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STRONG 2-CALIBRATIONS ON $\mathbf{R}^{2n}$

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Abstract

Most of the classical calibrations possess a property which does not seem to be recognized. Making this property explicit, we define what we call the strong calibrations and prove that a strong 2-calibration field on $\mathbf{R}^{2n}$ is a constant field.

1. Introduction

The notion of calibration was defined by Harvey and Lawson in their 1982 paper "Calibrated Geometries" [2]. This notion has been attracting continued interest because it provides a powerful way of identifying area minimizing surfaces.

Let's first define the notion of (point-wise) calibration on $\mathbf{R}^m$ equipped with the standard inner product (in the following, $\{e_1, e_2, \dots, e_m\}$ will denote the standard basis and $\{dx_1, dx_2, \dots, dx_m\}$ will denote the corresponding dual basis of $\mathbf{R}^m$). Let $\Lambda^p(\mathbf{R}^m)^*$ denote the space of all p-forms on the vector space $\mathbf{R}^m$.

Definition 1.1. A p-form $\varphi \in \Lambda^p(\mathbf{R}^m)^*$ is called a (point-wise) p-calibration on $\mathbf{R}^m$ if it satisfies the following conditions:

1. $|\varphi(u_1, u_2, \dots, u_p)| \le 1$ for any orthonormal set $\{u_1, u_2, \dots, u_p\}$ of vectors $u_k \in \mathbf{R}^m$, $k = 1, 2, \dots, p$.
2. There is at least one orthonormal set $\{u_1, u_2, \dots, u_p\}$ such that $|\varphi(u_1, u_2, \dots, u_p)| = 1$.

Example 1.2. The 2-form $\varphi = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ is a 2-calibration on $\mathbf{R}^4$.

Example 1.3. The 3-form

$$\varphi = dx_{123} - dx_{167} + dx_{257} - dx_{356} + dx_{145} + dx_{246} + dx_{347}$$

where dx_{ijk} is a shorthand for $dx_i \wedge dx_j \wedge dx_k$, is a 3-calibration on $\mathbf{R}^7$. (It is known as the fundamental 3-form or the associative calibration on $\mathbf{R}^7$, s. [2].)

In this article, we will focus on the case $p = 2$ in even dimensions. A 2-form φ on $\mathbf{R}^{2n}$ can be represented by a $2n \times 2n$ dimensional skew-symmetric

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real matrix $A = (a^{ij})$ with $a^{ij} = \varphi(e_i, e_j)$, which has pure imaginary eigenvalues that are of the form $\pm i\lambda_1, \pm i\lambda_2, \dots, \pm i\lambda_n$. There is an orthonormal basis $\{f_1, f_2, \dots, f_{2n}\}$ of $\mathbf{R}^{2n}$ such that the corresponding skew-symmetric matrix $B = (\varphi(f_i, f_j)) = P^t A P = P^{-1} A P$ (P being the base-change matrix) is of the form

$$B = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_n \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_n & 0 \end{pmatrix}_{2n \times 2n}.$$

The 2-form φ can then be written as

$$\varphi = \lambda_1 \, dy_1 \wedge dy_2 + \lambda_2 \, dy_3 \wedge dy_4 + \cdots + \lambda_n \, dy_{2n-1} \wedge dy_{2n},$$

where $\{dy_1, dy_2, \dots, dy_{2n}\}$ is the corresponding dual basis of $\mathbf{R}^{2n}$. It can easily be seen that $\varphi \in \Lambda^2(\mathbf{R}^{2n})^*$ is a calibration if and only if

$$max\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\} = 1.$$

Example 1.4. *The forms given by*

$$\varphi_1 = dx_1 \wedge dx_2 + \frac{1}{2} \, dx_3 \wedge dx_4$$

$$\varphi_2 = dx_1 \wedge dx_2 + \frac{1}{5} \, dx_3 \wedge dx_4 + dx_5 \wedge dx_6$$

are 2*-calibrations on* $\mathbf{R}^4$ *and* $\mathbf{R}^6$ *respectively.*

Now we define the notion of a calibration field on $\mathbf{R}^m$, where $\mathbf{R}^m$ is understood as a Riemannian manifold with the usual metric. (In the literature these calibration fields on Riemannian manifolds are called "calibrations". We distinguish between point-wise calibrations on vector spaces and calibration fields on Riemannian manifolds.)

