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CONFORMAL CLASSIFICATION OF (k, μ)-CONTACT MANIFOLDS

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Abstract

First we improve a result of Tanno that says "If a conformal vector field on a contact metric manifold M is a strictly infinitesimal contact transformation, then it is an infinitesimal automorphism of M" by waiving the "strictness" in the hypothesis. Next, we prove that a (k, μ)-contact manifold admitting a non-Killing conformal vector field is either Sasakian or has $k = -n - 1$, $\mu = 1$ in dimension > 3; and Sasakian or flat in dimension 3. In particular, we show that (i) among all compact simply connected (k, μ)-contact manifolds of dimension > 3, only the unit sphere S^{2n+1} admits a non-Killing conformal vector field, and (ii) a conformal vector field on the unit tangent bundle of a space-form of dimension > 2 is necessarily Killing.

1. Introduction

An m-dimensional Riemannian manifold (M, g) admitting a maximal, i.e. an $(m+1)(m+2)/2$-parameter group of conformal motions is conformally flat. We know (Okumura [8]) that a conformally flat Sasakian (normal contact metric) manifold is of constant curvature 1. Hence the existence of a maximal conformal group places a severe restriction on the Sasakian manifold. Therefore one would like to examine the effect of the existence of a 1-parameter group of conformal motions generated by a conformal vector field on a Sasakian manifold, and more generally on a contact metric manifold $(M, \eta, \xi, \varphi, g)$, where η is a contact 1-form, ξ the Reeb vector field, φ the fundamental collineation tensor and g an associated metric. We will assume M to be connected throughout this paper. We first recall the following definition (Tanno [11]): A vector field V on a contact manifold (M, η) is said to be an infinitesimal contact transformation if

$$\pounds_V \eta = \sigma \eta \tag{1}$$

where $\pounds_V$ denotes Lie-derivative operator along V and σ a smooth function on M. V is strictly infinitesimal contact transformation when $\sigma = 0$. We also say that a vector field on a contact metric manifold is an infitesimal automorphism if it leaves η, ξ, g and φ invariant. In [11] Tanno proved the following result.

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THEOREM (Tanno). *If a conformal vector field on a contact metric manifold M is a strictly infinitesimal contact transformation, then it is an infinitesimal automorphism of M.*

In this paper we generalize this result and prove

THEOREM 1. *Let V be a conformal vector field on a contact metric manifold M. If V is an infinitesimal contact transformation, then V is an infinitesimal automorphism of M.*

Next we recall the following result of Okumura [9], which shows that the existence of a non-Killing vector field places a very severe condition on a Sasakian manifold of dimension > 3.

THEOREM (Okumura). *Let M be a Sasakian manifold of dimension > 3, admitting a non-Killing conformal vector field V. Then V is special concircular. If, in addition, M is complete and connected, then it is isometric to a unit sphere.*

Its proof uses the following result of Obata [6].

THEOREM (Obata). *A necessary and sufficient condition for a complete connected Riemannian manifold (M, g) to be isometric to a Euclidean sphere of radius $\frac{1}{c}$ is that it admits a non-trivial solution f of the system of differential equations $\nabla\nabla f = -c^2 fg$.*

Okumura's result motivates us to examine the existence of a conformal vector field on a more general class of contact metric manifolds $M(\eta, \xi, \varphi, g)$ satisfying the following nullity condition:

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2}$$

where X, Y are arbitrary vector fields on M, R the curvature tensor, k, μ real constants and $h = \frac{1}{2}\pounds_\xi\varphi$ is a self-adjoint trace-free tensor of type $(1,1)$. Such manifolds were introduced by Blair, Koufogiorgos and Papantoniou [3], and are called (k,μ)-contact manifolds. These manifolds include Sasakian manifolds (for which $k = 1$ and $h = 0$) and the trivial sphere bundle $E^{n+1} \times S^n(4)$. Such manifolds are invariant under a D-homothetic deformation: $\bar{\eta} = a\eta$, $\bar{\xi} = \frac{1}{a}\xi$, $\bar{\varphi} = \varphi$, $\bar{g} = ag + a(a-1)\eta \otimes \eta$, which deforms a contact metric structure (η, ξ, φ, g) to another contact metric structure $(\bar{\eta}, \bar{\xi}, \bar{\varphi}, \bar{g})$ (see Tanno [12]). The case when $(k, 0)$-contact manifold admits a conformal vector field was covered by Sharma and Blair [10] who proved the following.

THEOREM (Sharma-Blair). *Let V be a non-Killing conformal vector field on a $(k, 0)$-contact manifold M. For dim.$M > 3$, M is Sasakian and V is concircular,* *and hence if M is complete then it is isometric to a unit sphere. For dim.*$M = 3$, *M is either Sasakian or flat.*

We note that the set of all $(k, 0)$*-contact manifolds is not closed under a D-homothetic deformation, whereas the set of* (k, μ)*-contact manifolds is closed under a D-homothetic deformation.* This fact intrigues us to generalize the result of Sharma and Blair on (k, μ)-contact manifolds. We accomplish it by proving the following conformal classification of such manifolds.

