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ON TEICHMÜLLER METRIC AND THE LENGTH SPECTRUMS OF TOPOLOGICALLY INFINITE RIEMANN SURFACES

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Abstract

We consider a metric d_L on the Teichmüller space $T(R_0)$ defined by the length spectrum of Riemann surfaces. H. Shiga proved that d_L defines the same topology as that of the Teichmüller metric d_T on $T(R_0)$ if a Riemann surface R_0 can be decomposed into pairs of pants such that the lengths of all their boundary components except punctures are uniformly bounded from above and below.

In this paper, we show that there exists a Riemann surface R_0 of infinite type such that R_0 cannot be decomposed into such pairs of pants, whereas the two metrics define the same topology on $T(R_0)$. We also give a sufficient condition for these metrics to have different topologies on $T(R_0)$, which is a generalization of a result given by Liu-Sun-Wei.

1. Introduction

Let R_0 be a hyperbolic Riemann surface. We consider a pair (R, f) of a Riemann surface R and a quasiconformal map $f : R_0 \to R$. Two such pairs (R_1, f_1) and (R_2, f_2) are called equivalent if $f_2 \circ f_1^{-1} : R_1 \to R_2$ is homotopic to a conformal map. We denote the equivalence class of (R, f) by $[R, f]$. The set of all equivalence classes is called *the Teichmüller space* of R_0; we denote it by $T(R_0)$.

The Teichmüller space $T(R_0)$ has a complete metric d_T called *the Teichmüller metric* which is defined by

$$d_T([R_1, f_1], [R_2, f_2]) = \inf_f \log K(f),$$

where the infimum is taken over all quasiconformal maps $f : R_1 \to R_2$ homotopic to $f_2 \circ f_1^{-1}$ and $K(f)$ is the maximal dilatation of f.

We introduce another metric on $T(R_0)$. Let Σ_{R_0} be the set of non-trivial closed geodesics in R_0. We define *the length spectrum metric* d_L by

$$d_L([R_1, f_1], [R_2, f_2]) = \log \sup_{\alpha \in \Sigma_{R_0}} \max\left\{ \frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))}, \frac{\ell_{R_2}(f_2(\alpha))}{\ell_{R_1}(f_1(\alpha))} \right\},$$

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where $\ell_{R_i}(f_i(\alpha))$ is the hyperbolic length of the closed geodesic on R_i freely homotopic to $f_i(\alpha)$.

PROPOSITION 1.1 (Thurston [9], Proposition 3.5). *Let Σ'_{R_0} be the set of non-trivial simple closed geodesics in R_0. Then*

$$\sup_{\alpha\in\Sigma_{R_0}} \max\left\{\frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))}, \frac{\ell_{R_2}(f_2(\alpha))}{\ell_{R_1}(f_1(\alpha))}\right\} = \sup_{\alpha\in\Sigma'_{R_0}} \max\left\{\frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))}, \frac{\ell_{R_2}(f_2(\alpha))}{\ell_{R_1}(f_1(\alpha))}\right\}$$

holds.

In 1972, Sorvali [8] defined d_L, and showed the following.

LEMMA 1.2 ([8]). *For any $[R_1, f_1], [R_2, f_2] \in T(R_0)$,*

$$d_L([R_1, f_1], [R_2, f_2]) \le d_T([R_1, f_1], [R_2, f_2])$$

holds.

He conjectured that d_L defines the same topolpgy as that of d_T on $T(R_0)$ if R_0 is a topologically finite Riemann surface. In 1986, Li [3] proved that the statement holds in the case where R_0 is a compact Riemann surface with genus ≥ 2. In 1999, Liu [4] proved that Sorvali's conjecture is true, and he asked whether the statement holds for any Riemann surface of infinite type. To this question, Shiga [7] gave a negative answer, that is, he showed that there exists a Riemann surface R_0 of infinite type such that d_L and d_T do not define the same topology on $T(R_0)$. Also, he gave a sufficient condition for these metrics to define the same topology on $T(R_0)$.

THEOREM 1.3 ([7]). *Let R_0 be a Riemann surface. Assume that there exists a pants decomposition $R_0 = \bigcup_{k=1}^{\infty} P_k$ satisfying the following conditions.*

(1) *Each connected component of ∂P_k $(k = 1, 2, 3 \dots)$ is either a puncture or a simple closed geodesic of R_0.*
(2) *There exists a constant $M > 0$ such that if α is a boundary curve of some P_k then*

$$0 < M^{-1} < l_{R_0}(\alpha) < M$$

holds.