Let $\Omega^p(\mathbf{R}^m)$ denote the space of all smooth p-form fields on $\mathbf{R}^m$. Given $\Phi \in \Omega^p(\mathbf{R}^m)$, we use the notation Φ_x for the p-form at x determined by Φ. We can give the definition of the notion of a calibration field on $\mathbf{R}^m$ as follows.

Definition 1.5. A closed p-form field $\Phi \in \Omega^p(\mathbf{R}^m)$ is called a p-calibration field on $\mathbf{R}^m$ if it satisfies the following conditions:

1. At each point $x \in \mathbf{R}^m$, $|\Phi_x(u_1, u_2, \dots, u_p)| \le 1$ for any orthonormal set of vectors $\{u_1, u_2, \dots, u_p\}$ of $T_x\mathbf{R}^m$.
2. For some x_0, there exists an orthonormal set $\{u_1, u_2, \dots, u_p\}$ of $T_{x_0}\mathbf{R}^m$ such that $|\Phi_{x_0}(u_1, u_2, \dots, u_p)| = 1$.

For the well-known examples of calibration fields we refer the reader to [2].

Most of the important calibrations used in the literature possess an important property, which does not seem to attract any attention. The second condition of Definition 1.1 asks for a single orthonormal set $\{u_1, u_2, \dots, u_p\}$ satisfying the property $|\varphi(u_1, u_2, \dots, u_p)| = 1$, but in all classical examples more is satisfied: For any orthonormal set of vectors $\{u_1, u_2, \dots, u_{p-1}\}$ of $\mathbf{R}^m$ there is at least one vector $u_p \in \mathbf{R}^m$, such that $\{u_1, u_2, \dots, u_{p-1}, u_p\}$ is orthonormal and $|\varphi(u_1, u_2, \dots, u_p)| = 1$. Demanding this as a condition, we define what we call a strong (point-wise) calibration:

Definition 1.6. A p-form $\varphi \in \Lambda^p(\mathbf{R}^m)^*$ (with $p > 1$) is called a strong (point-wise) p-calibration if it satisfies the following conditions:

1. $|\varphi(u_1, u_2, \dots, u_p)| \leq 1$ for any orthonormal set of vectors $\{u_1, u_2, \dots, u_p\}$ of $\mathbf{R}^m$.
2. For any orthonormal set of vectors $\{u_1, u_2, \dots, u_{p-1}\}$ of $\mathbf{R}^m$ there is at least one vector $u_p \in \mathbf{R}^m$, such that $\{u_1, u_2, \dots, u_{p-1}, u_p\}$ is orthonormal and $|\varphi(u_1, u_2, \dots, u_p)| = 1$.

We furthermore want to remark that in the usual definition 1.5 of calibration fields, it is only required that Φ_x is a calibration (in the sense of Definition 1.1) for some x_0. Again, in important classical examples all Φ_x are calibrations. Taking this into account and the above definition of strong calibrations, we give the following definition of strong calibration fields:

Definition 1.7. A closed p-form field $\Phi \in \Omega^p(\mathbf{R}^m)$ (with $p > 1$) is called a strong p-calibration field on (the Riemannian manifold) $\mathbf{R}^m$ if the p-form $\Phi_x \in \Lambda^p(T_x\mathbf{R}^m)^*$ is a strong p-calibration for all $x \in \mathbf{R}^m$.