Theorem 2. *Let a* (k, μ)*-contact manifold* $(M, \eta, \xi, \varphi, g)$ *admit a non-Killing conformal vector field V. For dim.*$M > 3$, (i) *M is Sasakian and V is concircular, in which case if M is complete then it is isometric to a unit sphere, or* (ii) $\mu = 1$, $k = -n - 1$. *In addition, if M is compact, then it is isometric to the unit sphere* S^{2n+1}.

Corollary. *Let* $T_1 M$ *be the unit tangent bundle over a Riemannian manifold M* (*dim.*$M > 2$) *of constant curvature c and g be the standard contact metric on* $T_1 M$. *Then a conformal vector field on* $(T_1 M, g)$ *is necessarily Killing.*

2. Review of contact manifolds and conformal vector fields

A $(2n+1)$-dimensional smooth manifold M is said to be a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. For a given contact 1-form η there exists a unique vector field ξ (the Reeb vector field) such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, one obtains a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 = -I + \eta \otimes \xi \tag{3}$$

g is called an associated metric of η and (φ, η, ξ, g) a contact metric structure. The tensor $h = \frac{1}{2}\pounds_\xi \varphi$ is known to be self-adjoint, anti-commutes with φ, and satisfies: $Tr.h = Tr.h\varphi = 0$. The contact structure on M is said to be normal if the almost complex structure on $M \times R$ defined by $J(X, f d/dt) = (\varphi X - f\xi, \eta(X) d/dt)$, where f is a real function on $M \times R$, is integrable. A normal contact metric manifold is called a Sasakian manifold. For a Sasakian manifold, we have

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{4}$$

For a (k, μ)-contact manifold defined in Section 1,

$$h^2 = (k-1)\varphi^2, \quad Q\xi = 2nk\xi \tag{5}$$

For a (k, μ)-contact manifold with $k < 1$, we also have the following formulas for Ricci tensor ([3])

$$\begin{aligned} Ric(X, Y) = {} & (2n - 2 - n\mu)g(X, Y) + (2n - 2 + \mu)g(hX, Y) \\ & + (2 - 2n + 2nk + n\mu)\eta(X)\eta(Y) \end{aligned} \tag{6}$$

and the scalar curvature

$$r = 2n(2n - 2 + k - n\mu) \tag{7}$$

For details we refer to Blair [1].

A vector field V on an m-dimensional Riemannian manifold (M, g) is said to be a conformal vector field if

$$£_V g = 2\rho g \tag{8}$$

for a smooth function ρ on M. Denoting the gradient vector field of ρ by $D\rho$ and the Laplacian $-div.D\rho$ by $\Delta\rho$ we have the following integrability conditions for the conformal vector field V (Yano [13]):

$$\begin{aligned} (£_V R)(X, Y, Z) = &-(\nabla\nabla\rho)(Y, Z)X + (\nabla\nabla\rho)(X, Z)Y \\ &- g(Y, Z)\nabla_X D\rho + g(X, Z)\nabla_Y D\rho \end{aligned} \tag{9}$$

$$(£_V Ric)(X, Y) = -(m - 2)(\nabla\nabla\rho)(X, Y) + (\Delta\rho)g(X, Y) \tag{10}$$

$$£_V r = 2(m - 1)\Delta\rho - 2r\rho \tag{11}$$

where $m = dim.M$ and X, Y, Z denote arbitrary vector fields on M.

3. Proofs of the theorems

First we prove

LEMMA 1. *If V is a conformal vector field on a contact metric manifold, then* (i) $(£_V\eta)(\xi) = \rho$ *and* (ii) $\eta(£_V\xi) = -\rho$.

Proof. Taking the Lie-derivative of $g(\xi, \xi) = 1$ along V and using equation (8) we get (ii). Lie-differentiating $\eta(\xi) = 1$ along V gives (i).

Proof of Theorem 1. Lie-differentiating $\eta(X) = g(X, \xi)$ along V and using the hypothesis, $£_V\eta = \sigma\eta$ provides $£_V\xi = (\sigma - 2\rho)\xi$. Operating this equation by η and using the part (ii) of Lemma 1 shows that $\sigma = \rho$. Thus we have

$$£_V\eta = \rho\eta, \quad £_V\xi = -\rho\xi \tag{12}$$

Operating the first equation in (12) by d, using the commutativity of d with $£_V$, and using the first equation in (3) gives

$$(£_V\, d\eta)(X, Y) = \frac{1}{2}[(X\rho)\eta(Y) - (Y\rho)\eta(X)] + \rho g(X, \varphi Y)$$

Now Lie-differentiating the first equation of (3) along V and using it in the above equation we obtain

$$(X\rho)\eta(Y) - (Y\rho)\eta(X) = 2[\rho g(X, \varphi Y) + g(X, (£_V\varphi)Y)] \tag{13}$$

Substituting ξ for Y and using the second equation in (12) we find that $d\rho = (\xi\rho)\eta$. Operating it by d, using Poincare lemma ($d^2 = 0$), and then taking wedge product with η yields $(\xi\rho)\eta \wedge (d\eta) = 0$. By definition of contact structure, $\eta \wedge (d\eta) \neq 0$ anywhere on M. Hence $\xi\rho = 0$. Consequently, $d\rho = 0$, i.e. ρ is constant. Thus equation (13) reduces to

$$\pounds_V \varphi = -\rho\varphi. \tag{14}$$

Next, taking Lie-derivative of the last equation in (3) along V, and using (12) and (14) we get $\rho\varphi^2 = 0$. Hence $\rho = 0$, i.e. V is Killing. It also follows from (12) that V leaves η and ξ invariant. This completes the proof.

