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ON INDIRECT SINGULAR POINTS FOR MEROMORPHIC FUNCTIONS

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Abstract

By using the potential theory and Pólya peaks, we will investigate the indirect singular points of meromorphic functions. This is a continuous work of M. Tsuji.

1. Introduction and results

We assume that the reader is familiar with the basic Nevanlinna notations. For example,

$$\overline{N}^{l)}(r,\Omega,f=a)=\int_{r_0}^{r}\frac{\bar{n}^{l)}(t,\Omega,f=a)}{t}\,dt,$$

where $\Omega=\{z:\alpha\le\arg z\le\beta,|z|<1\}$. $\bar{n}^{l)}(t,\Omega,f=a)$ is the number of distinct zeros with multiplicity $\le l$ of $f(z)=a$ in $\Omega\cap\{r_0<|z|<t\}$ counted only once. The lower order μ and the order ρ of $f(z)$:

$$\mu(f)=\liminf_{r\to1-}\frac{\log T(r,f)}{\log\dfrac{1}{1-r}},\quad \rho(f)=\limsup_{r\to1-}\frac{\log T(r,f)}{\log\dfrac{1}{1-r}}.$$

For a meromorphic function in $|z|<1$, Tsuji [4] proved the analogue of Biernacki-Rauch's theorem.

Theorem 1.1. *Let $f(z)$ be a meromorphic function of finite order $\rho>0$ in $|z|<1$. Then there exist a point z_0 on $|z|=1$ and a line J through z_0, directed inward of $|z|<1$, which may coincide with the tangent of $|z|=1$ at z_0 and satisfied the following condition.*

Let ω be any small angular domain, which contains J and is bounded by two lines through z_0 and $g(z)$ be a meromorphic function in $|z|<1$ and $\{z_v(f=g,\omega)\}$ be zero points of $f(z)=g(z)$ in ω, multiple zeros being counted only once.

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(i) *If $g(z)$ is of order $< \rho$, then*

$$\Sigma_v(1 - |z_v(f = g, \omega)|)^{\rho+1-\varepsilon} = \infty, \quad \varepsilon > 0,$$

with at most two exceptions for g.

(ii) *If $f(z)$ is of divergence type and $\int_0^1 T(r, g)(1 - r)^{\rho-1}\, dr < \infty$, then*

$$\Sigma_v(1 - |z_v(f = g, \omega)|)^{\rho+1} = \infty,$$

with at most two exceptions for g.

The point z_0 in Theorem 1.1 is called the indirect singular point. In this paper, we will prove the existence of singular points dealing with multiple values for $f(z)$ with $0 \le \rho(f) \le \infty$. For the positive order case, we will prove Theorem 1.2.

THEOREM 1.2. *Let $f(z)$ be a meromorphic function with the order $0 < \rho \le \infty$ and the lower order $0 \le \mu < \infty$ defined in $|z| < 1$. Then there exist a point z_0 on $|z| = 1$ and a line J through z_0, directed inward of $|z| < 1$, which may coincide with the tangent of $|z| = 1$ at z_0 and satisfied the following condition.*

Let ω be any small angular domain, which contains J and is bounded by two lines through z_0 and $g(z)$ be a meromorphic function in $|z| < 1$ with $T(r, g) = o(T(r, f))$ as $r \to 1-$, and $l(\ge 3)$ be a positive integer. Then

$$\limsup_{r \to 1-} \frac{\overline{N}^{l)}(r, \omega, f = g)}{T(r, f)} > 0, \tag{1.1}$$

with two possible exceptions for g, and z_0 is called an indirect T point of $f(z)$ dealing with multiple value.

Next, we consider some subclass of order zero. Suppose that $f(z)$ satisfies the following

$$\limsup_{r \to 1-} \frac{T(r, f)}{\log \frac{1}{1 - r}} = \infty, \quad \text{and} \quad \limsup_{r \to 1-} \frac{\log T(r, f)}{\log \frac{1}{1 - r}} = 0. \tag{1.2}$$

Set $X = \log \frac{1}{1 - r}$. By Valiron's results (see [3]), there exists a type function $W(X)$, which has a non-decreasing and positive continuous derivative $W'(X)$, and satisfies