With these modified definitions, the following property holds, which is the main result of our article:

Theorem 1.8. *Let $\Phi \in \Omega^2(\mathbf{R}^{2n})$ be a strong 2-calibration field on $\mathbf{R}^{2n}$. Then Φ is a constant field.*

A corresponding property holds for 1-calibration fields on $\mathbf{R}^m$. For $p = 1$, the second condition of Def.1.6 does not impose anything more on a 1-calibration, so we excluded this case from the definition of strong calibrations. A (point-wise) 1-calibration on $\mathbf{R}^m$ (according to Def.1.1) can be seen to be a 1-form $\varphi = a^1\, dx_1 + a^2\, dx_2 + \cdots + a^m\, dx_m \in \Lambda^1(\mathbf{R}^m)^* = (\mathbf{R}^m)^*$ with $(a^1)^2 + (a^2)^2 + \cdots + (a^m)^2 = 1$. A 1-calibration field on $\mathbf{R}^m$ is a closed field $\Phi = a^1\, dx_1 + a^2\, dx_2 + \cdots + a^m\, dx_m$ with a^i smooth functions on $\mathbf{R}^m$ satisfying $(a^1)^2 + (a^2)^2 + \cdots + (a^m)^2 = 1$. The closed 1-form field Φ on $\mathbf{R}^m$ is exact, $\Phi = df$. The function f has constant gradient norm, hence it is an affine function [5]. Consequently, Φ is a constant field:

Remark 1.9. A 1-calibration field $\Phi \in \Omega^1(\mathbf{R}^m)$ is constant.

We were unable to prove this property for higher strong p-calibration fields $(p \geq 3)$. The geometrically very reach associative 3-calibration field on $\mathbf{R}^7$ and the Cayley 4-calibration field on $\mathbf{R}^8$ (s. [2]) are defined as constant calibration fields and they are strong calibration fields in our sense. So, a generalization of this property to higher strong calibration fields seems probable and of interest.

In the next section we will discuss some technicalities of strong calibrations and in the third section we will give a proof of Theorem 1.8.

2. Some properties of strong calibrations

A strong 2-calibration on $\mathbf{R}^4$ is almost the same object as a self/antiself-dual 2-form (in the Hodge sense) on $\mathbf{R}^4$ as the following Lemma shows. This gives another support for the meaningfulness of the notion of strong calibrations.

Lemma 2.1. *Let $\varphi \in \Lambda^2(\mathbf{R}^4)^*$ be a 2-form on $\mathbf{R}^4$ given by*

$$\varphi = a^{12}\, dx_1 \wedge dx_2 + a^{13}\, dx_1 \wedge dx_3 + a^{14}\, dx_1 \wedge dx_4 + a^{23}\, dx_2 \wedge dx_3 + a^{24}\, dx_2 \wedge dx_4 + a^{34}\, dx_3 \wedge dx_4$$

where $a^{12}, a^{13}, a^{14}, a^{23}, a^{24}, a^{34} \in \mathbf{R}$. Then, φ is a strong 2-calibration iff

$$(a^{12})^2 + (a^{13})^2 + (a^{14})^2 = 1 \quad \text{and} \quad a^{12} = a^{34},\ a^{14} = a^{23},\ a^{13} = -a^{24}$$

or

$$(a^{12})^2 + (a^{13})^2 + (a^{14})^2 = 1 \quad \text{and} \quad a^{12} = -a^{34},\ a^{14} = -a^{23},\ a^{13} = a^{24}.$$

We omit the easy proof.

We know that if the matrix of a 2-form φ on $\mathbf{R}^{2n}$ with respect to an orthonormal basis has the eigenvalues $\pm i\lambda_1, \pm i\lambda_2, \ldots, \pm i\lambda_n$, then φ is a calibration if and only if

$$max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\} = 1.$$

Using (1.0.1) and the stronger condition imposed upon a calibration by Definition 1.6, we can obtain the following characterizations of strong 2-calibrations:

Lemma 2.2. *Let $\varphi \in \Lambda^2(\mathbf{R}^{2n})^*$ and $\pm i\lambda_k$, $k = 1, 2, \ldots, n$ be the eigenvalues of the corresponding skew-symmetric matrix with respect to some orthonormal basis. Then, φ is a strong 2-calibration if and only if*