In order to prove Theorem 2 we need the following lemmas.

LEMMA 2. *Let M be a (k,μ)-contact manifold admitting a conformal vector field V, and (e_i) a local orthonormal frame on M. Then* (i) $g((\pounds_V h)e_i, e_i) = 0$, (ii) $g((\pounds_V h)he_i, e_i) = 0$ *(i is summed over $1,\dots,2n+1$).*

Proof. As h is trace-free, we have $g(he_i, e_i) = 0$. Taking its Lie-derivative along V gives

$$g((\pounds_V h)e_i, e_i) + 2g(h\pounds_V e_i, e_i) = 0 \tag{15}$$

At this point, we let (e_i) be a φ-adapted frame $(e_a, \varphi e_a, e_{2n+1} = \xi)$ $(a = 1,\dots n)$ such that $he_a = \lambda e_a$, $h\varphi e_a = -\lambda\varphi e_a$ (where $\lambda = \sqrt{1-k}$). This setting makes sense in view of the first equation in (5). Then $g(h\pounds_V e_i, e_i) = g(h\pounds_V e_a, e_a) + g(h\pounds_V \varphi e_a, \varphi e_a) = \lambda[g(\pounds_V e_a, e_a) - g(\pounds_V \varphi e_a, \varphi e_a)] = 0$. Using this in (15) we obtain (i). For (ii) we first note that $g((\pounds_V h)X, Y) = g((\pounds_V h)Y, X)$, as V is conformal and h is self-adjoint. Now, $g((\pounds_V h)he_i, e_i) = g((\pounds_V h)e_i, he_i) = g(\pounds_V he_i, he_i) - g(\pounds_V e_i, h^2 e_i)$. Using the first equation in (5) we find that $g(\pounds_V he_i, he_i) = 2n(k-1)\rho$ and $g(\pounds_V e_i, h^2 e_i) = -2n(k-1)\rho$. Summing up, we get $g((\pounds_V h)he_i, e_i) = 0$ completing the proof.

LEMMA 3. *On a (k,μ)-contact manifold M,*

$$R(\varphi e_i, e_i)Y = -2(k + n\mu)\varphi Y \tag{16}$$

for an arbitrary vector field Y on M.

Proof. We first recall the formula from [3]:

$$\begin{aligned} R(X,Y)\varphi Z - \varphi R(X,Y)Z = {} & [(1-k)(\eta(X)g(\varphi Y, Z) - \eta(Y)g(\varphi X, Z)) \\ & + (1-\mu)(\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z))]\xi \\ & - g(Y + hY, Z)(\varphi X + \varphi hX) \\ & + g(X + hX, Z)(\varphi Y + \varphi hY) \\ & - g(\varphi Y + \varphi hY, Z)(X + hX) \\ & + g(\varphi X + \varphi hX, Z)(Y + hY) \\ & - \eta(Z)[(1-k)(\eta(X)\varphi Y - \eta(Y)\varphi X) \\ & + (1-\mu)(\eta(X)\varphi hY - \eta(Y)\varphi hX)] \end{aligned}$$

Substituting $X = e_i$ in the above equation, taking inner product with e_i and summing over i we get

$$\begin{aligned} (17) \qquad & Ric(Y, \varphi Z) - g(\varphi R((e_i, Y)Z, e_i) \\ & \quad = (2 - 2n - k)g(\varphi Y, Z) + (2 - 2n - \mu)g(\varphi hY, Z) \end{aligned}$$

Now, $g(\varphi R(e_i, Y)Z, e_i) = -g(R(e_i, Y)Z, \varphi e_i)$ which can be expressed in terms of the h-eigen basis $(e_a, \varphi e_a, e_{2n+1} = \xi$ such that $he_a = \lambda e_a$, $h\varphi e_a = -\lambda \varphi e_a$ where $\lambda = \sqrt{1-k})$ as $g(R(e_a, Y)Z, \varphi e_a) - g(R(\varphi e_a, Y)Z, e_a) = -g(R(e_a, Y)\varphi e_a, Z) - g(R(Y, \varphi e_a)e_a, Z) = g(R(\varphi e_a, e_a)Y, Z)$ (through Bianchi identity). We also note that $g(R(\varphi e_i, e_i)Y, Z) = 2g(R(\varphi e_a, e_a)Y, Z)$. Hence, we get $g(\varphi R(e_i, Y)Z, e_i) = -\frac{1}{2}g(R(\varphi e_i, e_i)Y, Z)$. Using this and (6) in (17) we obtain (16), completing the proof.

