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SOME PROPERTIES OF NORDEN-WALKER METRICS[1]

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Abstract

The main purpose of the present paper is to study almost Norden structures on 4-dimensional Walker manifolds. We discuss the integrability and Kähler (holomorphic) conditions for these structures. The curvature properties for Norden-Walker metrics is also investigated. Then examples Norden-Walker metrics are constructed from an arbitrary harmonic function of two variables.

1. Introduction

Let M_{2n} be a Riemannian manifold with a neutral metric, i.e., with a pseudo-Riemannian metric g of signature (n, n). We denote by $\Im_q^p(M_{2n})$ the set of all tensor fields of type (p, q) on M_{2n}. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

Let (M_{2n}, φ) be an almost complex manifold with almost complex structure φ. This structure is said to be integrable if the matrix $\varphi = (\varphi_j^i)$ is reduced to the constant form in a certain holonomic natural frame in a neighborhood U_x of every point $x \in M_{2n}$. In order that the almost complex structure φ be integrable, it is necessary and sufficient that there exists a torsion-free affine connection ∇ with respect to which the structure tensor φ is covariantly constant, i.e., $\nabla\varphi = 0$. Also, we know that the integrability of φ is equivalent to the vanishing of the Nijenhuis tensor $N_\varphi \in \Im_2^1(M_{2n})$ [8, p. 124]. If φ is integrable, then φ is a complex structure and moreover M_{2n} is a $\mathbf{C}$-holomorphic manifold $X_n(\mathbf{C})$ whose transition functions are holomorphic mappings.

1.1. Norden metrics

A metric g is a Norden metric [14] if

$$g(\varphi X, \varphi Y) = -g(X, Y)$$

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or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_{2n})$. Metrics of this kind have been also studied under the other names: pure, anti-Hermitian and B-metrics (see [4], [5], [9], [15], [19], [21]). If (M_{2n}, φ) is an almost complex manifold with Norden metric g, we say that (M_{2n}, φ, g) is an almost Norden manifold. If φ is integrable, we say that (M_{2n}, φ, g) is a Norden manifold.

1.2. Holomorphic (almost holomorphic) tensor fields

Let $\overset{*}{t}$ be a complex tensor field on a **C**-holomorphic manifold $X_n(\mathbf{C})$. The real model of such a tensor field is a tensor field on M_{2n} of the same order irrespective of whether its vector or covector arguments is subject to the action of the affinor structure φ. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e.g., [9], [16], [17], [19], [20], [21], [23]). In particular, for a $(0, q)$ -tensor field ω, the purity means that for any $X_1, \dots, X_q \in \Im_0^1(M_{2n})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, \dots, X_q) = \omega(X_1, \varphi X_2, \dots, X_q) = \cdots = \omega(X_1, X_2, \dots, \varphi X_q).$$

We define an operator

$$\Phi_\varphi : \Im_q^0(M_{2n}) \to \Im_{q+1}^0(M_{2n})$$

applied to the pure tensor field ω by (see [23])

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, \dots, Y_q) = (\varphi X)(\omega(Y_1, Y_2, \dots, Y_q)) - X(\omega(\varphi Y_1, Y_2, \dots, Y_q))$$
$$+ \omega((L_{Y_1}\varphi)X, Y_2, \dots, Y_q) + \cdots + \omega(Y_1, Y_2, \dots, (L_{Y_q}\varphi)X),$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a complex structure on M_{2n} and the tensor field $\Phi_\varphi \omega$ vanishes, the complex tensor field $\overset{*}{\omega}$ on $X_n(\mathbf{C})$ is said to be holomorphic (see [9], [19], [23]). Thus a holomorphic tensor field $\overset{*}{\omega}$ on $X_n(\mathbf{C})$ is realized on M_{2n} in the form of a pure tensor field ω, such that

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, \dots, Y_q) = 0$$

for any $X, Y_1, \dots, Y_q \in \Im_0^1(M_{2n})$. Therefore such a tensor field ω on M_{2n} is also called holomorphic tensor field. When φ is an almost complex structure on M_{2n}, a tensor field ω satisfying $\Phi_\varphi \omega = 0$ is said to be almost holomorphic.