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n| = 1.$$

Lemma 2.3. *Let $\varphi \in \Lambda^2(\mathbf{R}^{2n})^*$ and A be the corresponding skew-symmetric matrix with respect to some orthonormal basis. Then, φ is a strong 2-calibration if and only if A is an orthogonal matrix.*

We want to remark that, in contrast to calibrations, there may not be a strong p-calibration in every dimension $m \geq p$. For example, there is no strong 2-calibration on an odd-dimensional $\mathbf{R}^m$ because of the degeneracy of a 2-form in odd dimensions. In this regard, the following property holds:

PROPOSITION 2.4. *Let $\varphi \in \Lambda^p(\mathbf{R}^m)^*$ $(p > 1)$ be a strong p-calibration. Then $m - p$ is an even integer.*

Proof. The form ψ defined as

$$\psi(x, y) := \varphi(e_1, e_2, \ldots, e_{p-2}, \bar{x}, \bar{y})$$

is a strong 2-calibration on $\mathbf{R}^{m+2-p}$ where $x = (x_1, \ldots, x_{m+2-p})$, $y = (y_1, \ldots, y_{m+2-p}) \in \mathbf{R}^{m+2-p}$ and $\bar{x} = (0, \ldots, 0, x_1, \ldots, x_{m+2-p})$, $\bar{y} = (0, \ldots, 0, y_1, \ldots, y_{m+2-p}) \in \mathbf{R}^m$. By the above remark, $m - p$ must be even. ■

3. Strong 2-calibration fields

In this section we will give a proof of Theorem 1.8. First we want to recall the notion of the Pfaffian of a skew-symmetric matrix and discuss some properties of it. Let $A = (a^{ij})$ be a $2n \times 2n$ skew-symmetric real matrix. The determinant of A can be written as the square of a polynomial in the entries of A [4]: $\det(A) = Pf(A)^2$. This polynomial (denoted by $Pf(A)$) is called the Pfaffian of A. Let $K = \{1, 2, 3, \ldots, 2n\}$, S_K the set of permutations of K and let

$$\Pi := \{(i_1 \ j_1 \ i_2 \ j_2 \ \cdots \ i_n \ j_n) \in S_K \mid i_1 < \cdots < i_n \text{ and } i_p < j_p \text{ for all } p\}.$$

Then the Pfaffian polynomial of A can be written as

$$Pf(A) = \sum_{\alpha \in \Pi} A_\alpha$$

where $A_\alpha = sgn(\alpha) a^{i_1 j_1} a^{i_2 j_2} \cdots a^{i_n j_n}$.

LEMMA 3.1. *Let $A = (a^{ij})$ be a $2n \times 2n$ skew-symmetric and orthogonal real matrix. Then*

$$a^{ij} = (-1)^{i+j-1} Pf(M_{ij,ij}) Pf(A) \tag{3.0.1}$$

for all $1 \leq i < j \leq 2n$ where $M_{ij,ij}$ is the matrix formed by removing from A the rows i and j, as well as the columns i and j.

Proof. If $A = (a^{ij})$ is skew-symmetric and orthogonal then $-A = A^{-1}$. Thus we have

$$-a^{ij} = (-1)^{i+j} \det(M_{ji}) \tag{3.0.2}$$

where M_{ji} is the matrix formed by removing from A its jth row and ith column. Moreover, it is proven in [3] that

$$(-1)^{i+j}\det(M_{ji}) = (-1)^{i+j}Pf(M_{ij,ij})Pf(A) \tag{3.0.3}$$

for $i < j$. Combining (3.0.2) and (3.0.3) verifies the assertion of the lemma. ■