Proof of Theorem 2. If $k = 1$, then M is Sasakian, in which case Okumura's theorem applies and the corresponding conclusion follows. So, let $k < 1$. Taking the Lie-derivative of (2) along V and using (9) we get

$$\begin{aligned} (18) \qquad R(X, Y)\pounds_V \xi = {} & g(\nabla_\xi D\rho, Y)X - g(\nabla_\xi D\rho, X)Y \\ & + \eta(Y)\nabla_X D\rho - \eta(X)\nabla_Y D\rho \\ & + k[2\rho g(\xi, Y)X + g(\pounds_V \xi, Y)X - 2\rho g(\xi, X)Y - g(\pounds_V \xi, X)Y] \\ & + \mu[2\rho g(\xi, Y)hX + g(\pounds_V \xi, Y)hX + g(\xi, Y)(\pounds_V h)X \\ & - 2\rho g(\xi, X)hY - g(\pounds_V \xi, X)hY - g(\xi, X)(\pounds_V h)Y] \end{aligned}$$

Substituting ξ for X and using (2) in the above equation yields

$$\begin{aligned} (19) \qquad \nabla_Y D\rho = {} & g(\nabla_\xi D\rho, Y)\xi - g(\nabla_\xi D\rho, \xi)Y + \eta(Y)\nabla_\xi D\rho \\ & - \mu g(h\pounds_V \xi, Y)\xi + 2k\rho\eta(Y)\xi - 2k\rho Y \\ & + \mu\eta(Y)(\pounds_V h)\xi - 2\mu\rho hY - \mu(\pounds_V h)Y \end{aligned}$$

Substituting e_i for Y, taking inner product with e_i, summing over $i = 1, \ldots, 2n+1$, and using part (i) of Lemma 2 we have

$$(20) \qquad \Delta\rho = -\sum_{i=1}^{2n+1} g(\nabla_{e_i} D\rho, e_i) = 4nk\rho + (2n-1)(\nabla\nabla\rho)(\xi, \xi)$$

The use of equation (19), first equation in (5) and part (ii) of Lemma 2 enables us to obtain

$$(21) \qquad \sum_{i=1}^{2n+1} g(\nabla_{he_i} D\rho, e_i) = 4n(k-1)\mu\rho$$

Now, Lie-differentiating (6) along V and using (10) we have

$$(22)\qquad (2n-1)g(\nabla_X D\rho, Y) = (2-2n+k+n\mu)\rho g(X,Y) + (2-2n-\mu)[2\rho g(hX,Y) + g((£_V h)X,Y)] + (2n-2-2nk-n\mu)[(£_V\eta)(X)\eta(Y) + (£_V\eta)(Y)\eta(X)]$$

Substituting $X = he_i$, $Y = e_i$ in the above equation and summing over i, using part (ii) of Lemma 2 and using (19) we obtain $2\rho(k-1)(n-1)(\mu-1) = 0$. As $k < 1$ and V is non-Killing by hypothesis, it follows that either (a) $\mu = 1$ or (b) $n = 1$, i.e. $dim.M = 3$. In the following discussion we will pursue the case (a). Let us go back to equation (18), substitute $X = \varphi e_i$, $Y = e_i$, and sum over i in order to get

$$R(\varphi e_i, e_i)£_V\xi - g(\nabla_\xi D\rho, e_i)\varphi e_i + g(\nabla_\xi D\rho, \varphi e_i)e_i = k[g(£_V\xi, e_i)\varphi e_i - g(£_V\xi, \varphi e_i)e_i] + [g(£_V\xi, e_i)h\varphi e_i - g(£_V\xi, \varphi e_i)he_i]$$

where the first term has summation over i. Simplifying the right hand side reduces the above equation to

$$R(\varphi e_i, e_i)£_V\xi = 2[\varphi\nabla_\xi D\rho + k\varphi £_V\xi + h\varphi £_V\xi]$$

Using Lemma 3 in the above equation yields

$$(23)\qquad (n+2k)\varphi £_V\xi = -\varphi\nabla_\xi D\rho - h\varphi £_V\xi$$

We also have from equation (22) that

$$(24)\qquad (1-2n)(\nabla\nabla\rho)(\xi,\xi) = (2-2n-k+n+4nk)\rho$$

Now substituting $X = e_i$ in equation (18), taking inner product with e_i, summing over i, and then using equation (6) and also Lemma 2, we get

$$(n-2-2nk)£_V\xi + (2n-1)h£_V\xi + [2n-4-6nk+k]\rho\xi + (1-2n)\nabla_\xi D\rho = 0$$

Operating the above equation by φ gives

$$(2n-1)[\varphi\nabla_\xi D\rho + h\varphi £_V\xi] = (n-2-2nk)\varphi £_V\xi$$

Now comparing this with (23) provides

$$[(n-2-2nk) + (2n-1)(2k+n)]\varphi £_V\xi = 0$$

which shows that either (i) the constant within brackets vanishes, which simplifies to $k = -n-1$, or (ii) $£_V\xi$ is a function multiple of ξ, which, by Theorem 1, implies that V is Killing, contradicting our hypothesis. Thus we conclude that case (a) has $k = -n-1$. To prove the last part which assumes M to be compact, we use formula (6) and the first equation in (5) to show that the norm of the Ricci tensor is constant. We also have from (7) that the scalar curvature is constant. Hence, by the following result of Lichnerowicz [5] "If a compact

Riemannian manifold of dimension $m > 2$ admitting a non-Killing conformal vector field has scalar curvature and norm of Ricci tensor both constant, then it is isometric to a sphere" we conclude that M is isometric to a sphere and hence of constant curvature. But a contact metric manifold of constant curvature is a Sasakian manifold of curvature 1 in dimension > 3 (Olszak [9]). Therefore, M is isometric to a unit sphere.