Then d_L defines the same topology as that of d_T on $T(R_0)$.

On the other hand, Liu-Sun-Wei [5] obtained a sufficient condition for these metrics to define different topologies on $T(R_0)$.

THEOREM 1.4 ([5]). *Let R_0 be a Riemann surface with a sequence $\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma'_{R_0}$ such that $\ell_{R_0}(\alpha_n) \to 0$ $(n \to \infty)$. Then d_L does not define the same topology as that of d_T on $T(R_0)$.*

The converse of Theorem 1.4 is not true. Indeed, a Rienann surface Shiga constructed in [7] is a counterexample.

In this paper, we show that the converse of Theorem 1.3 is not true by giving a counterexample. Also, we give a new sufficient condition for these metrics to define different topologies on $T(R_0)$ as follows.

THEOREM 1.5. *Let R_0 be a Riemann surface. Suppose that there exists a sequence $\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma'_{R_0}$ such that for an arbitrary sequence $\{\beta_n\}_{n=1}^{\infty} \subset \Sigma'_{R_0}$ with $\alpha_n \cap \beta_n \neq \emptyset$ $(n = 1, 2, \dots)$,*

$$\frac{\ell_{R_0}(\beta_n)}{\#(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)} \to \infty \quad (n \to \infty).$$

Then d_L does not define the same topology as that of d_T on $T(R_0)$.

We will explain in Section 3 that the above theorem is a generalization of Theorem 1.4.

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2. A counterexample

In this section, we show that the converse of Theorem 1.3 is not true. We use the following lemmas due to Bishop [2].

LEMMA 2.1 ([2], Lemma 3.1). *Let $T_1, T_2 \subset \mathbf{D}$ (the unit disk) be two hyperbolic triangles with sides (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively. Suppose all their angles are bounded below by $\theta > 0$ and*

$$\varepsilon := \max\left(\left|\log \frac{a_1}{a_2}\right|, \left|\log \frac{b_1}{b_2}\right|, \left|\log \frac{c_1}{c_2}\right|\right) \leq A.$$

Then there is a constant $C = C(\theta, A)$ and a $(1 + C\varepsilon)$-quasiconformal map $\varphi : T_1 \to T_2$ such that φ maps each vertex to the corresponding vertex and φ is affine on the edge of T_1.

LEMMA 2.2 ([2], Corollary 3.3). *Let $H, H' \subset \mathbf{D}$ be two hyperbolic hexagons with sides $(a_1, \dots, a_6)$ and $(b_1, \dots, b_6)$ respectively. Suppose $a_1, \dots, a_6$ and $b_1, \dots, b_6$ are $\leq B$ and are comparable with a constant B. Also assume that three alternating angles of H and the corresponding angles of H' are $\pi/2$ and the remaining angles are bounded below by $\theta > 0$ and above by $\pi - \theta$. If $\varepsilon = \max_i |\log a_i/b_i| \leq 2$, then there is a constant $C = C(\theta, B)$ and a $(1 + C\varepsilon)$-quasiconformal map $\varphi : H \to H'$ such that φ maps each vertex to the corresponding vertex and φ is affine on the edge of H.*

Lemma 2.3 ([2], Lemma 6.2). *Let P_1 and P_2 be pants with boundary lengths (a_1, b_1, c_1) and (a_2, b_1, c_1) respectively. Suppose $a_1, a_2, b_1, c_1 \le L$ (punctures count as length zero). Assume that $\varepsilon := |\log a_1/a_2| \le 2$, where we define $|\log a_1/a_2| = 0$ if $a_1 = a_2 = 0$ and $|\log a_1/a_2| = +\infty$ if one is zero and the other is not. Then there is a constant $C = C(L)$ and a $(1 + C\varepsilon)$-quasiconformal map $\varphi : P_1 \to P_2$ such that φ is affine on each of the boundary components.*

Also we note the following lemma (cf. Beardon [1]).