1.3. Holomorphic Norden (Kähler-Norden) metrics

In a Norden manifold a Norden metric g is called a *holomorphic* if

$$(\Phi_\varphi g)(X, Y, Z) = 0 \qquad (1)$$

for any $X, Y, Z \in \Im_0^1(M_{2n})$.

By setting $X = \partial_k$, $Y = \partial_i$, $Z = \partial_j$ in the equation (1), we see that the components $(\Phi_\varphi g)_{kij}$ of $\Phi_\varphi g$ with respect to a local coordinate system $x^1, \dots, x^n$ may be expressed as follows:

$$(\Phi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im} \partial_j \varphi_k^m.$$

If (M_{2n}, φ, g) is a Norden manifold with holomorphic Norden metric g, we say that (M_{2n}, φ, g) is a *holomorphic Norden manifold.*

In some aspects, holomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is analogue to the next known result: An almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

Theorem 1 [6] (For paracomplex version see [18]). *For an almost complex manifold with Norden metric g, the condition $\Phi_\varphi g = 0$ is equivalent to $\nabla \varphi = 0$, where ∇ is the Levi-Civita connection of g.*

A *Kähler-Norden* manifold can be defined as a triple (M_{2n}, φ, g) which consists of a manifold M_{2n} endowed with an almost complex structure φ and a pseudo-Riemannian metric g such that $\nabla \varphi = 0$, where ∇ is the Levi-Civita connection of g and the metric g is assumed to be Nordenian. Therefore, there exists a one-to-one correspondence between *Kähler-Norden* manifolds and Norden manifolds with a *holomorphic metric*. Recall that in such a manifold, the Riemannian curvature tensor is pure and holomorphic, also the curvature scalar is locally holomorphic function (see [6], [15]).

Remark 1. We know that the integrability of the almost complex structure φ is equivalent to the existing a torsion-free affine connection with respect to which the equation $\nabla \varphi = 0$ holds. Since the Levi-Civita connection ∇ of g is a torsion-free affine connection, we have: If $\Phi_\varphi g = 0$, then φ is integrable. Thus, almost Norden manifold with conditions $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$, i.e., *almost holomorphic Norden manifolds* (*analogues of the almost Kähler manifolds with closed Kähler form*) *does not exist*.

In the present paper, we shall focus our attention to the Norden manifolds in dimension four. Using the Walker metric we construct a new Norden-Walker metrics together with a proper almost complex structure [11].

2. Norden-Walker metrics

2.1. Walker metric g

A neutral metric g on a 4-manifold M_4 is said to be Walker metric if there exists a 2-dimensional null distribution D on M_4, which is parallel with respect to g. From Walker's theorem [22], there is a system of coordinates with respect to which g takes the local canonical form

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{2}$$

where a, b, c are smooth functions of the coordinates (x, y, z, t). The paralel null 2-plane D is spanned locally by $\{\partial_x, \partial_y\}$, where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$.

In [11, Fact 1], a proper almost complex structure with respect to g is defined as a g-orthogonal almost complex structure J so that J is a standard generator of a positive $\frac{\pi}{2}$ rotation on D, i.e., $J\partial_x = \partial_y$ and $J\partial_y = -\partial_x$. Then for the Walker metric g, such a proper almost complex structure J is determined uniquely as

$$\begin{pmatrix} 0 & -1 & -c & \frac{1}{2}(a-b) \\ 1 & 0 & \frac{1}{2}(a-b) & c \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

In [1], for such a proper almost complex structure J on Walker 4-manifold M, an almost Norden structure (g^{N+}, J) is constructed, where g^{N+} is a metric on M, with properties $g^{N+}(JX, JY) = -g^{N+}(X, Y)$. In fact, as one of these examples, such a metric takes the form (see Proposition 6 in [1]):

$$g^{N+} = \begin{pmatrix} 0 & -2 & 0 & -b \\ -2 & 0 & -a & -2c \\ 0 & -a & 0 & \frac{1}{2}(1-ab) \\ -b & -2c & \frac{1}{2}(1-ab) & -2c \end{pmatrix}.$$

We may call this an almost Norden-Walker metric. The construction of such a structure in [1] is to find a Norden metric for a given almost complex structure, which is different form the Walker metric.