Proof of the Corollary. We know [3] that if the base manifold M has constant curvature c, then T_1M is a (k, μ)-contact manifold with $k = c(2 - c)$, $\mu = -2c$. If T_1M has a non-Killing conformal vector field, then Theorem 1 implies that either (i) it is a Sasakian $(c = 1)$ manifold of dimension > 3 and isometric to the unit sphere, or (ii) $k = -n - 1$, $\mu = 1$. In case (i) T_1M is conformally flat and hence M has dimension 2, by a theorem of Blair and Koufogiorgos [2]. Thus case (i) is ruled out. Case (ii) is not compatible with the condition $k = c(2 - c)$, $\mu = -2c$, and hence ruled out. This completes the proof.

Remark. The conclusion (ii) of Theorem 2 seems accidental and pathological, and gets ruled out in special cases, for example when M is compact. There is another situation in which (ii) can be ruled out. Under the conclusion (ii), equation (7) reduces to $r = -6n$. Use of this in (11) shows $\Delta\rho = -3\rho$. Now

$$\begin{aligned}\Delta\rho^2 &= -\nabla^i\nabla_i\rho^2 = -2[(\nabla^i\rho)(\nabla_i\rho) + \rho(\nabla^i\nabla_i\rho] \\ &= -2[|D\rho|^2 - \rho\Delta\rho]\end{aligned}$$

Hence we obtain

$$\Delta\rho^2 = 2[|D\rho|^2 + 3\rho^2] \tag{25}$$

If we assume M to be complete, and $|D\rho|$ and ρ both L^2-integrable over M, then by Gaffney' theorem [4] we have $\int_M \Delta\rho^2\, dv = 0$ (dv denotes the volume element of M). Hence, integrating (25) over M we conclude that $\rho = 0$, contradicting our hypothesis.

In general, the contact metric in case (ii) can be D-homothetically deformed to the standard metric of a unit tangent bundle of a space of constant curvature $c = \dfrac{1 - 2\sqrt{n+2}}{1 + 2\sqrt{n+2}}$. This can be shown by using the fact that a (k, μ)-contact metric gets D-homothetically deformed to a $(\bar{k}, \bar{\mu})$-contact metric such that $\bar{k} = \dfrac{k + a^2 - 1}{a^2}$, $\bar{\mu} = \dfrac{\mu + 2a - 2}{a}$ (see [3]). Substituting $k = -n - 1$, $\mu = 1$ and requiring $(\bar{k}, \bar{\mu})$-contact metric to be the standard metric on the unit tangent bundle of a space of constant curvature c, so that $\bar{k} = c(2 - c)$, $\bar{\mu} = -2c$, we obtain the aforementioned value of c. We note that in this case c lies in the open interval $(-1, 0)$.

So far we have confined our attention to contact (k, μ)-manifolds M of $dim.M > 3$. The next section addresses the 3-dimensional case.

4. The three dimensional case

In this section we resolve the 3-dimensional case using Lie-algebra theoretic approach. We take a vector field E_1 corresponding to the positive eigenvalue of $h = \frac{1}{2}\pounds_V \varphi$. We take $E_2 = \varphi E_1$ and $E_3 = \xi$. It follows immediately that

$$(26) \quad [E_1, E_2] = 2E_3 \quad [E_3, E_1] = \left(1 + \lambda - \frac{1}{2}\mu\right)E_2 \quad [E_2, E_3] = \left(1 - \lambda - \frac{1}{2}\mu\right)E_1,$$

where $\lambda = \sqrt{1-k}$. Throughout this section we will assume that M is not a Sasakian space form. Hence we assume that $k < 1$ and therefore $\lambda > 0$.

As $\{E_1, E_2, E_3\}$ form an orthonormal basis of the tangent space, applying the Koszul formula gives:

$$\begin{array}{lll} \nabla_{E_1} E_1 = 0 & \nabla_{E_1} E_2 = (1+\lambda)E_3 & \nabla_{E_1} E_3 = -(1+\lambda)E_2 \\ \nabla_{E_2} E_1 = (-1+\lambda)E_3 & \nabla_{E_2} E_2 = 0 & \nabla_{E_2} E_3 = (1-\lambda)E_1 \\ \nabla_{E_3} E_1 = -\frac{1}{2}\mu E_2 & \nabla_{E_3} E_2 = \frac{1}{2}\mu E_1 & \nabla_{E_3} E_3 = 0 \end{array}$$