Lemma 2.4 ([1]). *For a hyperbolic right hexagon with the edge lengths a_1, b_3, a_2, b_1, a_3, b_2 (in counterclockwise direction),*

$$\cosh b_2 = \frac{\cosh a_2 + \cosh a_1 \cosh a_3}{\sinh a_1 \sinh a_3},$$

$$\frac{\sinh a_1}{\sinh b_1} = \frac{\sinh a_2}{\sinh b_2} = \frac{\sinh a_3}{\sinh b_3}.$$

Especially, a right hexagon is determined by the lengths of three alternating sides.

Now we give a counterexample to the converse of Theorem 1.3.

Example. Let Γ be a hyperbolic triangle group of signature $(2, 4, 8)$ acting on $\mathbf{D}$ and let P be a fundamental domain for Γ with angles $(\pi, \pi/4, \pi/4, \pi/4)$. Let O, a, b, c denote the vertices of P, where the angle at O is π (as in Figure 1).

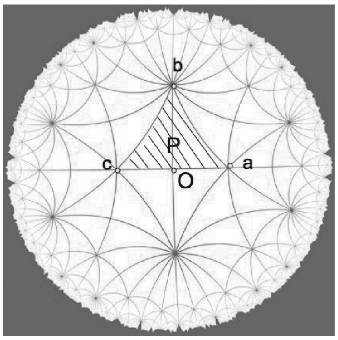

Figure 1. Tessellation by the $(2, 4, 8)$ group.

Now, take a sufficiently small number $\varepsilon > 0$. Let b' the point on the segment $[Ob]$ whose hyperbolic distance from b is ε. Similarly, we take a' and c' in P. See Figure 2.

We define a Riemann surface R_0 by removing the Γ-orbits of a', b', c' from the unit disk $\mathbf{D}$. See Figure 3.

It is clear that R_0 does not satisfy the assumption of Theorem 1.3. Indeed, for an arbitrary pants decomposition $R_0 = \bigcup_{k=1}^{\infty} P_k$, there is a sequence $\{\alpha_N\}$

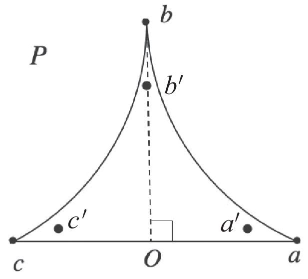

FIGURE 2. $b' \in [Ob]$. $\rho_{\mathbf{D}}(b, b') = \varepsilon$.

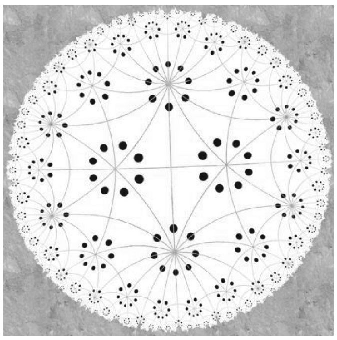

FIGURE 3. $R_0 := \mathbf{D} - \{\gamma(a'), \gamma(b'), \gamma(c') \mid \gamma \in \Gamma\}$.

of simple closed curves in $\{\partial P_k\}_{k=1}^{\infty}$ such that $\ell_{\mathbf{D}}(\alpha_N) \to \infty$ $(N \to \infty)$. Since $\ell_{\mathbf{D}}(\alpha_N) \leq \ell_{R_0}(\alpha_N)$ by Schwarz Lemma, we have $\ell_{R_0}(\alpha_N) \to \infty$ $(N \to \infty)$.

We show that d_L defines the same topology as that of d_T on $T(R_0)$. From Lemma 1.2, it suffices to show that for any sequence $\{p_n\}_{n=o}^{\infty} \subset T(R_0)$ with $d_L(p_n, p_0) \to 0$ $(n \to \infty)$, $d_T(p_n, p_0)$ converges to 0 as $n \to \infty$.

We assume that $p_0 = [R_0, id]$. Put $p_n = [R_n, f_n]$.

We divide R_0 into punctured disks and right hexagons. For $a \in P \subset \mathbf{D}$ (in Figure 1), take $\gamma_i \in \Gamma$ $(i = 1, \dots, 8)$ such that $\gamma_i(P) \cap \{a\} \neq \emptyset$. Let α_a be the shortest geodesic in Σ'_{R_0} that surrounds eight punctures $\gamma_i(a')$. Take α_b and α_c similarly. Next, let α_{ab} be the shortest geodesic in Σ'_{R_0} that surrounds α_a and α_b. Take α_{bc} and α_{ca} similarly.

