^n of a complex hyperbolic space, we obtain γ has two distinct points at infinity.

We next study the case $\zeta(\kappa, r) > 0$ for κ with $|\kappa| > \tanh r$. In view of (6.3) we find it has three distinct real solutions if and only if $|\tau_T(\kappa; r)| < 1$. This means that the original characteristic equation (6.1) has 3 distinct pure imaginary solutions if and only if $|\tau_T(\kappa; r)| < 1$. Thus if $|\tau_T(\kappa; r)| \ge 1$, which occurs when both $2\{1-(\cosh r)^{-1}\} \le \kappa^2 \le 2\{1+(\cosh r)^{-1}\}$ and $\zeta(\kappa; r) > 0$ hold, our circular trajectory γ is unbounded in both directions. We also find that if $|\tau_T(\kappa; r)| < 1$ it is bounded.

When $|\tau_T(\kappa; r)| > 1$, the solutions of (6.3) are of the form $-2\alpha_\kappa$, $\alpha_\kappa \pm \sqrt{-1}\beta_\kappa$ with real numbers α_κ, β_κ satisfying $2(3\alpha_\kappa^2 - \beta_\kappa^2) = 3$ and $2\sqrt{2}\alpha_\kappa(\alpha_\kappa^2 + \beta_\kappa^2) = \tau_T(\kappa; r)$. In particular, we have $\beta_\kappa \ne 0$. Thus we find that a horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ is of the form

$$\begin{aligned}\hat{\gamma}(t) &= Ae^{-\sqrt{-1}\{2\sqrt{2\zeta(\kappa;r)}\alpha_\kappa - (\kappa+\kappa^{-1})\}t/3} \\ &\quad + (Be^{\sqrt{2\zeta(\kappa;r)}\beta_\kappa t/3} + Ce^{-\sqrt{2\zeta(\kappa;r)}\beta_\kappa t/3})e^{\sqrt{-1}\{\sqrt{2\zeta(\kappa;r)}\alpha_\kappa + (\kappa+\kappa^{-1})\}t/3}\end{aligned}$$

with $\mathbf{C}$-linearly independent $A, B, C \in \mathbf{C}^{n+1}$. We obtain γ has two distinct points at infinity in this case. When $\tau_T(\kappa; r) = \pm 1$, we find that a horizontal lift $\hat{\gamma}$ of the extrinsic shape of γ is of the form

$$\hat{\gamma}(t) = Ae^{\sqrt{-1}\{\mp 2\sqrt{\zeta(\kappa;r)} + (\kappa+\kappa^{-1})\}t/3} + (B + Ct)e^{\sqrt{-1}\{\pm\sqrt{\zeta(\kappa;r)} + (\kappa+\kappa^{-1})\}t/3}$$

with $A, B, C \in \mathbf{C}^{n+1}$. Thus we find γ has a single point at infinity in this case. We hence get the conclusion. $\square$

Since trajectories are defined by their initial velocity vectors, it is clear that unbounded trajectories are open. We hence study whether bounded circular trajectories are closed or not on tubes around totally geodesic $\mathbf{C}H^{n-1}$ in $\mathbf{C}H^n$.

THEOREM 5. *Let γ be a bounded circular trajectory for $\mathbf{F}_\kappa$ on a tube $T(r)$ of radius r around $\mathbf{C}H^{n-1}(-4)$ in $\mathbf{C}H^n(-4)$.*

(1) *When $r \leq \log(\sqrt{2}+1)$ and $\kappa^2 = \{4 + 3\sqrt{2}(\cosh r)^{-1}\}/2$, it is closed of length $2\pi\sqrt{\cosh r(4\cosh r + 3\sqrt{2})}$.*

(2) *When $r > \log(\sqrt{2}+1)$ and $\kappa^2 = \{4 \pm 3\sqrt{2}(\cosh r)^{-1}\}/2$, it is closed of length $2\pi\sqrt{\cosh r(4\cosh r \pm 3\sqrt{2})}$, where double signs take the same signatures.*