(1) $T(r, f) \le W(X)$, and there exists a sequence $X_n = \log \frac{1}{1 - r_n} \to \infty$ $(n \to \infty)$, such that $\frac{1}{2} W(X_n) < T(r_n, f)$,

$$\lim_{X \to \infty} \frac{W(X)}{X} = \infty,$$

(2) $\dfrac{W'(X)}{W(X)}$ tending to 0 decreasingly,

(3) $W(X+\log h) < K(h)W(X)$, $W'(X+\log h) < K(h)W'(X)$, where $K(h)$ is a constant dependent on h,

$$\lim_{r\to 1-} \frac{W\left(\log \dfrac{h}{1-r}\right)}{W\left(\log \dfrac{1}{1-r}\right)} = 1, \quad \text{for any } h > 1.$$

THEOREM 1.3. *Let $w = f(z)$ be a meromorphic function in $|z| < 1$ and satisfy (1.2). Then there exist a point $z_0 \in \{|z| = 1\}$, and a line J through z_0, directed inward of the unit disk, which may coincide with the tangent of $|z| = 1$ at z_0 and satisfying the following condition.*

Let ω be any small angular domain, which contains J and is bounded by two lines through z_0 and $g(z)$ be a meromorphic function in $|z| < 1$ with $T(r,g) = O(1)$, $l(\geq 3)$ be a positive integer. Then

$$\limsup_{r\to 1-} \frac{\overline{N}^{l)}(r,\omega,f(z)=g)}{T(r,f)} > 0,$$

with at most two possible exceptions for $g(z)$.

THEOREM 1.4. *Let $w = f(z)$ be a meromorphic function in $|z| < 1$ and satisfy (1.2). Then there exists a point $z_0 \in \{|z| = 1\}$, and a line J through z_0, directed inward of the unit disk, which may coincide with the tangent of $|z| = 1$ at z_0 and satisfying the following*:

Let ω be any small angular domain, which contains J and is bounded by two lines through z_0 and $g(z)$ be a meromorphic function in $|z| < 1$ with $T(r,g) = o\left(W\left(\log \dfrac{1}{1-r}\right)\right)$, $l(\geq 3)$ be a positive integer. Then

$$\limsup_{r\to 1-} \frac{\log \bar{n}^{l)}(r,\omega,f=g)}{\log W\left(\log \dfrac{1}{1-r}\right) + \log \dfrac{1}{1-r}} = 1, \tag{1.3}$$

with at most two possible exceptions for $g(z)$, where $W\left(\log \dfrac{1}{1-r}\right)$ is the type function of $T(r,f)$, and z_0 is called an indirect maximum type Borel point of $f(z)$ dealing with multiple value.

2. Some lemmas

First, we recall the Ahlfors-Shimuzi characteristic function of $f(z)$ defined in the sector Ω.

$$\mathscr{S}(r,\Omega,f) = \frac{1}{\pi}\int_{r_0}^{r}\int_{\alpha}^{\beta}\frac{|f'(te^{i\theta})|^2}{(1+|f(te^{i\theta})|)^2}t\,dtd\theta,$$

$$\mathscr{T}(r,\Omega,f) = \int_{r_0}^{r}\frac{\mathscr{S}(t,\Omega,f)}{t}\,dt.$$

We write $\mathscr{S}(r,\Omega,f) = S(r,f)$ and $\mathscr{T}(r,\Omega,f) = T(r,f)$ when $\Omega = \{z : 0 \le \arg z < 2\pi\}$.

Now we give some lemmas which will be used in the proof of the theorems.

LEMMA 2.1 ([2]). *Let $f(z)$ be a meromorphic function in $|z| < R$, $a_1, a_2, \dots, a_q$ $(q \ge 3)$ be q different points on $\hat{\mathbf{C}}$ and the spherical distance between any two points in them $|a_i, a_j| \ge \delta$, $\delta \in \left(0, \frac{1}{2}\right)$, $l(\ge 3)$ be a positive integer. Then for any $r \in (0, R)$, we have*

$$\left(q-2-\frac{2}{l}\right)S(r,f) \le \sum_{j=1}^{q}\bar{n}^{l)}(R,a_j) + A\frac{R}{R-r},$$

where A is a constant number.

Combing Lemma 2.1 with [5], we can prove Lemmas 2.2–2.4. For the completeness, we give the proof of Lemma 2.2.