The purpose of the present paper is to find also an almost Norden-Walker structure (g, F), where the metric is nothing but the Walker metric g, with an appropriate almost complex structure F, to be determined. That is, for a fixed metric g, we will find an almost complex structure F which satisfy $g(FX, FY) = -g(X, Y)$.

In [1], for a given almost complex structure, a metric is constructed. Our method is, however, for a given metric, an almost complex structure is constructed.

2.2. Almost Norden-Walker manifolds

Let F be an almost complex structure on a Walker manifold M_4, which satisfies

i) $F^2 = -I$,
ii) $g(FX, Y) = g(X, FY)$ (Nordenian property),
iii) $F\partial_x = \partial_y$, $F\partial_y = -\partial_x$ (F induces a positive $\frac{\pi}{2}$-rotation on D).

We easily see that these three properties define F non-uniquely, i.e.,

$$\begin{cases} F\partial_x = \partial_y, \\ F\partial_y = -\partial_x, \\ F\partial_z = \alpha\partial_x + \frac{1}{2}(a+b)\partial_y - \partial_t, \\ F\partial_t = -\frac{1}{2}(a+b)\partial_x + \alpha\partial_y + \partial_z \end{cases}$$

and F has the local components

$$F = (F_j^i) = \begin{pmatrix} 0 & -1 & \alpha & -\frac{1}{2}(a+b) \\ 1 & 0 & \frac{1}{2}(a+b) & \alpha \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

with respect to the natural frame $\{\partial_x, \partial_y, \partial_z, \partial_t\}$, where $\alpha = \alpha(x, y, z, t)$ is an arbitrary function.

We must note that the proper almost complex structure J as in (3) is determined uniquely. In our case of the almost Norden-Walker structure, the almost complex structure F just obtained contains an arbitrary function $\alpha(x, y, z, t)$.

Our purpose is to find a nontrivial almost Norden-Walker structure with the Walker metric g explicitly.

Therefore, we now put $\alpha = c$. Then g defines a unique almost complex structure

$$\varphi = (\varphi_j^i) = \begin{pmatrix} 0 & -1 & c & -\frac{1}{2}(a+b) \\ 1 & 0 & \frac{1}{2}(a+b) & c \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4}$$

The triple (M_4, φ, g) is called almost Norden-Walker manifold. In conformity with the terminology of [2], [3], [10], [11], [13] we call φ the proper almost complex structure.

Remark 2. From (4) we immediately see that in the case $a = -b$ and $c = 0$, φ is integrable.

2.3. Integrability of φ

We consider the general case for integrability.

The almost complex structure φ on almost Norden-Walker manifolds is integrable if and only if

$$(N_\varphi)_{jk}^i = \varphi_j^m \partial_m \varphi_k^i - \varphi_k^m \partial_m \varphi_j^i - \varphi_m^i \partial_j \varphi_k^m + \varphi_m^i \partial_k \varphi_j^m = 0. \tag{5}$$

From (4) and (5) find the following integrability condition.

Theorem 2. *The proper almost complex structure φ on almost Norden-Walker manifolds is integrable if and only if the following PDEs hold*:

$$\begin{cases} a_x + b_x + 2c_y = 0, \\ a_y + b_y - 2c_x = 0. \end{cases} \tag{6}$$

From this theorem, we easily see that if $a = -b$ and $c = 0$, then φ is integrable (see Remark 2).