Let $\Phi \in \Omega^2(\mathbf{R}^{2n})$ be a strong 2-calibration field on $\mathbf{R}^{2n}$. At each point $x = (x_1, x_2, \ldots, x_{2n}) \in \mathbf{R}^{2n}$, Φ_x can be written as

$$\Phi_x = \sum_{1 \le i < j \le 2n} a^{ij}(x_1, x_2, \ldots, x_{2n})\, dx_i \wedge dx_j,$$

where $a^{ij} \in C^\infty(\mathbf{R}^{2n})$ are the smooth coefficient functions of Φ. For notational simplicity, we denote by a^{ij} also its value $a^{ij}(x_1, x_2, \ldots, x_{2n})$ at a point if it is understood from the context. For a given x, the corresponding skew-symmetric matrix of Φ_x is of the form

$$A = \begin{pmatrix} 0 & a^{12} & a^{13} & \cdots & a^{1\ 2n} \\ -a^{12} & 0 & a^{23} & \cdots & a^{2\ 2n} \\ -a^{13} & -a^{23} & 0 & & \vdots \\ \vdots & \vdots & & \ddots & a^{2n-1\ 2n} \\ -a^{1\ 2n} & -a^{2\ 2n} & \cdots & -a^{2n-1\ 2n} & 0 \end{pmatrix}.$$

For each $x \in \mathbf{R}^{2n}$, the skew-symmetric matrix A is orthogonal by Lemma 2.3, hence $Pf(A) = \pm 1$. By connectivity of $\mathbf{R}^{2n}$, either $Pf(A) \equiv 1$ or $Pf(A) \equiv -1$. There is no loss of generality in assuming that $Pf(A) = 1$. Combining this with (3.0.1) yields

$$a^{ij} = (-1)^{i+j-1}Pf(M_{ij,ij}) \tag{3.0.4}$$

for $1 \le i < j \le 2n$. From now on, we focus on the coefficient a^{12}. Is the Laplacian of the function a^{12} zero?

For simplicity of notation, we write a_k^{ij} for the kth partial derivative $\dfrac{\partial}{\partial_k} a^{ij}$.

Let $K^{1k} = \{1, 2, \ldots, 2n-1, 2n\} \setminus \{1, k\}$ and

$$\Pi^{1k} = \{(i_1\ \ j_1\ \ \cdots\ \ i_{n-1}\ \ j_{n-1}) \in S_{K^{1k}} \mid i_1 < \cdots < i_{n-1} \text{ and } i_p < j_p \text{ for } p = 1, \ldots, n-1\}$$

where k is a fixed integer with $2 \le k \le 2n$. Then the Pfaffian polynomial of $M_{1k,1k}$ can be written as

$$Pf(M_{1k,1k}) = \sum_{\alpha^{1k} \in \Pi^{1k}} A_{\alpha^{1k}} \tag{3.0.5}$$

where $A_{\alpha^{1k}} = sgn(\alpha^{1k}) a^{i_1 j_1} a^{i_2 j_2} \cdots a^{i_{n-1} j_{n-1}}$.

Similarly, Let $K^{1kij} = \{1, 2, \ldots, 2n-1, 2n\} \backslash \{1, k, i, j\}$ and

$$\Pi^{1kij} = \{(i_1 \ \ j_1 \ \cdots \ i_{n-2} \ \ j_{n-2}) \in S_{K^{1kij}} \mid i_1 < \cdots < i_{n-2} \text{ and } i_p < j_p \text{ for } p = 1, \ldots, n-2\}$$

where the fixed integers k, i, j with $2 \leq k, i, j \leq 2n$ differ from each other. Then the Pfaffian polynomial of $M_{1kij, 1kij}$ can be written as

$$Pf(M_{1kij, 1kij}) = \sum_{\alpha^{1kij} \in \Pi^{1kij}} A_{\alpha^{1kij}} \tag{3.0.6}$$

where $A_{\alpha^{1kij}} = sgn(\alpha^{1kij}) a^{i_1 j_1} a^{i_2 j_2} \cdots a^{i_{n-2} j_{n-2}}$.