LEMMA 2.2. *Let $f(z)$ be meromorphic in $|z| < 1$ and $\Delta \subset \Delta_0$ be two angular domains (see figure 1), each of which is bounded by two lines, through $z = 1$, which do not touch $|z| = 1$ and $\Delta(r)$, $\Delta_0(r)$ be the part of Δ, Δ_0, which lies in $r_0 \le |z| \le r < 1$ respectively, where $r_0 \ge \frac{1}{2}$ is so chosen that the circle $|z| = r_0$ meets the both sides of Δ and Δ_0, $l(\ge 3)$ be a positive integer. Then we have*

$$\left(q-2-\frac{2}{l}\right)\mathscr{S}(r,\Delta,f) \le 3\sum_{j=1}^{q}\bar{n}^{l)}\left(\frac{r+3}{4},a_i,\Delta_0\right) + O\left(\log\frac{1}{1-r}\right),$$

$$\left(q-2-\frac{2}{l}\right)\mathscr{T}(r,\Delta,f) \le 21\sum_{j=1}^{q}\overline{N}^{l)}\left(\frac{r+3}{4},a_i,\Delta_0\right) + O(1).$$

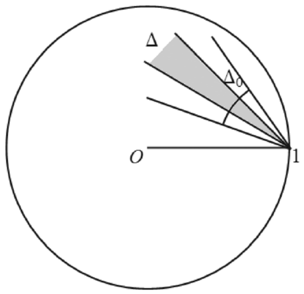

FIGURE 1

Proof. Let $r_n = 1 - \dfrac{1 - r_0}{2^n}$ $(n = 1, 2, \dots)$ and for $n \geq 2$, Δ_n be the part of Δ, which lies in $r_{n-1} \leq |z| \leq r_n$ and Δ_n^0 be that of Δ_0, which lies in $r_{n-2} \leq |z| \leq r_{n+1}$ and

$$S_n = \frac{1}{\pi} \int\int_{\Delta_n} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 r \, dr d\theta. \tag{2.1}$$

Let N_n^0 be the number of zero points of $\prod_{i=1}^q (f(z) - a_i)$ in Δ_n^0, multiple zeros being counted $\leq l$.

We map Δ_n^0 conformally on $|\zeta| < 1$, such that the point $z_n \left(|z_n| = \dfrac{r_n + r_{n-1}}{2} \right)$ on the bisector of Δ becomes $\zeta = 0$, then the image of Δ_n is contained in $|\zeta| < \kappa < 1$, where κ is a constant, independent of n.

If we apply Lemma 2.1, then $(q - 2 - 2/l) S_n < N_n^0 + K$ $(K = \text{const})$, so that

$$\left(q - 2 - \frac{2}{l} \right) \sum_{n=2}^m S_n \leq \sum_{n=2}^m N_n^0 + Km = \sum_{n=2}^m N_n^0 + O\left(\log \frac{1}{1 - r_m} \right). \tag{2.2}$$

Since $\sum_{n=2}^m S_n = S(r_m, \Delta) - S(r_1, \Delta)$ and Δ_n^0 overlap at most 3-times, $\sum_{n=2}^m N_n^0 \leq 3 \sum_{i=1}^q \bar{n}^{l)}(r_{m+1}, \Delta_0, a_i)$, so that by (2.2),

$$\left(q - 2 - \frac{2}{l} \right) \mathscr{S}(r_n, \Delta, f) \leq 3 \sum_{i=1}^q \bar{n}^{l)}(r_{n+1}, \Delta_0, a_i) + O\left(\log \frac{1}{1 - r_n} \right).$$

If $r_{m-1} \leq r \leq r_m$, then $\mathscr{S}(r, \Delta, f) \leq \mathscr{S}(r_m, \Delta, f)$ and $r_{m+1} = \dfrac{r_{m-1} + 3}{4} \leq \dfrac{r + 3}{4}$, hence

$$\left(q - 2 - \frac{2}{l} \right) \mathscr{S}(r, \Delta, f) \leq 3 \sum_{i=1}^q \bar{n}^{l)} \left(\frac{r + 3}{4}, \Delta_0, a_i \right) + O\left(\log \frac{1}{1 - r} \right), \tag{2.3}$$

so that

$$\left(q - 2 - \frac{2}{l} \right) \mathscr{T}(r, \Delta, f) \leq 12 \sum_{i=1}^q \int_{t_0}^{(r+3)/4} \frac{\bar{n}^{l)}(t, \Delta_0, a_i)}{4t - 3} dt + O(1), \quad t_0 = \frac{r_0 + 3}{4} \geq \frac{7}{8}.$$