Now we consider a pair of pants P_{ab} whose boundaries are α_a, α_b and α_{ab}. There are three lines which divide P_{ab} into two isometric right hexagons. Let β_{ab} be a line connecting α_a and α_b in those. (See Figure 4.) Note that the length of β_{ab} depends only on the lengths of α_a, α_b, α_{ab} from Lemma 2.4. Take β_{bc} and β_{ca} similarly.

Then we obtain a right hexagon bounded by β_{ab}, β_{bc}, β_{ca} and subarcs of α_a, α_b, α_c. Note that the lengths of subarcs of α_a, α_b, α_c depends only on the lengths

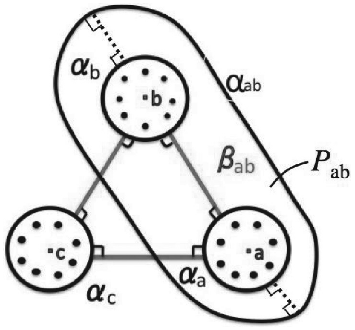

FIGURE 4. β_{ab} is one of lines dividing P_{ab} into two isometric right hexagons.

of β_{ab}, β_{bc}, β_{ca} from Lemma 2.4. Hence the right hexagon is determined by the lengths of α_a, α_b, α_c and α_{ab}, α_{bc}, α_{ca}.

Continue to take right hexagons as above, then R_0 is divided into eight-times punctured disks and right hexagons.

Next, we consider the division of R_n. For any $\alpha \in \Sigma'_{R_0}$, there is a simple closed geodesic in R_n homotopic to $f_n(\alpha)$. We denote it by $[f_n(\alpha)]$.

Take points $e_a \in \alpha_a \cap \beta_{ab}$ and $e_b \in \alpha_b \cap \beta_{ab}$. Let $f_n(e_a)'$ be a point on $[f_n(\alpha_a)]$ corresponding to $f_n(e_a) \in f_n(\alpha_a)$ about the continuous map Φ_a giving homotopy from $f_n(\alpha_a)$ to $[f_n(\alpha_a)]$, that is, for the homotopy map $\Phi_a : [0,1] \times [0,1] \to R_n$ with $\Phi_a(0, t_0) = f_n(e_a)$, we put $f_n(e_a)' := \Phi_a(1, t_0)$. Take $f_n(e_b)' \in [f_n(\alpha_b)]$ similarly. (See Figure 5.) Connect $f_n(e_a)$ and $f_n(e_a)'$ by a curve $\{\Phi_a(s, t_0) \mid 0 \leq s \leq 1\}$. Similarly connect $f_n(e_b)$ and $f_n(e_b)'$. Let $\widehat{f_n(\beta_{ab})}$ denote a curve from $f_n(e_a)'$ to $f_n(e_b)'$ given by connecting them. Take the shortest geodesic segment $[\widehat{f_n(\beta_{ab})}]$ homotopic to $\widehat{f_n(\beta_{ab})}$, where the homotopy map moves endpoints $f_n(e_a)'$ and $f_n(e_b)'$ on $[f_n(\alpha_a)]$ and $[f_n(\alpha_b)]$ respectively.

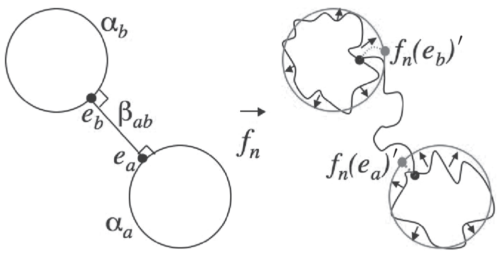

FIGURE 5. $f_n(e_a)' \in [f_n(\alpha_a)]$. $f_n(e_b)' \in [f_n(\alpha_b)]$.

On the other hand, we consider a pair of pants P^n_{ab} bounded by $[f_n(\alpha_a)]$, $[f_n(\alpha_b)]$ and $[f_n(\alpha_{ab})]$. There are three lines which divide P^n_{ab} into two isometric right hexagons. Let β^n_{ab} be a line connecting $[f_n(\alpha_a)]$ and $[f_n(\alpha_b)]$ in those. Then $[\widehat{f_n(\beta_{ab})}] = \beta^n_{ab}$.