The following property can be easily checked:

LEMMA 3.2. *Let $\alpha^{1k} \in \Pi^{1k}$ be the element given by*

$$\alpha^{1k} = (i_1 \ \ j_1 \ \cdots \ i_{p-1} \ \ j_{p-1} \ \ i_p \ \ j_p \ \ i_{p+1} \ \ j_{p+1} \ \cdots \ i_{n-1} \ \ j_{n-1}).$$

Then

$$sgn(\alpha^{1k}) = (-1)^{i_p + j_p - 1} \, sgn[(i_p - k)(j_p - k)] \, sgn(\alpha^{1k i_p j_p})$$

where

$$\alpha^{1k i_p j_p} = (i_1 \ \ j_1 \ \cdots \ i_{p-1} \ \ j_{p-1} \ \ i_{p+1} \ \ j_{p+1} \ \cdots \ i_{n-1} \ \ j_{n-1}) \in \Pi^{1k i_p j_p}.$$

We need also the following technical lemma:

LEMMA 3.3. *Let k be an integer such that $2 \leq k \leq 2n$. Then*

$$\frac{\partial}{\partial_k} Pf(M_{1k, 1k}) = \sum_{2 \leq i < j \leq 2n} (-1)^{i+j-1} \, sgn[(i-k)(j-k)] a_k^{ij} Pf(M_{1kij, 1kij}). \tag{3.0.7}$$

Proof. Using the equality (3.0.5) we can write

$$\frac{\partial}{\partial k} Pf(M_{1k, 1k}) = \frac{\partial}{\partial k} \sum_{\alpha^{1k} \in \Pi^{1k}} A_{\alpha^{1k}} = \sum_{\alpha^{1k} \in \Pi^{1k}} \frac{\partial}{\partial_k} A_{\alpha^{1k}}.$$

Let r and s be integers with $r < s$, which are different from 1 and k. Let T^{rs} denote the sum of the terms in which $\frac{\partial}{\partial_k} a^{rs}$ appears as a factor. For a given

$$\alpha^{1k} = (i_1 \ \ j_1 \ \cdots \ i_p = r \ \ j_p = s \ \cdots \ i_{n-1} \ \ j_{n-1}) \in \Pi^{1k}$$

consider the term $A_{\alpha^{1k}} = sgn(\alpha^{1k}) a^{i_1 j_1} \cdots a^{rs} \cdots a^{i_{n-1} j_{n-1}}$. Using the Lemma 3.2, the contribution of the kth partial derivative of $A_{\alpha^{1k}}$ to T^{rs} is

$$(3.0.8) \quad (-1)^{r+s-1}\, sgn[(r-k)(s-k)]\, sgn(\alpha^{1krs}) \frac{\partial}{\partial_k} a^{rs} \prod_{q \in \{1,2,3,\ldots,n-1\} - \{p\}} a^{i_q j_q}.$$

Let Π_{rs}^{1k} denote the set of all permutations $\alpha^{1k} = (i_1 \; j_1 \; \cdots \; i_p = r \; j_p = s \; \cdots \; i_{n-1} \; j_{n-1}) \in \Pi^{1k}$ in which $i_p = r$ and $j_p = s$ for some $p \in \{1, 2, 3, \ldots, n-1\}$, write p_α. Taking the sum of the terms (3.0.8) over Π_{rs}^{1k}, we thus get

$$\begin{aligned} T^{rs} &= \sum_{\alpha_{rs}^{1k} \in \Pi_{rs}^{1k}} \left((-1)^{r+s-1}\, sgn[(r-k)(s-k)]\, sgn(\alpha^{1krs}) \frac{\partial}{\partial_k} a^{rs} \prod_{q \in \{1,2,3,\ldots,n-1\} - \{p_\alpha\}} a^{i_q j_q} \right) \\ &= (-1)^{r+s-1}\, sgn[(r-k)(s-k)] \frac{\partial}{\partial_k} a^{rs} \left(\sum_{\alpha_{rs}^{1k} \in \Pi_{rs}^{1k}} sgn(\alpha^{1krs}) \prod_{q \in \{1,2,3,\ldots,n-1\} - \{p_\alpha\}} a^{i_q j_q} \right). \end{aligned}$$