Since $4t - 3 \geq \dfrac{4t}{7}$, if $t \geq 7/8$, we have

$$\left(q - 2 - \frac{2}{l} \right) \mathscr{T}(r, \Delta, f) \leq 21 \sum_{i=1}^q \bar{N}^{l)} \left(\frac{r + 3}{4}, \Delta_0, a_i \right) + O(1). \tag{2.4}$$

□

Lemma 2.3. *Let $f(z)$ be meromorphic in $|z| < 1$ and $\Delta \subset \Delta_0$ be two sectors, as shown in the figure 2, where $0 < \rho_0 < \rho < 1$ $\Delta(r)$, $\Delta_0(r)$ be the part of Δ, Δ_0, which lies in $r_0 \leq |z| \leq r < 1$ respectively, where $r_0 = \dfrac{1 + \rho}{2}$, $l (\geq 3)$ be a positive integer. Then*

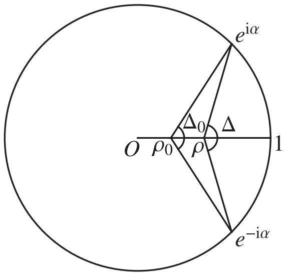

FIGURE 2

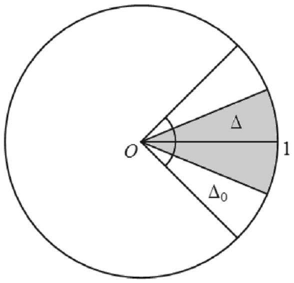

FIGURE 3

$$\left(q-2-\frac{2}{l}\right)\mathscr{S}(r,\Delta,f)\le 9\sum_{j=1}^{q}\bar{n}^{l)}\left(\frac{r+3}{4},a_i,\Delta_0\right)+O\left(\frac{1}{1-r}\right),$$

$$\left(q-2-\frac{2}{l}\right)\mathscr{T}(r,\Delta,f)\le 63\sum_{j=1}^{q}\overline{N}^{l)}\left(\frac{r+3}{4},a_i,\Delta_0\right)+O\left(\log\frac{1}{1-r}\right).$$

LEMMA 2.4. *Let $f(z)$ be meromorphic in $|z|<1$ and $\Delta\subset\Delta_0$ be two domains, as shown in the figure* 3

$$\Delta:|\arg z|<\alpha,\quad \Delta_0:|\arg z|<\alpha_0\quad(\alpha<\alpha_0).$$

Let $\Delta(r)$, $\Delta_0(r)$ be the part of Δ, Δ_0, which lies in $|z|\le r<1$ respectively, $l(\ge 3)$ be a positive integer. Then

$$\left(q-2-\frac{2}{l}\right)\mathscr{S}(r,\Delta,f)\le 9\sum_{j=1}^{q}\bar{n}^{l)}\left(\frac{r+3}{4},a_i,\Delta_0\right)+O\left(\frac{1}{1-r}\right),$$

$$\left(q-2-\frac{2}{l}\right)\mathscr{T}(r,\Delta,f)\le 63\sum_{j=1}^{q}\overline{N}^{l)}\left(\frac{r+3}{4},a_i,\Delta_0\right)+O\left(\log\frac{1}{1-r}\right).$$

The following lemma is applicable in the discussion of angular distribution of a meromorphic function dealing with small functions.

Lemma 2.5. *Let* $w(z)$, $g_i(z)$ $(i = 1, 2, 3, 4)$ *be meromorphic in* $|z| < 1$. *Set*

$$f(z) = \frac{g_1(z)w(z) + g_2(z)}{g_3(z)w(z) + g_4(z)}.$$

Let $\Delta \subset \Delta_0$ *be two sectors defined in Lemma* 2.2. *Then*

$$\mathcal{S}(r, \Delta, f) \le 27\mathcal{S}\left(\frac{r+63}{64}, \Delta_0, w\right) + O\left(\int_0^{(r+127)/128} \frac{T(r,g)}{(1-r)^2}\, dr\right),$$

where $T(r,g) = \sum_{i=1}^{4} T(r, g_i)$.

The same relation holds, if $\Delta \subset \Delta_0$ *are sectors of Lemma* 2.3, *where* 27 *should be replaced by* 729.