Let (M_4, φ, g) be a Norden-Walker manifolds ($N_\varphi = 0$) and $a = b$. Then the equation (6) reduces to

$$\begin{cases} a_x = -c_y, \\ a_y = c_x, \end{cases} \tag{7}$$

from which follows

$$\begin{aligned} a_{xx} + a_{yy} = 0, \\ c_{xx} + c_{yy} = 0, \end{aligned} \tag{8}$$

e.g., the functions a and c are harmonic with respect to the arguments x and y.

Thus we have

Theorem 3. *If the triple (M_4, φ, g) is Norden-Walker and $a = b$, then a and c are all harmonic with respect to the arguments x, y.*

2.4. Example of Norden-Walker metric

We now apply the Theorem 3 to establish the existence of special types of Norden-Walker metrics. In our arguments, the harmonic function plays an important part.

Let $a = b$ and $h(x, y)$ be a harmonic function of variables x and y, for example $h(x, y) = e^x \cos y$. We put

$$a = a(x, y, z, t) = h(x, y) + \alpha(z, t) = e^x \cos y + \alpha(z, t)$$

where α is an arbitrary smooth function of z and t. Then, a is also hormonic with respect to x and y. We have

$$a_x = e^x \cos y,$$
$$a_y = -e^x \sin y.$$

From (7), we have PDEs for c to satisfy as

$$c_x = a_y = -e^x \sin x,$$
$$c_y = -a_x = -e^x \cos y.$$

For these PDEs, we have solutions

$$c = -e^x \sin y + \beta(z, t),$$

where β is arbitrary smooth function of z and t. Thus the Norden-Walker metric has components of the form

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & e^x \cos y + \alpha(z,t) & -e^x \sin y + \beta(z,t) \\ 0 & 1 & -e^x \sin y + \beta(z,t) & e^x \cos y + \alpha(z,t) \end{pmatrix}$$

3. Holomorphic Norden-Walker (Kähler-Norden-Walker) metrics

Let (M_4, φ, g) be an almost Norden-Walker manifold. If

$$(\Phi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im} \partial_j \varphi_k^m = 0, \tag{9}$$

then by virtue of Theorem 1 φ is integrable and the triple (M_4, φ, g) is called a holomorphic Norden-Walker or a Kähler-Norden-Walker manifold. Taking account of Remark 1, we see that almost Kähler-Norden-Walker manifold with condition $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$ does not exist.

Substituting (2) and (4) in (9), we obtain

$$\begin{aligned}
&(\Phi_\varphi g)_{xzz} = a_y, \quad (\Phi_\varphi g)_{xzt} = (\Phi_\varphi g)_{xtz} = \tfrac{1}{2}(b_x - a_x) + c_y, \\
&(\Phi_\varphi g)_{xtt} = b_y - 2c_x, \quad (\Phi_\varphi g)_{yzz} = -a_x, \\
&(\Phi_\varphi g)_{yzt} = (\Phi_\varphi g)_{ytz} = \tfrac{1}{2}(b_y - a_y) - c_x, \\
&(\Phi_\varphi g)_{ytt} = -b_x - 2c_y, \\
&(\Phi_\varphi g)_{zxz} = (\Phi_\varphi g)_{zzx} = (\Phi_\varphi g)_{txt} = (\Phi_\varphi g)_{ttx} = c_x, \\
&(\Phi_\varphi g)_{zxt} = (\Phi_\varphi g)_{ztx} = -(\Phi_\varphi g)_{txz} = -(\Phi_\varphi g)_{tzx} = \tfrac{1}{2}(a_x + b_x), \\
&(\Phi_\varphi g)_{zyz} = (\Phi_\varphi g)_{zzy} = (\Phi_\varphi g)_{tyt} = (\Phi_\varphi g)_{tty} = c_y, \\
&(\Phi_\varphi g)_{zyt} = (\Phi_\varphi g)_{zty} = -(\Phi_\varphi g)_{tyz} = -(\Phi_\varphi g)_{tzy} = \tfrac{1}{2}(a_y + b_y), \\
&(\Phi_\varphi g)_{zzz} = ca_x - a_t + 2c_z + \tfrac{1}{2}(a+b)a_y, \\
&(\Phi_\varphi g)_{zzt} = (\Phi_\varphi g)_{ztz} = cc_x + b_z + \tfrac{1}{2}(a+b)c_y, \\
&(\Phi_\varphi g)_{ztt} = cb_x + a_t - 2c_z + \tfrac{1}{2}(a+b)b_y, \\
&(\Phi_\varphi g)_{tzz} = ca_y - b_z - \tfrac{1}{2}(a+b)a_x, \\
&(\Phi_\varphi g)_{tzt} = (\Phi_\varphi g)_{ttz} = cc_y - a_t + 2c_z - \tfrac{1}{2}(a+b)c_x, \\
&(\Phi_\varphi g)_{ttt} = cb_y + b_z - \tfrac{1}{2}(a+b)b_x.
\end{aligned}$$