There is a one to one relation between the terms of Π_{rs}^{1k} and the terms of Π^{1krs}. Moreover, for a given $\alpha_{rs}^{1k} \in \Pi_{rs}^{1k}$

$$\prod_{q \in \{1,2,3,\ldots,n-1\} - \{p_\alpha\}} a^{i_q j_q}$$

does not contain the term a^{rs} as a factor. So we have

$$\sum_{\alpha_{rs}^{1k} \in \Pi_{rs}^{1k}} \left(sgn(\alpha^{1krs}) \prod_{q \in \{1,2,3,\ldots,n-1\} - \{p_\alpha\}} a^{i_q j_q} \right) = \sum_{\alpha^{1krs} \in \Pi^{1krs}} \left(sgn(\alpha^{1krs}) \prod_{q \in \{1,2,3,\ldots,n-2\}} a^{i_q j_q} \right)$$

which yields

$$\begin{aligned} T^{rs} &= (-1)^{r+s-1}\, sgn[(r-k)(s-k)] \frac{\partial}{\partial_k} a^{rs} \left(\sum_{\alpha_{rs}^{1k} \in \Pi_{rs}^{1k}} sgn(\alpha^{1krs}) \prod_{q \in \{1,2,3,\ldots,n-1\} - \{p_\alpha\}} a^{i_q j_q} \right) \\ &= (-1)^{r+s-1}\, sgn[(r-k)(s-k)] \frac{\partial}{\partial_k} a^{rs} \left(\sum_{\alpha^{1krs} \in \Pi^{1krs}} sgn(\alpha^{1krs}) \prod_{q \in \{1,2,3,\ldots,n-2\}} a^{i_q j_q} \right) \\ &= (-1)^{r+s-1}\, sgn[(r-k)(s-k)] \frac{\partial}{\partial_k} a^{rs} \left(\sum_{\alpha^{1krs} \in \Pi^{1krs}} A_{\alpha^{1krs}} \right) \\ &= (-1)^{r+s-1}\, sgn[(r-k)(s-k)] \frac{\partial}{\partial_k} a^{rs} Pf(M_{1krs,1krs}). \end{aligned}$$

We can get $\frac{\partial}{\partial k} Pf(M_{1k,1k})$ by taking the sum of T^{rs} over all possible pair (r,s) where $2 \le r < s \le 2n$. Then we get

$$\begin{aligned}\frac{\partial}{\partial k} Pf(M_{1k,1k}) &= \sum_{\alpha^{1k} \in \Pi^{1k}} sgn(\alpha^{1k}) \frac{\partial}{\partial_k} (a^{i_1 j_1} a^{i_2 j_2} \cdots a^{i_{n-1} j_{n-1}}) \\ &= \sum_{2 \le r < s \le 2n} T^{rs} \\ &= \sum_{2 \le r < s \le 2n} (-1)^{r+s-1} \, sgn[(r-k)(s-k)] a_k^{rs} Pf(M_{1krs,1krs})\end{aligned}$$

which is the desired conclusion. ■

Now we will prove Theorem 1.8 with the help of the above Lemma.

Proof. The proof will be based on the fact that a bounded harmonic function on $\mathbf{R}^m$ is constant [1]. Since the coefficient functions are bounded (by calibration condition), it will be enough to see that they are harmonic on $\mathbf{R}^{2n}$.

Since the 2-form field Φ is closed, we get

$$a_k^{ij} - a_j^{ik} + a_i^{jk} = 0 \tag{3.0.9}$$

for all $1 \le i < j < k \le 2n$. Substituting $i = 1$ and $j = 2$ into (3.0.9) we obtain

$$a_k^{12} - a_2^{1k} + a_1^{2k} = 0 \Rightarrow a_k^{12} = a_2^{1k} - a_1^{2k}$$

for all $3 \le k \le 2n$. Applying $\frac{\partial}{\partial_k}$ to a_k^{12} yields