(3) *If κ satisfies either $\tanh^2 r < \kappa^2 < 2\{1 - (\cosh r)^{-1}\}$ or $\kappa^2 > 2\{1 + (\cosh r)^{-1}\}$ and is not in the cases of (1) and (2), it is closed if and only if*

$$\frac{|\kappa^2 - 2|\,|2\kappa^4 - 8\kappa^2 + 9\tanh^2 r - 1|}{2(\kappa^4 - 4\kappa^2 + 3\tanh^2 r + 1)^{3/2}} = \frac{q(9p^2 - q^2)}{(3p^2 + q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q satisfying $p > q$. In this case its length is given as $\pi\delta(p,q)|\kappa| \cdot \sqrt{(3p^2+q^2)/(\kappa^4 - 4\kappa^2 + 3\tanh^2 r + 1)}$, where $\delta(p,q) = 1$ when pq is odd and $\delta(p,q) = 2$ when pq is even.

Proof. We need to consider the case that three conditions $\zeta(\kappa; r) > 0$, $|\tau_T(\kappa; r)| \leq 1$ and $|\kappa| > \tanh r$ hold. We compare (6.3) with the characteristic equation for circles on $\mathbf{C}P^n(4)$ of geodesic curvature $1/\sqrt{2}$ and complex torsion $\tau_{12} = \tau_T(\kappa; r)$ (see (5.1) in [8]). If we consider the case $\tau_T(\kappa; r) = 0$ we obtain the first and the second assertions. If we consider the case $0 < |\tau_T(\kappa; r)| < 1$, we obtain the third, and complete the proof. □

Summarizing Theorems 4 and 5 up we obtain the following result on our tubes in a complex hyperbolic space of constant holomorphic sectional curvature $-c$.

PROPOSITION 5. *Let γ be a circular trajectory for $\mathbf{F}_\kappa$ on a tube $T(r)$ of radius r around $\mathbf{C}H^{n-1}(-c)$ in $\mathbf{C}H^n(-c)$.*

(1) *If κ satisfies $(c/2)\{1 - (\cosh(\sqrt{c}r/2))^{-1}\} < \kappa^2 < (c/2)\{1 + (\cosh(\sqrt{c}r/2))^{-1}\}$, it is unbounded in both directions and has two distinct points at infinity.*

(2) *When $\kappa^2 = (c/2)\{1 \pm (\cosh(\sqrt{c}r/2))^{-1}\}$, it is also unbounded in both directions but has a single point at infinity.*

(3) *If κ satisfies either $(c/4)\tanh^2(\sqrt{c}r/2) < \kappa^2 < (c/2)\{1 - (\cosh(\sqrt{c}r/2))^{-1}\}$ or $\kappa^2 > (c/2)\{1 + (\cosh(\sqrt{c}r/2))^{-1}\}$, then it is bounded and satisfies the following*:

1) *When* $r \le (2/\sqrt{c})\log(\sqrt{2}+1)$ *and* $\kappa^2 = c\{4+3\sqrt{2}(\cosh(\sqrt{c}r/2))^{-1}\}/8$, *it is closed of length* $4\pi\sqrt{\cosh r(4\cosh r + 3\sqrt{2})/c}$.
2) *When* $r > (2/\sqrt{c})\log(\sqrt{2}+1)$ *and* $\kappa^2 = c\{4 \pm 3\sqrt{2}(\cosh r)^{-1}\}/8$, *it is closed of length* $4\pi\sqrt{\cosh r(4\cosh r \pm 3\sqrt{2})/c}$, *where double signs take the same signatures.*
3) *If* κ *satisfies either* $(c/4)\tanh^2(\sqrt{c}r/2) < \kappa^2 < (c/2)\{1-(\cosh(\sqrt{c}r/2))^{-1}\}$ *or* $\kappa^2 > (c/2)\{1+(\cosh(\sqrt{c}r/2))^{-1}\}$ *and is not in the cases of* 3–1) *and* 3–2), *it is closed if and only if*