Indeed, we consider a closed curve $C := [f_n(\alpha_a)] \cdot [\widehat{f_n(\beta_{ab})}] \cdot [f_n(\alpha_b)] \cdot [\widehat{f_n(\beta_{ab})}]^{-1}$. Since f_n is a homeomorphism, C is freely homotopic to $f_n(\alpha_{ab})$. On the other hand, if we consider another closed curve $C' := [f_n(\alpha_a)] \cdot \beta_{ab}^n \cdot [f_n(\alpha_b)] \cdot (\beta_{ab}^n)^{-1}$, then C', also, is freely homotopic to $f_n(\alpha_{ab})$. Hence C is freely homotopic to C'. Since both $[\widehat{f_n(\beta_{ab})}]$ and β_{ab}^n are the shortest of all the curves connecting $[f_n(\alpha_a)]$ and $[f_n(\alpha_b)]$, we have $[\widehat{f_n(\beta_{ab})}] = \beta_{ab}^n$.

By the definition of β_{ab}^n, the length of β_{ab}^n depends only on the lengths of $[f_n(\alpha_a)]$, $[f_n(\alpha_b)]$ and $[f_n(\alpha_{ab})]$. Since $d_L(p_n, p_0) < \varepsilon$ for sufficiently large number n, the lengths of β_{ab} and $\beta_{ab}^n = [\widehat{f_n(\beta_{ab}^n)}]$ are almost the same.

Let H_{abc} be a right hexagon in R_0 bounded by β_{ab}, β_{bc}, β_{ca} and subarcs of α_a, α_b, α_c (See Figure 4). On the other hand, let H_{abc}^n be a right hexagon in R_n bounded by $[\widehat{f_n(\beta_{ab})}]$, $[\widehat{f_n(\beta_{bc})}]$, $[\widehat{f_n(\beta_{ca})}]$ and subarcs of $[f_n(\alpha_a)]$, $[f_n(\alpha_b)]$, $[f_n(\alpha_c)]$. By Lemma 2.2, we obtain a quasiconformal map from H_{abc} to H_{abc}^n.

Let $\{H_i\}_{i=1}^{\infty} \subset R_0$ be the set of all the right hexagons. For each $H_i \subset R_0$, we take a right hexagon $H_i^n \subset R_n$ in the same way we took H_{abc}^n for H_{abc}. Put $R_0' := \bigcup_{i=1}^{\infty} H_i$ and $R_n' := \bigcup_{i=1}^{\infty} H_i^n$.

By Lemma 2.2, we obtain a quasiconformal map $g_n : R_0' \to R_n'$. We claim that f_n is homotopic to g_n on R_0', where the homotopy map does not necessarily keep points of ∂R_0 fixed.

For an arbitrary simple closed geodesic $\alpha \subset R_0'$, we consider $f_n(\alpha)$ and $g_n(\alpha)$. Let $\{H_{i(j)}\}_{j \in J} \subset R_0'$ be the set of all the right hexagons such that $H_{i(j)} \cap \alpha \neq \emptyset$. Then for each $j \in J$, a curve $g_n(\alpha \cap H_{i(j)})$ is homotopic to a curve $[f_n(\alpha)] \cap H_{i(j)}^n$ in $H_{i(j)}^n$, where the homotopy map does not necessarily fix endpoints. Hence $f_n(\alpha)$ is homotopic to $g_n(\alpha)$, so we verified the claim.

Next, we consider a quasiconformal map of the connected set $R_0^a \subset R_0 - R_0'$ bounded by α_a with eight punctures. We decompose R_0^a into seven pairs of pants as in Figure 6.

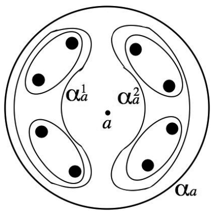

FIGURE 6. The pants decomposition of R_0^a.

Let α_a^1 and α_a^2 be simple closed geodesics in R_0^a surrounding four punctures (See Figure 6), and let $D_{\alpha_a^i}$ be an interior of α_a^i $(i = 1, 2)$. By the lemmas of Bishop, we obtain quasiconformal maps g_n on $D_{\alpha_a^i}$ $(i = 1, 2)$. However, g_n on R_0' is locally affine on α_a, so we will construct a quasiconformal map on a pair of pants P_{α_a} bounded by α_a^1, α_a^2 and α_a.