$$a_{kk}^{12} = a_{k2}^{1k} - a_{k1}^{2k}$$

for all $3 \le k \le 2n$. Thus we can write the Laplacian of a^{12} as

$$\begin{aligned}\Delta a^{12} &= a_{11}^{12} + a_{22}^{12} + a_{33}^{12} + a_{44}^{12} + \cdots + a_{2n\,2n}^{12} \\ &= a_{11}^{12} + a_{22}^{12} + (a_{32}^{13} - a_{31}^{23}) + (a_{42}^{14} - a_{41}^{24}) + \cdots + (a_{2n\,2}^{1\,2n} - a_{2n\,1}^{2\,2n}) \\ &= (a_{11}^{12} - a_{31}^{23} - a_{41}^{24} - \cdots - a_{2n\,1}^{2\,2n}) + (a_{22}^{12} + a_{32}^{13} + a_{42}^{14} + \cdots + a_{2n\,2}^{1\,2n}) \\ &= \frac{\partial}{\partial_2}(a_2^{12} + a_3^{13} + a_4^{14} + \cdots + a_{2n}^{1\,2n}) - \frac{\partial}{\partial_1}(-a_1^{12} + a_3^{23} + a_4^{24} + \cdots + a_{2n}^{2\,2n}).\end{aligned}$$

Let $T = (a_2^{12} + a_3^{13} + a_4^{14} + \cdots + a_{2n}^{1\,2n})$. Combining (3.0.4) and Lemma (3.3), we get

$$\begin{aligned}
T &= \sum_{k=2}^{2n} a_k^{1k} \\
&= \sum_{k=2}^{2n} \frac{\partial}{\partial k}[(-1)^{1+k-1} Pf(M_{1k,1k})] \\
&= \sum_{k=2}^{2n} (-1)^k \left[\sum_{1 \le i < j \le 2n} (-1)^{i+j-1}\ sgn[(i-k)(j-k)] a_k^{ij} Pf(M_{1kij,1kij}) \right] \\
&= \sum_{k=2}^{2n} \left[\sum_{2 \le i < j \le 2n} (-1)^k (-1)^{i+j-1}\ sgn[(i-k)(j-k)] a_k^{ij} Pf(M_{1kij,1kij}) \right].
\end{aligned}$$

From the fact

$$Pf(M_{1kij,1kij}) = Pf(M_{1ijk,1ijk}) = Pf(M_{1jik,1jik})$$

and by the definiton of sign function we obtain

$$\begin{aligned}
T = &\sum_{2 \le k < i < j \le 2n} (-1)^{k+i+j-1} a_k^{ij} Pf(M_{1kij,1kij}) + \sum_{2 \le i < k < j \le 2n} (-1)^{k+i+j} a_k^{ij} Pf(M_{1kij,1kij}) \\
&+ \sum_{2 \le i < j < k \le 2n} (-1)^{k+i+j-1} a_k^{ij} Pf(M_{1kij,1kij}) \\
= &\sum_{2 \le i < j < k \le 2n} (-1)^{k+i+j-1} a_i^{jk} Pf(M_{1ijk,1ijk}) + \sum_{2 \le i < j < k \le 2n} (-1)^{k+i+j} a_j^{ik} Pf(M_{1jik,1jik}) \\
&+ \sum_{2 \le i < j < k \le 2n} (-1)^{k+i+j-1} a_k^{ij} Pf(M_{1kij,1kij}) \\
= &\sum_{2 \le i < j < k \le 2n} (-1)^{k+i+j-1} (a_k^{ij} - a_j^{ik} + a_i^{jk}) Pf(M_{1ijk,1ijk})
\end{aligned}$$

Thus, from (3.0.9) we have $T = 0$. Similarly, $(a_1^{21} + a_3^{23} + a_4^{24} + \cdots + a_{2n}^{2\,2n})$ is seen to be 0. Thus we get $\Delta a^{12} = 0$ which means a^{12} is harmonic.

By a change of basis, a^{rs} can be brought to the position of a^{12} for all $1 \le r < s \le 2n$. Repeating the same argument in this orthonormal basis, we obtain $\Delta a^{rs} = 0$ which means that a^{rs} is harmonic. This completes the proof. ∎

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