$$\frac{|2\kappa^2 - c|\,|32\kappa^4 - 32c\kappa^2 + c^2(9\tanh^2(\sqrt{c}r/2)-1)|}{\{16\kappa^4 - 16c\kappa^2 + c^2(3\tanh^2(\sqrt{c}r/2)+1)\}^{3/2}} = \frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

holds with some relatively prime positive integers p, q *satisfying* $p > q$. *In this case its length is given as*

$$4\pi\delta(p,q)|\kappa|\sqrt{(3p^2+q^2)/\{16\kappa^4 - 16c\kappa^2 + c^2(3\tanh^2(\sqrt{c}r/2)+1)\}},$$

where $\delta(p,q) = 1$ *when* pq *is odd and* $\delta(p,q) = 2$ *when* pq *is even.*

Next we study length spectrums of circular trajectories on tubes around complex hypersurfaces in $\mathbf{C}H^n$. We only consider the case $c = 4$. We define functions $f, g : I_1 \cup I_2 \to \mathbf{R}$ on the union of two intervals by

$$f(s) = \frac{(s-2)(2s^2 - 8s + 9\tanh^2 r - 1)}{2(s^2 - 4s + 3\tanh^2 r + 1)^{3/2}}, \quad g(s) = \frac{s}{s^2 - 4s + 3\tanh^2 r + 1},$$

where $I_1 = (\tanh^2 r, 2 - 2\sqrt{1-\tanh^2 r})$ and $I_2 = (2 + 2\sqrt{1-\tanh^2 r}, \infty)$. We then find these functions satisfy the following properties:

i) The function f is monotone increasing on each interval I_1, I_2, hence satisfies

$$-1 < \frac{(\tanh^2 r + 1)(2 - \tanh^2 r)(1 - 2\tanh^2 r)}{2(\tanh^4 r - \tanh^2 r + 1)^{3/2}} < f(s) < 1 \quad \text{on } I_1$$

and $-1 < f(s) < 1$ on I_2 and $f(\{4 + 3\sqrt{2}(\cosh r)^{-1}\}/2) = 0$;

ii) g is monotone increasing on I_1, hence satisfies

$$\sinh^2 r\cosh^2 r/(\sinh^4 r + \sinh^2 r + 1) < g(s) < 2\cosh r(\cosh r - 1);$$

iii) g is monotone decreasing on I_2, hence satisfies

$$2\cosh r(\cosh r + 1) > g(s) > 0.$$

Therefore we obtain the following:

Lemma 5. *The length of a circular trajectory* γ *for* $\mathbf{F}_\kappa$ *on a tube* $T(r)$ *around totally geodesic* $\mathbf{C}H^{n-1}$ *in* $\mathbf{C}H^n(-4)$ *satisfying either* $f(\kappa^2) = \mu(p,q)$ *or* $f(\kappa^2) = -\mu(p,q)$ *is roughly estimated from above as*

$$\text{length}(\gamma) < \pi\delta(p,q)\sqrt{2(3p^2+q^2)\cosh r(\cosh r + 1)}.$$

LEMMA 6. *The length of a circular trajectory γ for $\mathbf{F}_\kappa$ on a tube $T(r)$ around totally geodesic $\mathbf{C}H^{n-1}$ in $\mathbf{C}H^n(-4)$ satisfying $f(\kappa^2) = -\mu(p,q)$ is roughly estimated from below as*

$$\mathrm{length}(\gamma) > \pi\delta(p,q)\sqrt{(3p^2+q^2)\cosh r(4\cosh r + 3\sqrt{2})/3}, \quad \textit{if } \ r \le \log(\sqrt{2}+1),$$

$$\mathrm{length}(\gamma) > \pi\delta(p,q)\sinh 2r\sqrt{(3p^2+q^2)/\{4(\sinh^4 r + \sinh^2 r + 1)\}}, \quad \textit{if } \ r > \log(\sqrt{2}+1).$$