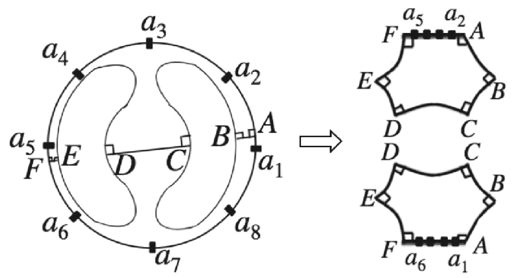

FIGURE 7. The division of P_{α_a}.

Let $a_1, \dots, a_8 \in \alpha_a$ be eight vertices of right-hexagons outside α_a, and let $A, \dots, F \in \partial P_{\alpha_a}$ be the vertices of two symmetric right-hexagons constructing P_{α_a} (See Figure 7). Suppose that A is on the segment $[a_1 a_2]$, and F is on the segment $[a_5 a_6]$. Let x_{12} be the length of $[a_1 a_2]$ and let x_{1A} be the length of the $[a_1 A]$. Then there is a number $t_0 \in [0, 1]$ such that $x_{1A} = t_0 x_{12}$. Similarly there is a number $s_0 \in [0, 1]$ such that $x_{5F} = s_0 x_{56}$.

On the other hand, let $a_1^n, \dots, a_8^n \in [f_n(\alpha_a)] \subset R_n$ be eight vertices of right-hexagons outside $[f_n(\alpha_a)]$. We take the points $g_n(A), \dots, g_n(F)$ on the geodesics $[f_n(\alpha_a)]$, $[f_n(\alpha_a^1)]$ and $[f_n(\alpha_a^2)]$.

We consider a hyperbolic hexagon with vertices $g_n(A), \dots, g_n(F)$. We claim that the angle formed by $[g_n(C)g_n(D)]$ and $[g_n(D)g_n(E)]$ is about $\pi/2$.

Indeed, let $R_0^{ab} \subset R_0$ be a connected set bounded by $\alpha_{ab} \subset \Sigma'_{R_0}$ and let $\widehat{R_0^{ab}}$ be the Nielsen extension of R_0^{ab}. We consider the Fenchel-Nielsen coordinates of the Teichmüller space $T(\widehat{R_0^{ab}})$. Then the twist parameter along $[f_n(\alpha_a^1)] = g_n(\alpha_a^1)$ is almost the same as that along α_a^1 (cf. Shiga [7], Lemma 4.1). Hence we verify the claim. The remaining angles are about $\pi/2$, similarly.

Let x_{ij}^n be the hyperbolic length of the segment $[a_i^n a_j^n]$ $(1 \le i, j \le 8)$, and let x_{i*}^n be the hyperbolic length of the segment $[a_i^n g_n(*)]$ $(* = A, F)$. Then, for $t_0 \in [0, 1]$ and $s_0 \in [0, 1]$ we took above, $x_{1g_n(A)}^n = t_0 x_{12}^n$ and $x_{5g_n(F)}^n = s_0 x_{56}^n$ hold, because g_n of R_0' is locally affine on α_a. Moreover, since for each $i = 1, 2$, g_n of $D_{\alpha_a^i}$ is affine on α_a^i, the lengths of six sides $[AB], \dots, [FA]$ and the lengths of six sides $[g_n(A)g_n(B)], \dots, [g_n(F)g_n(A)]$ are almost the same respectively. Hence the right-hexagon $A, \dots, F$ and the hexagon $g_n(A), \dots, g_n(F)$ are almost congruous.

We triangulate these hexagons as in Figure 8.

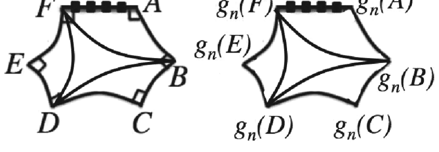

FIGURE 8. Triangulation.

From the First Cosine Rule (for hyperbolic geometry), the length of the new sides are determined by the sides and angles of the hexagon. Quasiconformal maps of the triangles BCD, DEF, BDF are obtained from Lemma 2.1.

We consider a quasiconformal map of the triangle ABF. In the triangle ABF, connect the points $a_2, \dots, a_5$ by geodesics segments to B. Similarly, in the triangle $g_n(A)g_n(B)g_n(F)$, connect the points $a_2^n, \dots, a_5^n$ by geodesics segments to $g_n(B)$. Then we obtain a quasiconformal map of the triangle ABF from Lemma 2.1.