From these equations we have

THEOREM 4. *The triple (M_4, φ, g) is Kähler-Norden-Walker if and only if the following PDEs hold*:

$$a_x = a_y = c_x = c_y = b_x = b_y = b_z = 0, \quad a_t - 2c_z = 0. \tag{10}$$

COROLLARY. *The triple (M_4, φ, g) with metric*

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(z) & 0 \\ 0 & 1 & 0 & b(t) \end{pmatrix}$$

is always Kähler-Norden-Walker.

4. Curvature properties of Norden-Walker manifolds

If R and r are respectively the curvature and the scalar curvature of the Walker metric, then the components of R and r have, respectively, expressions (see [11], Appendix A and C)

$$\begin{aligned}
&R_{xzxz} = -\tfrac{1}{2}a_{xx}, \quad R_{xzxt} = -\tfrac{1}{2}c_{xx}, \quad R_{xzyz} = -\tfrac{1}{2}a_{xy}, \quad R_{xzyt} = -\tfrac{1}{2}c_{xy},\\
&R_{xzzt} = \tfrac{1}{2}a_{xt} - \tfrac{1}{2}c_{xz} - \tfrac{1}{4}a_y b_x + \tfrac{1}{4}c_x c_y, \quad R_{xtxt} = -\tfrac{1}{2}b_{xx}, \quad R_{xtyz} = -\tfrac{1}{2}c_{xy},\\
&R_{xtyt} = -\tfrac{1}{2}b_{xy}, \quad R_{xtzt} = \tfrac{1}{2}c_{xt} - \tfrac{1}{2}b_{xz} - \tfrac{1}{4}(c_x)^2 + \tfrac{1}{4}a_x b_x - \tfrac{1}{4}b_x c_y + \tfrac{1}{4}b_y c_x,\\
&R_{yzyz} = -\tfrac{1}{2}a_{yy}, \quad R_{yzyt} = -\tfrac{1}{2}c_{yy},\\
&R_{yzzt} = \tfrac{1}{2}a_{yt} - \tfrac{1}{2}c_{yz} - \tfrac{1}{4}a_x c_y + \tfrac{1}{4}a_y c_x - \tfrac{1}{4}a_y b_y + \tfrac{1}{4}(c_y)^2, \quad R_{ytyt} = -\tfrac{1}{2}b_{yy},\\
&R_{ytzt} = \tfrac{1}{2}c_{yt} - \tfrac{1}{2}b_{yz} - \tfrac{1}{4}c_x c_y + \tfrac{1}{4}a_y b_x,\\
&R_{ztzt} = c_{zt} - \tfrac{1}{2}a_{tt} - \tfrac{1}{2}b_{zz} - \tfrac{1}{4}a(c_x)^2 + \tfrac{1}{4}aa_x b_x + \tfrac{1}{4}ca_x b_y - \tfrac{1}{2}cc_x c_y - \tfrac{1}{2}a_t c_x\\
&\qquad + \tfrac{1}{2}a_x c_t - \tfrac{1}{4}a_x b_z + \tfrac{1}{4}ca_y b_x + \tfrac{1}{4}ba_y b_y - \tfrac{1}{4}b(c_y)^2 - \tfrac{1}{2}b_z c_y\\
&\qquad + \tfrac{1}{4}a_y b_t + \tfrac{1}{4}a_z b_x + \tfrac{1}{2}b_y c_z - \tfrac{1}{4}a_t b_y
\end{aligned} \tag{11}$$