Proof. For given τ $(0 < |\tau| < 1)$ we denote by $s_\tau^{(1)} \in I_1$, $s_\tau^{(2)} \in I_2$ the solutions for the equation $f(s) = \tau$ if they exist. For negative τ, it is clear that $s_\tau^{(2)} < 2 + (3\sqrt{2})/(2\cosh r)$. It is also clear that $s_\tau^{(1)}$ does not exist when $\tanh^2 r \le 1/2$ and that $s_\tau^{(1)} > \tanh^2 r$ when $\tanh^2 r > 1/2$ and if it exists. Thus we have

$$g(s_\tau^{(1)}) > \frac{\sinh^2 r\cosh^2 r}{\sinh^4 r + \sinh^2 r + 1}, \quad g(s_\tau^{(2)}) > \frac{\cosh r(4\cosh r + 3\sqrt{2})}{3},$$

hence get the conclusion. □

For a pair of relatively prime positive integers p, q satisfying $p > q$, we set $\varepsilon_3(p,q) = (3p^2+q^2)\{1 - \mu(p.q)\}$.

LEMMA 7. *The length of a circular trajectory γ for $\mathbf{F}_\kappa$ on a tube $T(r)$ in $\mathbf{C}H^n(-4)$ satisfying $f(\kappa^2) = \mu(p,q)$ is roughly estimated from below as follows: When $r \le \log(\sqrt{2}+1)$, we have*

$$\begin{aligned}\mathrm{length}(\gamma) &> \pi\delta(p,q)\\ &\quad\times\min\left\{\frac{1}{2}\sinh 2r\sqrt{\frac{3p^2+q^2}{\sinh^4 r + \sinh^2 r + 1}},\right.\\ &\qquad\left.\sqrt{\frac{1}{3}\cosh r(4\cosh r + 2\sqrt{3})\varepsilon_3(p,q)}\right\},\end{aligned}$$

and when $r > \log(\sqrt{2}+1)$, we have

$$\begin{aligned}\mathrm{length}(\gamma) &> \pi\delta(p,q)\\ &\quad\times\min\left\{\sqrt{\frac{1}{3}(3p^2+q^2)\cosh r(4\cosh r - 3\sqrt{2})},\right.\\ &\qquad\left.\sqrt{\frac{1}{3}\cosh r(4\cosh r + 2\sqrt{3})\varepsilon_3(p,q)}\right\}.\end{aligned}$$

Proof. We use the notations in the proof of Lemma 6. We have $s_\tau^{(1)} > \tanh^2 r$ when $\tanh^2 r \le 1/2$ and if it exists, and $s_\tau^{(1)} > \{4 - 3\sqrt{2}(\cosh r)^{-1}\}/2$ when $\tanh^2 r > 1/2$. On the other hand, we have $s_\tau^{(2)} > \{4 + 3\sqrt{2}(\cosh r)^{-1}\}/2$. We put $u_\tau = 2 + \sqrt{3(3-2\tau)(1-\tanh^2 r)/\{2(1-\tau)\}}$, which is the solution of the equation

$$(2s^2 - 8s + 9\tanh^2 r - 1)/\{2(s^2 - 4s + 3\tanh^2 r + 1)\} = \tau$$

with $u_\tau > 2$. As we have

$$f(s) > (2s^2 - 8s + 9\tanh^2 r - 1)/\{2(s^2 - 4s + 3\tanh^2 r + 1)\}$$

when $s > 2 + 3\sqrt{(1-\tanh^2 r)/2}$, we obtain $s_\tau^{(2)} < u_\tau$. Since we have

$$g(\{4 - 3\sqrt{2}(\cosh r)^{-1}\}/2) = \cosh r(4\cosh r - 3\sqrt{2})/3,$$

$$\begin{aligned} g(u_\tau) &= \cosh r\{4(1-\tau)\cosh r + \sqrt{6(3-2\tau)(1-\tau)}\}/3 \\ &> \cosh r(4\cosh r + 2\sqrt{3})(1-\tau)/3, \end{aligned}$$

we obtain the conclusion. □

Theorem 6. *The length spectrum* $\mathrm{LSpec}_\phi(T(r))$ *of circular trajectories on a tube* $T(r)$ *around totally geodesic* $\mathbf{C}H^{n-1}(-c)$ *in* $\mathbf{C}H^n(-c)$ *has the following properties.*