Hence we obtain a quasiconformal map g_n of the whole of R_0 such that g_n is homotopic to f_n and $K(g_n) \to 1$ $(n \to \infty)$. Thus $d_T(p_n, p_0) \to 0$ $(n \to \infty)$.

In the case where $p_0 \neq [R_0, id]$, we can show that $d_T(p_n, p_0) \to 0$ $(n \to \infty)$ similarly. Indeed, K-quasiconformal map f $(1 \leq K < \infty)$ does not crush any triangle with the sides of bounded lengths. Since any right hexagon in R_0 can be divided into such triangles, any Riemann surface $f(R_0)$ in $T(R_0)$ can be divided into punctured regions and hexagons whose the lengths of the sides are uniformly bounded by a constant depending on K. □

3. Examples and proof of Theorem 1.5

First, we give examples of Riemann surfaces satisfying the assumption of Theorem 1.5.

Examples. (a) Any Riemann surface R_0 satisfying the assumption of Theorem 1.4 satisfies the assumption of Theorem 1.5. Indeed, let $\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma'_{R_0}$ be a sequence such that $\ell_{R_0}(\alpha_n) \to 0$ $(n \to \infty)$. Then for any sequence $\{\beta_n\}_{n=1}^{\infty} \subset \Sigma'_{R_0}$ with $\alpha_n \cap \beta_n \neq \emptyset$ $(n = 1, 2, \dots)$, we have $\ell_{R_0}(\beta_n) \to \infty$ $(n \to \infty)$ by the collar lemma. Hence Theorem 1.5 extends Theorem 1.4.

(b) The Riemann surface R_0 constructed by Shiga ([7], pp. 317–319) satisfies the assumption of Theorem 1.5. (The Riemann surface does not satisfy the assumption of Theorem 1.4.) In this case, some sequence $\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma'_{R_0}$ with $\ell_{R_0}(\alpha_n) \to \infty$ $(n \to \infty)$ satisfies the condition.

(c) We construct a Riemann surface R_0 such that R_0 satisfies the assumption of Theorem 1.5 and the lengths of $\{\alpha_n\}_{n=1}^{\infty} \subset \Sigma'_{R_0}$ are uniformly bounded from above and below.

First, let P_0 be a pair of pants with boundary lengths (a_0, b_0, b_0). Make countable copies of P_0 and glue them along the boundaries of length b_0.

Next, we take a monotone divergent sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers. Let P_n be a pair of pants with boundary lengths (a_0, a_n, a_n). Make two copies of P_n and glue each copy with the union of the copies of P_0 along the boundaries of length a_0 as in Figure 9. Let R'_0 denote a Riemann surface with boundary we have obtained.

Let R_0 be the Nielsen extension of R'_0 and put $\alpha_n := P_0 \cap P_n$. Then R_0 satisfies the assumption of Theorem 1.5. Also, if we take some pants $\{Q_m\}_{m=1}^{\infty}$ and define $R_0 := R'_0 \cup \bigcup_{m=1}^{\infty} Q_m$, then R_0 satisfies the assumption of Theorem 1.5. □

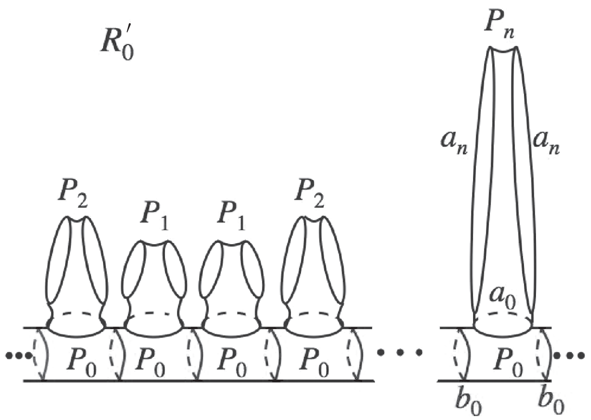

FIGURE 9. $R_0' = P_0 \cup P_0 \cup \cdots \cup P_1 \cup P_1 \cup \cdots \cup P_n \cup P_n \cup \cdots$.

To prove Theorem 1.5, we use the following theorem, which is a partial result of Theorem 1 in Matsuzaki [6].