We now assume from now on that M admits a conformal vector field V. Using the previously defined frame we can write $V = a_1 E_1 + a_2 E_2 + a_3 E_3$, where a_1, a_2, a_3 are well defined functions on M. Using the conformal equation (8) we find that the functions a_1, a_2, a_3 satisfy the following system of differential equations:

$$\begin{aligned} &E_1(a_1) = E_2(a_2) = E_3(a_3) = \rho \\ &E_2(a_1) + E_1(a_2) = 2a_3\lambda \\ &E_3(a_1) + E_1(a_3) + a_2\left(1 + \lambda + \frac{1}{2}\mu\right) = 0 \\ &E_2(a_3) + E_3(a_2) + a_1\left(-1 + \lambda - \frac{1}{2}\mu\right) = 0. \end{aligned}$$

Hence introducing local functions b_1, b_2, b_3 by

$$\begin{aligned} b_1 &= E_2(a_1) - a_3\lambda \\ b_2 &= E_3(a_1) + \frac{1}{2}a_2\left(1 + \lambda + \frac{1}{2}\mu\right) \\ b_3 &= E_3(a_2) + \frac{1}{2}a_1\left(-1 + \lambda - \frac{1}{2}\mu\right) \end{aligned}$$

we find that the functions a_1, a_2, a_3 satisfy the following system of differential equations:

$$E_1(a_1) = \rho \qquad E_1(a_2) = -b_1 + a_3\lambda \qquad E_1(a_3) = -b_2 - \frac{1}{2}a_2A$$
$$E_2(a_1) = b_1 + a_3\lambda \qquad E_2(a_2) = \rho \qquad E_2(a_3) = -b_3 - \frac{1}{2}a_1B$$
$$E_3(a_1) = b_2 - \frac{1}{2}a_2A \qquad E_3(a_2) = b_3 - \frac{1}{2}a_1B \qquad E_3(a_3) = \rho$$

where $A = 1 + \lambda + \frac{\mu}{2}$ and $B = -1 + \lambda - \frac{\mu}{2}$. We now introduce another set of auxiliary functions $\rho_1,\ \rho_2,\ \rho_3$ through the conditions

$$E_1(\rho) = r_1, \quad E_2(\rho) = r_2, \quad E_3(\rho) = r_3.$$

Then we have the following lemma:

LEMMA 3. *The integrability conditions for the differential equations for the functions $a_1,\ a_2,\ a_3$ imply that the functions $b_1,\ b_2,\ b_3$ satisfy the following system of differential equations*:

$$E_1(b_1) = \frac{1}{4}(4(r_2 + b_2(2+\lambda)) + a_2(\lambda - 2)(2 + 2\lambda + \mu))$$
$$E_2(b_1) = \frac{1}{4}(-4(r_1 + b_3(\lambda - 2)) - a_1(2+\lambda)(-2 + 2\lambda - \mu)$$
$$E_3(b_1) = -\rho$$
$$E_1(b_2) = \frac{1}{4}(4r_3 + b_1(-6 - 6\lambda + \mu) + a_3\lambda(-2 - 2\lambda + 3\mu))$$
$$E_2(b_2) = \frac{1}{4}(2 + 2\lambda - \mu)\rho$$
$$E_3(b_2) = \frac{1}{16}(-16r_1 + 4b_3(2 + 2\lambda - 3\mu) + a_1(-12 + 12\lambda^2 - 4\mu - 8\lambda\mu + \mu^2))$$
$$E_1(b_3) = \frac{1}{4}(-2 + 2\lambda + \mu)\rho$$
$$E_2(b_3) = \frac{1}{4}(4r_3 + a_3\lambda(2 - 2\lambda - 3\mu) + b_1(-6 + 6\lambda + \mu))$$
$$E_3(b_3) = \frac{1}{16}(-16r_2 + 4b_2(-2 + 2\lambda + 3\mu) + a_2(-12 + 12\lambda^2 - 4\mu + 8\lambda\mu + \mu^2))$$

Proof. Using the fact that both λ and μ are constant, we find that

$$E_2(E_1(a_3)) = -E_2(b_2) - \frac{1}{4}(2 + 2\lambda + \mu)\rho$$
$$E_1(E_2(a_3)) = -E_1(b_3) + \frac{1}{4}(2 - 2\lambda + \mu)\rho.$$