3. Proof of the theorems

We introduce for the first time the Pólya peaks for a $T(r)$ in $(0,1)$.

Definition 3.1. A sequence of positive numbers $\{r_n\}$ is called a sequence of Pólya peaks for $T(r)$ of order β (outside a set E) provided that there exist four sequence $\{r_n'\}$, $\{r_n''\}$, $\{\varepsilon_n\}$ and $\{\varepsilon_n'\}$ such that

(1) $r_n \notin E$, $0 < r_n' < r_n < r_n'' < 1$, $r_n' \to 1-$, $\dfrac{1-r_n'}{1-r_n} \to \infty$, $\dfrac{1-r_n}{1-r_n''} \to \infty$, $\varepsilon_n \to 0$, $\varepsilon_n' \to 0$ $(n \to \infty)$;

(2) $\liminf_{n\to\infty} \dfrac{\log T(r_n)}{\log \dfrac{1}{1-r_n}} \ge \beta$;

(3) $T(t) < (1+\varepsilon_n)\left(\dfrac{1-r_n}{1-t}\right)^{\beta} T(r_n)$, $t \in [r_n', r_n'']$;

(4) $T(t) \le KT(r_n)\left(\dfrac{1-r_n}{1-t}\right)^{\beta-\varepsilon_n'}$, $0 < t \le r_n''$ and for a positive constant K.

Lemma 3.1. *Let* $T(r)$ *be a non-negative and non-decreasing continuous function in* $(0,1)$ *with* $0 \le \mu(T) = \liminf_{r\to\infty} \dfrac{\log T(r)}{\log \dfrac{1}{1-r}} < \infty$ *and* $0 < \rho(T) = \limsup_{r\to\infty} \dfrac{\log T(r)}{\log \dfrac{1}{1-r}} \le \infty$. *Then for arbitrary finite and positive number* β

satisfying $\mu \le \beta \le \lambda$ *and a set* F *with* $\int_F \frac{dt}{1-t} < \infty$, *there exists a sequence of the Pólya peaks of order* β *outside* F.

Proof. Zheng [6] proved that for a non-negative and non-decreasing function $T(r)$ in $0 < r < \infty$ with $0 \le \mu(T) = \liminf_{r\to\infty} \frac{\log T(r)}{\log r} < \infty$ and $0 < \rho(T) = \limsup_{r\to\infty} \frac{\log T(r)}{\log r} \le \infty$. Then for arbitrary finite and positive number β satisfying $\mu \le \beta \le \lambda$ and a set F with finite logarithmic measure, that is $\int_F \frac{dt}{t} < \infty$, there exists a sequence of the Pólya peaks of order β outside F.

For our purpose, we set $t = \frac{1}{1-r} \in (0, +\infty)$ for $r \in (0,1)$. Then using Zheng's result to the function $T(r) = T\left(1 - \frac{1}{t}\right)$ implies the lemma. □

The following lemma comes from Gao [1].

Lemma 3.2 [1]. *Let* f *be meromorphic in* $|z| < 1$ *of zero order satisfying* (1.2). *Then we have*

$$\limsup_{r\to 1-} \frac{\log T(r,f)}{\log W\left(\log \frac{1}{1-r}\right)} = \limsup_{r\to 1-} \frac{\log S(r,f)}{\log W\left(\log \frac{1}{1-r}\right) + \log \frac{1}{1-r}} = 1, \tag{3.1}$$

and there exists a $\arg z = \theta$, *such that for any* $\varepsilon > 0$, *we have*

$$\limsup_{r\to 1-} \frac{\log \mathscr{S}(r, Z_\varepsilon(\theta), f)}{\log W\left(\log \frac{1}{1-r}\right) + \log \frac{1}{1-r}} = 1. \tag{3.2}$$

Lemma 3.3. *Let* f *be meromorphic in* $|z| < 1$ *and satisfy* (1.2). *Then there exists a half line* $\arg z = \theta$, *such that for each* $\varepsilon > 0$ *small enough, we have*

$$\limsup_{r\to 1-} \frac{\mathscr{T}(r, Z_\varepsilon(\theta), f)}{W\left(\log \frac{1}{1-r}\right)} > 0,$$

where $Z_\varepsilon(\theta) = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon\}$.