and

$$r = a_{xx} + 2c_{xy} + b_{yy}. \tag{12}$$

Suppose that the triple (M_4, φ, g) is Kähler-Norden-Walker. Then from the last equation in (10) and (11), we see that

$$R_{ztzt} = c_{zt} - \tfrac{1}{2}a_{tt} = -\tfrac{1}{2}(a_t - 2c_z)_t = 0.$$

From (10) we easily we see that the another components of R in (11) directly all vanish. Thus we have

THEOREM 5. *If a Norden-Walker manifold* (M_4, φ, g) *is Kähler-Norden-Walker, then* M_4 *is flat.*

Remark 3. We note that a Kähler-Norden manifold is non-flat, in such manifold curvature tensor pure and holomorphic [6].

Let (M_4, φ, g) be a Norden-Walker manifold with the integrable proper structure φ, i.e., $N_\varphi = 0$. If $a = b$, then from proof of the Theorem 3 we see that the equation (7) hold. If $c = c(y, z, t)$ and $c = c(x, z, t)$, then $c_{xy} = (c_x)_y = (c_y)_x = 0$. In these cases, by virtue of (7) we find $a = a(x, z, t)$ and $a = (y, z, t)$ respectively. Using of $c_{xy} = 0$ and $a_{xx} + b_{yy} = 0$ (see (8)), we from (12) obtain $r = 0$. Thus we have

THEOREM 6. *If* (M_4, φ, g) *is a Norden-Walker non-Kähler manifold with metrics*

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x,z,t) & c(y,z,t) \\ 0 & 1 & c(y,z,t) & a(x,z,t) \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(y,z,t) & c(x,z,t) \\ 0 & 1 & c(x,z,t) & a(y,z,t) \end{pmatrix}$$

then M_4 *is scalar flat.*

5. On a relation between the Goldberg conjecture of almost Norden-Walker and Kähler-Norden-Walker manifolds

Let (M_{2n}, φ, g) be an almost Norden manifold, and choose a φ-compatible 2-form $\Omega_\varphi(X, Y) = h(\varphi X, Y)$ on M_{2n}, where $h(X, Y) = \tilde{g}(X, Y) + \tilde{g}(\varphi X, \varphi Y)$ is Hermitian metric for any Riemannian metric $\tilde{g}$, which exists provided M_{2n} is paracompact [7, p. 60]. Then we can propose an almost Norden version of Goldberg conjecture as follows [12]: (G_1) if M_{2n} is compact and (G_2) g is Einstein, and (G_3) if a φ-compatible 2-form is closed, then φ must be integrable.

We now define two subfamilles in the set of all compact Norden-Walker 4-manifolds:

$$KNW = \{(M_4, \varphi, g) : \Phi_\varphi g = 0\},$$
$$GNW = \{(M_4, \varphi, g) : M_4 \text{ with conditions } (G_2), (G_3)\}.$$

THEOREM 7. *Let* $M_4 \in KNW$. *Then* M_4 *must be a manifold with condition* (G_2) *of* GNW, *and* φ *is integrable.*

Proof. Suppose that $M_4 \in KNW$. Then from Theorem 5 we see that M_4 is flat, and therefore g is Einstein. By virtue of Remark 1, we see that φ is integrable, if $\Phi_\varphi g = 0$. Thus the proof is completed.

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