(1) *It is an unbounded set.*
(2) *For every positive* T, *the set* $\{[\gamma] \in \mathscr{T}_\phi(T(r)) \mid \mathscr{L}([\gamma]) \le T\}$ *is a finite set.*

Proof. The first assertion is clear by Lemma 6. In order to show the second assertion we are enough to show that the number of pairs (p,q) satisfying $\varepsilon_3(p,q) \le T_1$ is finite for an arbitrary positive T_1. As we can easily check that $\varepsilon_3(p,q) > \sqrt{3p^2+q^2}$, we get the conclusion. □

Proposition 6. *The bottom* $\lambda_0(T(r))$ *of the length spectrum* $\mathrm{LSpec}_\phi(T(r))$ *of circular trajectories on a tube* $T(r)$ *around totally geodesic* $\mathbf{C}H^{n-1}$ *in* $\mathbf{C}H^n(-4)$ *is roughly estimated from below as* $\lambda_0(G(r)) > \pi \sinh 2r\sqrt{7/(\sinh^4 r + \sinh^2 r + 1)\}}$. *It is roughly estimated from above as*

$$\begin{cases} \lambda_0(G(r)) \le 2\pi\sqrt{\cosh r(4\cosh r + 3\sqrt{2})}, & \text{if } r \le \log(\sqrt{2}+1), \\ \lambda_0(G(r)) \le 2\pi\sqrt{\cosh r(4\cosh r - 3\sqrt{2})}, & \text{if } r > \log(\sqrt{2}+1). \end{cases}$$

Proof. By calculating differentials of corresponding functions, we can easily find that

i) $\varepsilon_3(p+1,q) > \varepsilon_3(p,q)$ for arbitrary pair (p,q) of integers with $p > q > 0$,
ii) $\varepsilon_3(q+1,q)$ is monotone increasing with respect to q,
iii) $4\varepsilon_3(2,1) = 4(13 - 35/\sqrt{13}) > 28 - 40/\sqrt{7} = \varepsilon_3(3,1)$.

We first consider to estimate $\lambda_0(T(r))$ from above. Since the function f takes all the value in the interval $(-1, 1)$, we compare lengths of circular trajectories with $f(\kappa^2) = \mu(3, 1)$ and those with $\kappa^2 = \{4 \pm 3\sqrt{2}(\cosh r)^{-1}\}/2$. We get our estimate by Lemma 5.

To get an estimate from below we need to use Lemmas 6 and 7. When $r \le \log(\sqrt{2} + 1)$, we have to compare

$$4 \cosh r(4 \cosh r + 3\sqrt{2}), \frac{7 \sinh^2 2r}{\sinh^4 r + \sinh^2 r + 1}, \frac{28\sqrt{7} - 40}{3\sqrt{7}} \cosh r(4 \cosh r + 2\sqrt{3}).$$

Clearly the first is larger than the third. Since $\sinh^2 r = \cosh^2 r - 1$, we find the second is the smallest. When $r > \log(\sqrt{2} + 1)$, we have to compare

$$4 \cosh r(4 \cosh r - 3\sqrt{2}), \frac{7 \sinh^2 2r}{\sinh^4 r + \sinh^2 r + 1}, \frac{28\sqrt{7} - 40}{3\sqrt{7}} \cosh r(4 \cosh r + 2\sqrt{3}).$$

As $\cosh r > \sqrt{2}$, we find the second is the smallest. We hence get the conclusion. □

If we make mention of trajectories with structure torsion ± 1 on $T(r)$, they are geodesics which are unbounded in both directions and have two points at infinity.