As

$$E_1(E_2(a_3)) - E_2(E_1(a_3)) = [E_1, E_2]a_3 = 2E_3(a_3) = 2\rho,$$

we find that

$$-E_1(b_3) + E_2(b_2) + \frac{1}{2}(\mu - 2)\rho = 0.$$

Similarly, we find that

$$-4(r_1 + b_3(\lambda - 2)) - a_1(2 + \lambda)(-2 + 2\lambda - \mu) - 4E_2(b_1) = 0$$
$$(-6 + 2\lambda + \mu)\rho - 4E_3(b_1) - 4E_1(b_3) = 0$$
$$(4r_3 + 2a_3\lambda - 2a_3\lambda^2 - 3a_3\lambda\mu + b_1(-6 + 6\lambda + \mu) - 4E_2(b_3) = 0$$
$$a_2(-12 + 12\lambda^2 - 4\mu + 8\lambda\mu + \mu^2) + 4(-4r_2 + b_2(-2 + 2\lambda + 3\mu)) - 16E_3(b_3) = 0$$
$$a_1(-12 + 12\lambda^2 - 4\mu - 8\lambda\mu + \mu^2) + 4(-4r_1 + b_3(2 + 2\lambda - 3\mu)) - 16E_3(b_2) = 0$$
$$(6 + 2\lambda - \mu)\rho + 4E_3(b_1) - 4E_2(b_2)$$
$$4(r_2 + b_2(\lambda + 2)) + a_2(2 - \lambda)(-2 - 2\lambda - \mu) - 4E_1(b_1) = 0$$
$$(4r_3 - 2a_3\lambda - 2a_3\lambda^2 + 3a_3\lambda\mu) + b_1(-6 - 6\lambda + \mu) - 4E_1(b_2) = 0$$

Looking at the above equations as a system of linear equations in the derivatives of the functions b_i and solving this system of equations explicitly completes the proof.

Note that the above equations can also be obtained using

$$(£_V\nabla)(X, Y) = d\rho(X)(Y) + d\rho(Y)X - \langle X, Y\rangle D\rho,$$

which is a straightforward consequence of the fact that V is a conformal vector field, see [13].

Similarly expressing in the same way the integrability conditions for the functions b_1, b_2, b_3, we find that r_1, r_2, r_3 satisfy the following system of differential equations:

$$E_1(r_1) = (1 - \lambda^2 + \mu - 2\lambda\mu)\rho$$
$$E_2(r_1) = r_3(\lambda - 1) + \lambda\mu(-2b_1 + a_3(\mu - 2))$$
$$E_3(r_1) = -\frac{1}{2}\mu r_2 - (\lambda - 1)((2 + 2\lambda + \mu)b_2$$
$$\qquad - \frac{1}{4}(-12 + 4\lambda^2 + 4\lambda(\mu - 2) - 4\mu + \mu^2)a_2)$$
$$E_1(r_2) = r_3(\lambda + 1) + \lambda\mu(-2b_1 + a_3(\mu - 2))$$
$$E_2(r_2) = (1 - \lambda^2 + \mu + 2\lambda\mu)\rho$$

$$E_3(r_2) = \frac{1}{2}\mu r_1 + (\lambda+1)((2-2\lambda+\mu)b_3$$
$$-\frac{1}{4}(-12+4\lambda^2-4\lambda(\mu-2)-4\mu+\mu^2)a_1)$$
$$E_1(r_3) = -r_2(1+\lambda) + (\lambda-1)(-(2+2\lambda+\mu)b_2$$
$$-\frac{1}{4}(-12+4\lambda^2+4\lambda(\mu-2)-4\mu+\mu^2)a_2)$$
$$E_2(r_3) = r_1(1-\lambda) + (1+\lambda)((\mu+2-2\lambda)b_3$$
$$-\frac{1}{4}(-12+4\lambda^2-4\lambda(\mu-2)-4\mu+\mu^2)a_1)$$
$$E_3(r_3) = (-3+3\lambda^2-\mu)\rho.$$

At this point it is easy to verify that the integrability conditions for the system of differential equations determining ρ are trivially satisfied. However, computing

$$E_2(E_1(r_3)) - E_1(E_2(r_3)) = -2E_3(r_3),$$

we find that

$$(2+\lambda^2(\mu-2)+\mu)\rho = 0.$$

As ρ was supposed to be non vanishing this implies that

$$\mu = \frac{2(\lambda^2-1)}{1+\lambda^2}.$$

Using

$$E_3(E_1(r_2)) - E_1(E_3(r_2)) = \left(1+\lambda-\frac{1}{2}\mu\right)E_2(r_2),$$

we find that

$$\lambda(\lambda^2-1)^2(3+\lambda^2)\rho = 0.$$

Hence, as we assumed that λ was positive, it follows that $\lambda = 1$. A straightforward computation shows that in the case that $\lambda = 1$, hence $k = 0$ and $\mu = 0$, all integrability conditions are satisfied. Thus we proved

THEOREM 3. *If a 3-dimensional non-Sasakian (k,μ)-contact manifold admits a non-Killing conformal vector field, then it is locally flat.*

We now construct all possible conformal vector fields in this case.

First, we notice, by a straightforward computation, that all components of the curvature tensor vanish identically, i.e. M is flat. Moreover, we see that the integrability conditions for the following function θ are satisfied:

$$E_1(\theta) = -2, \quad E_2(\theta) = E_3(\theta) = 0.$$

Hence for any initial condition θ a local solution exists. Let θ be such a solution. Then it follows that

$$[\cos\theta E_2 + \sin\theta E_3, -\sin\theta E_2 + \cos\theta E_3] = [E_2, E_3] = 0$$
$$[E_1, \cos\theta E_2 + \sin\theta E_3] = E_1(\theta)(-\sin\theta E_2 + \cos\theta E_3]) + \cos\theta[E_1, E_2] + \sin\theta[E_1, E_3] = 0$$
$$[E_1, -\sin\theta E_2 + \cos\theta E_3] = 0.$$

This means that there exist local coordinates u, v, w on M such that $\frac{\partial}{\partial u} = E_1$, $\frac{\partial}{\partial v} = \cos\theta E_2 + \sin\theta E_3$ and $\frac{\partial}{\partial w} = -\sin\theta E_2 + \cos\theta E_3$. From the above differential equations for θ it follows that we can take $\theta = -2u$. Note that as these coordinates are mutually orthogonal coordinates, we can identify these coordinates with the standard coordinates of $\mathbf{R}^3$. We also have

$$E_2 = \cos 2u\frac{\partial}{\partial v} + \sin 2u\frac{\partial}{\partial w}, \quad E_3 = -\sin 2u\frac{\partial}{\partial v} + \cos 2u\frac{\partial}{\partial w}$$

which allows us to express our structure in terms of the usual coordinates.