Proof. Suppose the lemma fails, that is, for any $\theta \in [0, 2\pi)$, there exists $\varepsilon_\theta > 0$

$$\mathscr{T}(r, Z_\varepsilon(\theta), f) = o\left(W\left(\log \frac{1}{1-r}\right)\right).$$

There exist a finite number of the radials $\arg z = \theta_j$ $(j = 1, 2, \ldots, m)$ and $\varepsilon_{\theta_j} > 0$, such that

$$[0, 2\pi) \subseteq \bigcup_{i=1}^{m} (\theta_i - \varepsilon_{\theta_i}, \theta_i + \varepsilon_{\theta_i}).$$

$$T(r, f) \le \sum_{i=1}^{m} \mathscr{T}(r, Z_\varepsilon(\theta_i), f) = o\left(W\left(\log \frac{1}{1-r} \right) \right).$$

This leads a contradiction with (3.1). □

Using the same method we also have the following lemma.

Lemma 3.4. *Let f be meromorphic in $|z| < 1$. Then for any sequence $\{r_n\} \to 1-$, there exists a* $\arg z = \theta$, *for each $\varepsilon > 0$ small enough*,

$$\limsup_{n\to\infty} \frac{\mathscr{T}(r_n, Z_\varepsilon(\theta), f)}{T(r_n, f)} > 0.$$

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. In view of Lemma 3.4, there exists a direction $L : \arg z = \theta_0$, for each small enough $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \frac{\mathscr{T}(r_n, Z_\varepsilon(\theta_0), f)}{T(r_n, f)} > 0. \tag{3.3}$$

We assume that $\theta_0 = 0$ and denote the sector: $|\arg z| < \delta$, $|z| < 1$ by the letter Δ. L_1, L_2 denote the half lines $\arg z = \delta$ and $\arg z = -\delta$ respectively, and $\xi_0 = e^{i\delta}$ $\left(0 < \delta < \dfrac{\pi}{2}\right)$ is the intersection point of L_1 and the unit circle. We draw two lines L_3, L_4 through $z = 1$ directly inward of the unit disk and symmetric with respect to the real axis, making an angle $< \dfrac{\pi}{2}$ with the negative real axis. Point a is the intersection point of L_1 and L_3. Let Ω be the angular domain, bounded by these two lines and we denote the common part of Ω and Δ by the same letter Ω. Then Δ consists of three parts: $\Delta = \Omega + \Omega' + \Omega''$, where Ω' bounded by L_1, L_3 and the unit circle, Ω'' bounded by L_2, L_4 and the unit circle (see figure 4). Then one of the following holds

$$\limsup_{n\to\infty} \frac{\mathscr{T}(r_n, \Omega, f)}{T(r_n, f)} > 0, \quad \limsup_{n\to\infty} \frac{\mathscr{T}(r_n, \Omega', f)}{T(r_n, f)} > 0, \quad \limsup_{n\to\infty} \frac{\mathscr{T}(r_n, \Omega'', f)}{T(r_n, f)} > 0.$$

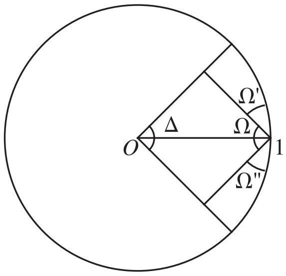

FIGURE 4

Case 1. First we suppose

$$\limsup_{n\to\infty} \frac{\mathcal{T}(r_n, \Omega, f)}{T(r_n, f)} > 0. \tag{3.4}$$

By dividing the angular domain Ω into 2^n equal parts by lines through $z = 1$, we see that there exists a line $J \in \Omega$ through $z = 1$, such that for any small angular domain ω, which contains J and is bounded by two lines through $z = 1$,

$$\limsup_{n\to\infty} \frac{\mathcal{T}(r_n, \omega, f)}{T(r_n, f)} > 0. \tag{3.5}$$

Let $\omega \subset \omega_1 \subset \omega_0$ be three angular domains, whose common vertex is at $z = 1$. Let $g_i(z)$ $(i = 1, 2, 3)$ be three meromorphic functions in $|z| < 1$, such that $T(r, g_i) = o(T(r, f))$.