Addendum: Circular trajectories on geodesic spheres in $\mathbf{C}P^n$

We here add a property on length spectrum of circular trajectories on geodesic spheres in a complex projective space $\mathbf{C}P^n$ to Theorem 2 in [7].

THEOREM 7. *The length spectrum* $\mathrm{LSpec}(G(r))$ *of circular trajectories on a geodesic sphere* $G(r)$ *in* $\mathbf{C}P^n$ *is a discrete set. At each point* $\lambda \in \mathrm{LSpec}(G(r))$, *the multiplicity of length spectrum* $\mathscr{L}$ *is finite.*

Proof. We use Lemmas 3 and 4 in [7]. For arbitrary positive T, it is clear that the number of pairs (p, q) of relatively prime positive integers satisfying $p > q > 0$ and $3p^2 + q^2 \le T$ is finite. In order to show our theorem, we are enough to see that the number of pairs (p, q) of relatively prime positive integers satisfying $p > q > 0$ and $\varepsilon_4(p, q) \le T$ with $\varepsilon_4(p, q) = 3p^2 + q^2 - q(9p^2 - q^2)(3p^2 + q^2)^{-1/2}$ is finite for arbitrary positive T. We can easily check the following properties on $\varepsilon_4(p, q)$:

i) $\varepsilon_4(p + 1, q) > \varepsilon_3(p, q)$ for arbitrary pair (p, q) of integers with $p > q > 0$,

ii) $\varepsilon_4(q + 1, q)$ is monotone increasing with respect to q.

As $\varepsilon_4(p, q) > \sqrt{3p^2 + q^2}$, we get our conclusion. □

REFERENCES

[1] T. ADACHI, Kähler magnetic flows on a manifold of constant holomorphic sectional curvature, Tokyo J. Math. **18** (1995), 473–483.

[2] T. Adachi, Trajectories on geodesic spheres in a non-flat complex space form, J. Geom. **90** (2008), 1–29.
[3] T. Adachi, Essential Killing helices of low order on a non-flat complex space form, to appear in J. Math. Soc. Japan.
[4] T. Adachi and S. Maeda, Length spectrum of circles in a complex projective space, Osaka J. Math. **35** (1998), 553–565.
[5] T. Adachi and S. Maeda, Sasakian curves on hypersurfaces in a nonflat complex space form, Results in Math. **56** (2009), 488–499.
[6] T. Adachi, S. Maeda and M. Yamagishi, Length spectrum of geodesic spheres in a non-flat complex space form, J. Math. Soc. Japan **54** (2002), 373–408.
[7] T. Bao and T. Adachi, Circular trajectories on real hypersurfaces in a nonflat complex space form, J. Geom. **96** (2009), 41–55.
[8] T. Bao and T. Adachi, Lengths of circular trajectories on geodesic spheres in a complex projective space, Kodai Math. J. **34** (2011), 257–271.
[9] P. Eberlein, Geometry of nonpositively curved manifolds, Univ. Chicago Press, Chicago and London, 1996.
[10] S. Maeda and T. Adachi, Holomorphic helices in a complex space form, Proc. A.M.S. **125** (1997), 1197–1202.
[11] S. Maeda and Y. Ohnita, Helical geodesic immersion into complex space form, Geom. Dedicata **30** (1983), 93–114.

Tuya Bao
Division of Mathematics and Mathematical Science
Graduate School of Engineering
Nagoya Institute of Technology
Gokiso, Nagoya, 466-8555
Japan

College of Mathematics
Inner Mongolia University for the Nationalities
536 Huo Lin He Street, Tongliao
Inner Mongolia, 028043
People's Republic of China
E-mail: btyngy@yahoo.co.jp

Toshiaki Adachi
Department of Mathematics
Nagoya Institute of Technology
Gokiso, Nagoya 466-8555
Japan
E-mail: adachi@nitech.ac.jp