THEOREM 3.1 ([6]). *Let α be a simple closed geodesic on a Riemann surface R_0 and let $f : R_0 \to R_0$ be the n-times Dehn twist along α. Then the maximal dilatation $K(f)$ of an extremal quasiconformal automorphism of f satisfies*

$$K(f) \geq \left\{ \left(\frac{(2|n| - 1)\ell_{R_0}(\alpha)}{\pi} \right)^2 + 1 \right\}^{1/2}.$$

Proof of Theorem 1.5. First, suppose that there exists a constant $c > 0$ such that $\ell_{R_0}(\alpha_n) > c$ for all $n \in \mathbf{N}$. Let f_n be a Dehn twist along α_n. Then we have

$$\ell_{R_0}(f_n(\beta_n)) \leq \ell_{R_0}(\beta_n) + \#(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n).$$

Thus,

$$\frac{\ell_{R_0}(f_n(\beta_n))}{\ell_{R_0}(\beta_n)} \leq 1 + \frac{\#(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \to 1 \quad (n \to \infty).$$

Since f_n^{-1} is also a Dehn twist, from the same argument as above, we have

$$\frac{\ell_{R_0}(\beta_n)}{\ell_{R_0}(f_n(\beta_n))} \leq \frac{\ell_{R_0}(\beta_n)}{\ell_{R_0}(\beta_n) - \#(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)} \to 1 \quad (n \to \infty).$$

Hence

$$\lim_{n \to \infty} d_L([R_0, f_n], [R_0, id]) = 0.$$

On the other hand, from Theorem 3.1 the maximal dilatation $K(f_n)$ of an extremal quasiconformal map of f_n satisfies

$$K(f_n) \geq \left\{ \left(\frac{\ell_{R_0}(\alpha_n)}{\pi} \right)^2 + 1 \right\}^{1/2} > \left\{ \left(\frac{c}{\pi} \right)^2 + 1 \right\}^{1/2}.$$

Hence

$$\varliminf_{n \to \infty} d_T([R_0, f_n], [R_0, id]) > 0.$$

Next, suppose that $\ell_{R_0}(\alpha_n) \to 0$ $(n \to \infty)$. We note that $\ell_{R_0}(\beta_n) \to \infty$ $(n \to \infty)$ by the collar lemma. Let f_n be the $[1/\ell_{R_0}(\alpha_n) + 1]$-times Dehn twist along α_n. Then, we have

$$\begin{aligned}
\frac{\ell_{R_0}(f_n(\beta_n))}{\ell_{R_0}(\beta_n)} &\leq \frac{\ell_{R_0}(\beta_n) + \#(\alpha_n \cap \beta_n)\left[\dfrac{1}{\ell_{R_0}(\alpha_n)} + 1\right]\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \\
&\leq 1 + \frac{\#(\alpha_n \cap \beta_n)\left(\dfrac{1}{\ell_{R_0}(\alpha_n)} + 1\right)\ell_{R_0}(\alpha_n)}{\ell_{R_0}(\beta_n)} \\
&\leq 1 + \frac{\#(\alpha_n \cap \beta_n)(1 + \ell_{R_0}(\alpha_n))}{\ell_{R_0}(\beta_n)} \quad \to 1 \quad (n \to \infty),
\end{aligned}$$

and

$$\frac{\ell_{R_0}(\beta_n)}{\ell_{R_0}(f_n(\beta_n))} \leq \frac{\ell_{R_0}(\beta_n)}{\ell_{R_0}(\beta_n) - \#(\alpha_n \cap \beta_n)\left[\dfrac{1}{\ell_{R_0}(\alpha_n)} + 1\right]\ell_{R_0}(\alpha_n)} \to 1 \quad (n \to \infty).$$

Hence

$$\lim_{n \to \infty} d_L([R_0, f_n], [R_0, id]) = 0.$$

On the other hand, from Theorem 3.1 the maximal dilatation $K(f_n)$ of an extremal quasiconformal map of f_n satisfies

$$K(f_n) \geq \left\{ \left(\frac{\left(2 \cdot \dfrac{1}{\ell_{R_0}(\alpha_n)} - 1\right)\ell_{R_0}(\alpha_n)}{\pi} \right)^2 + 1 \right\}^{1/2} \to \left\{ \left(\frac{2}{\pi} \right)^2 + 1 \right\}^{1/2}$$

as $n \to \infty$. Hence

$$\varliminf_{n \to \infty} d_T([R_0, f_n], [R_0, id]) > 0. \qquad \square$$

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