Hence if we put $T(r, g) = \sum_{i=1}^{3} T(r, g_i)$, then $T(r, g) = o(T(r, f))$. We put

$$w(z) = \frac{f(z) - g_1(z)}{f(z) - g_3(z)} \frac{g_2(z) - g_3(z)}{g_2(z) - g_1(z)}, \quad f(z) = \frac{h_1(z)w(z) + h_2(z)}{h_3(z)w(z) + h_4(z)},$$

then $T(r, h_i) = O(T(r, g))$ $(i = 1, 2, 3, 4)$. In view of Lemma 2.5, we have

$$\mathcal{T}(r_n, \omega, f) \le const.\mathcal{T}\left(\frac{r_n + 63}{64}, \omega_1, w\right) + O\left(\int_0^{r_n} \int_0^{(r+127)/128} \frac{T(t, g)}{(1 - t)^2}\, dtdr\right),$$

By Lemma 3.1, there exists a sequence of Pólya peaks $($*of* $T(r, f))$ $\{r_n\}$ with finite positive order σ between μ and ρ such that

$$T\left(\frac{t + 127}{128}, f\right) \le K\left(\frac{1 - r_n}{1 - t}\right)^{\sigma} T\left(\frac{r_n + 127}{128}, f\right),$$

for $0 \le t \le r_n$. Thus we have

$$\int_0^{r_n}\int_0^{(r+127)/128}\frac{T(t,f)}{(1-t)^2}\,dtdr = \int_0^{(r_n+127)/128}\int_{128t-127}^{r_n}\frac{T(t,f)}{(1-t)^2}\,drdt$$
$$\le \int_0^{(r_n+127)/128}\int_{128t-127}^{1}\frac{T(t,f)}{(1-t)^2}\,drdt$$
$$= \int_0^{(r_n+127)/128}\frac{T(t,f)}{1-t}\,dt$$
$$\le \int_0^{r_n}\frac{T\left(\frac{t+127}{128},f\right)}{1-t}\,dt$$
$$\le \sigma KT\left(\frac{r_n+127}{128},f\right)$$
$$\le 2\sigma K 128^{\sigma} T(r_n,f).$$

Hence

$$\mathscr{T}(r_n,\omega,f) \le const.\mathscr{T}\left(\frac{r_n+63}{64},\omega_1,w\right) + o(T(r_n,f)).$$

In view of Lemma 2.2, we have

$$\left(1-\frac{2}{l}\right)\mathscr{T}(r_n,\omega,f) \le const.\sum_{i=1}^{3}\overline{N}^{l)}\left(\frac{r_n+255}{256},\omega_0,f=g_i\right) + o(T(r_n,f)).$$

By noting that $\{r_n\}$ is a sequence of Pólya peaks, we have $T\left(\frac{r_n+255}{256},f\right) \le KT(r_n,f)$ and this implies that we have

$$\limsup_{n\to\infty}\frac{\sum_{i=1}^{3}\overline{N}^{l)}\left(\frac{r_n+255}{256},\omega_0,f=g_i\right)}{T\left(\frac{r_n+255}{256},f\right)} > 0.$$

Case 2. We assume

$$\limsup_{n\to\infty}\frac{\mathscr{T}(r_n,\Omega',f)}{T(r_n,f)} > 0.$$

Let ξ_0 and a defined as before, and ξ_1 be a point on $|z|=1$, which lies symmetric to $z=1$ with respect to the line $O\xi_0$.

Let a_0 be a point on the line $O\xi_0$, such that $a = \frac{a_0+\xi_0}{2}$.

Let Σ_0, Σ be the sectors, defined as in the figure 5.

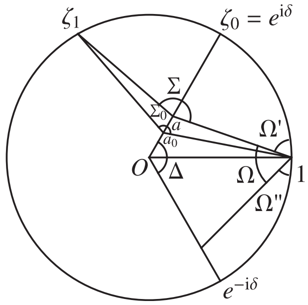

FIGURE 5

Then we have

$$\limsup_{n\to\infty}\frac{\mathcal{T}(r_n,\Sigma,f)}{T(r_n,f)}>0.$$

Using Lemma 2.3 and the method similar to Case 1, we can obtain

$$\limsup_{r\to 1-}\frac{\overline{N}^{l)}(r,\Sigma_0,f=g)}{T(r,f)}>0, \tag{3.6}$$

with at most two possible exceptions for g.