Specialising the system of differential equations for the functions r_1, r_2, r_3 we find that the only non vanishing derivatives are given by

$$\frac{\partial r_2}{\partial u} = 2r_3, \quad \frac{\partial r_3}{\partial u} = -2r_2$$

So we see that there exists constants c_1, c_2, c_3 such that $r_1 = c_1$, $r_2 = c_2\cos 2u + c_3\sin 2u$ and $r_3 = -c_2\sin 2u + c_3\cos 2u$. This means that the differential equations for ρ, b_1, b_2, b_3 now reduce respectively to

$$\frac{\partial\rho}{\partial u} = r_1 = c_1$$
$$\frac{\partial\rho}{\partial v} = \cos 2ur_2 - \sin 2ur_3 = c_2$$
$$\frac{\partial\rho}{\partial w} = \sin 2ur_2 + \cos 2ur_3 = c_3$$
$$\frac{\partial b_1}{\partial u} = -a_2 + 3b_2 + c_2\cos 2u + c_3\sin 2u$$
$$\frac{\partial b_1}{\partial v} = (b_3 - c_1)\cos 2u + \rho\sin 2u$$
$$\frac{\partial b_1}{\partial w} = -\rho\cos 2u - (c_1 - b_3)\sin 2u$$

$$\frac{\partial b_2}{\partial u} = -a_3 - 3b_1 + c_3 \cos 2u - c_2 \sin 2u$$
$$\frac{\partial b_2}{\partial v} = \rho \cos 2u - (b_3 - c_1) \sin 2u$$
$$\frac{\partial b_2}{\partial w} = (b_3 - c_1) \cos 2u + \rho \sin 2u$$
$$\frac{\partial b_3}{\partial u} = 0$$
$$\frac{\partial b_3}{\partial v} = c_3$$
$$\frac{\partial b_3}{\partial w} = -c_2$$

Similarly the differential equations for the functions a_1, a_2, a_3 reduce to

$$\frac{\partial a_1}{\partial u} = \rho$$
$$\frac{\partial a_1}{\partial v} = (a_3 + b_1) \cos 2u + (a_2 - b_2) \sin 2u$$
$$\frac{\partial a_1}{\partial w} = -(a_2 - b_2) \cos 2u + (a_3 + b_1) \sin 2u$$
$$\frac{\partial a_2}{\partial u} = a_3 - b_1$$
$$\frac{\partial a_2}{\partial v} = \rho \cos 2u - b_3 \sin 2u$$
$$\frac{\partial a_2}{\partial w} = b_3 \cos 2u + \rho \sin 2u$$
$$\frac{\partial a_3}{\partial u} = -a_2 - b_2$$
$$\frac{\partial a_3}{\partial v} = -b_3 \cos 2u - \rho \sin 2u$$
$$\frac{\partial a_3}{\partial w} = \rho \cos 2u - b_3 \sin 2u.$$

In order to solve these equations, we first note that $\rho = c_1 u + c_2 v + c_3 w + c_4$ and $b_3 = c_3 v - c_2 w + c_5$. Next looking at the differential equations for $a_3 + b_1$ and $a_2 - b_2$, we find that there exist constants c_6 and c_7 such that

$$b_1 = -a_3 + (c_2 u - c_1 v + c_6) \cos 2u + (-c_1 w + c_3 u + c_7) \sin 2u$$
$$b_2 = a_2 + (c_1 v - c_2 u - c_6) \sin 2u + (-c_1 w + c_3 u + c_7) \cos 2u$$

Finally solving the final differential equations we get that there exist constants c_8, c_9, c_{10} such that

$$a_1 = (c_8 + c_4 u + c_6 v + c_2 uv + c_7 w + c_3 uw + \frac{1}{2} c_1 (u^2 - v^2 - w^2)$$

$$a_2 = C(u, v, w) \cos(2u) + D(u, v, w) \sin(2u)$$

$$a_3 = -C(u, v, w) \sin(2u) + D(u, v, w) \cos(2u)$$

where $C(u, v, w) = c_9 - c_6 u + c_4 v + c_1 uv + c_5 w + c_3 vw - \frac{1}{2} c_2 (u^2 - v^2 + w^2)$ and $D(u, v, w) = c_{10} - c_7 u - c_5 v + c_4 w + c_1 uw + c_2 vw - \frac{1}{2} c_3 (u^2 + v^2 - w^2)$.

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