Case 3. We denote the angular magnitude of Ω by $|\Omega|$, then $0<|\Omega|<\pi$. Let $\Omega_1\subseteq\Omega_2\subseteq\cdots\subseteq\cdots$ be angular domains as Ω, such that $0<|\Omega_n|\to\pi$ and suppose that for $n=1,2,\ldots$,

$$\limsup_{r\to 1-}\frac{\mathcal{T}(r,\Omega_n,f)}{T(r,f)}>0.$$

Without loss of generality, we assume that

$$\limsup_{r\to 1-}\frac{\mathcal{T}(r,\Omega_n',f)}{T(r,f)}>0\quad(n=1,2,\ldots).$$

Let J be the positive tangent of $|z|<1$ at $z=1$. Then from (3.6) we see that for each small angular domain ω, which contains J,

$$\limsup_{r\to 1-}\frac{\overline{N}^{l)}(r,\omega,f=g)}{T(r,f)}>0,$$

with two possible exceptions for g.

Theorem 1.2 follows.

Proofs of Theorem 1.3 *and Theorem* 1.4. We can prove Theorem 1.3 and Theorem 1.4 with the same method of Theorem 1.2, here we only give a sketch of the proofs of them.

Sketch of proof of Theorem 1.3. In view of Lemma 2.2, Lemma 2.3 and Lemma 2.5, we have

$$\left(1-\frac{2}{l}\right)\mathscr{T}(r,\omega,f) \le K\sum_{i=1}^{3}\overline{N}^{l)}\left(\frac{r+255}{256},\omega_0,f=g_i(z)\right) + O\left(\int_0^r\int_0^{(r+127)/128}\frac{T(t,g)}{(1-t)^2}\,dtdr\right),$$

where $T(r,g)=\sum_{j=1}^{3}T(r,g_j)$, and K is a constant.

The similar to the proof of Theorem 1.2 implies that

$$\limsup_{r\to 1-}\frac{\mathscr{T}(r,\omega,f)}{W\left(\log\left(\frac{1}{1-r}\right)\right)}>0.$$

Since $T(r,g)=O(1)$, we have

$$\int_0^r\int_0^{(s+127)/128}\frac{T(t,g)}{(1-t)^2}\,dtds = O\left(\int_0^r\int_0^{(s+127)/128}\frac{1}{(1-t)^2}\,dtds\right) = O\left(\log\frac{1}{1-r}\right)=o\left(W\left(\log\left(\frac{1}{1-r}\right)\right)\right).$$

Moreover, by the property of the type function, we have

$$\limsup_{r\to 1-}\frac{\sum_{i=1}^{3}\overline{N}^{l)}\left(\frac{r+255}{256},\omega_0,f(z)=g_i(z)\right)}{W\left(\log\left(\frac{256}{1-r}\right)\right)}>0.$$

Thus

$$\limsup_{r\to 1-}\frac{\overline{N}^{l)}(r,\omega_0,f(z)=g(z))}{T(r,f)}>0$$

with at most two possible exceptions for g.

Sketch of proof of Theorem 1.4. (3.2) implies that

$$\left(1-\frac{2}{l}\right)\mathscr{S}(r,\omega,f)\le K\sum_{j=1}^{3}\bar{n}^{l)}\left(\frac{r+255}{256},\omega_0,f=a_j(z)\right) + O\left(\int_{r_0}^{(r+127)/128}\frac{T(r,g)}{(1-r)^2}\,dr\right). \tag{3.7}$$

Now we treat with the last term of (3.7)

$$O\left(\int_{r_0}^{(r+127)/128}\frac{T(r,g)}{(1-r)^2}\,dr\right)=o\left(\int_{r_0}^{(r+127)/128}\frac{W\left(\log\frac{1}{1-r}\right)}{(1-r)^2}\,dr\right) = o\left(W\left(\log\frac{1}{1-r}\right)\frac{1}{1-r}\right).$$

Hence,

$$\limsup_{r\to 1-}\sum_{j=1}^{3}\frac{\log\bar{n}^{l)}(r,\omega,f(z)=a_j(z))}{\log W\left(\log\frac{1}{1-r}\right)+\log\frac{1}{1-r}}\ge 1.$$

On the other hand,

$$\limsup_{r\to 1-}\sum_{j=1}^{3}\frac{\log\bar{n}^{l)}(r,\omega,f(z)=a_j(z))}{\log W\left(\log\frac{1}{1-r}\right)+\log\frac{1}{1-r}}\le 1.$$

Thus Theorem 1.4 follows.

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