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ON THE ZEROS OF SOLUTIONS OF A CLASS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS*†§

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Abstract

In this paper, we investigate the exponent of convergence of the zero-sequence of solutions of the second order linear differential equation

$$f'' + \left(\sum_{j=1}^{l} Q_j(z)e^{P_j(z)}\right)f = 0,$$

where $P_j(z)$ $(j = 1, 2, \dots, l \geq 3)$ are polynomials of degree $n \geq 1$, $Q_j(z)$ are entire functions of order less than n, and obtain some results which improve and generalize the previous results in [8, 9, 13].

1. Introduction and results

We shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see [7, 10]). We will use the notation $\rho(f)$ to denote the order of growth of meromorphic function $f(z)$, $\lambda(f)$ to denote the exponent of convergence of the zero-sequence of $f(z)$.

For second order linear differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an entire function, many authors have investigated the growth and the convergence of the zero-sequence of solutions of (1.1), and have achieved many results (see [1, 2, 3, 11]). When $A(z) = e^{P_1(z)} + e^{P_2(z)} + Q_0(z)$, for the following second order linear differential equation

$$f'' + (e^{P_1(z)} + e^{P_2(z)} + Q_0(z))f = 0, \tag{1.2}$$

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where $P_1(z)$, $P_2(z)$ are non-constant polynomials

$$P_1(z) = \zeta_1 z^n + \cdots, \quad P_2(z) = \zeta_2 z^m + \cdots, \quad \zeta_1\zeta_2 \neq 0, \quad (n, m \in N).$$

and $Q_0(z)$ is an entire function of order less than $\max\{n, m\}$. If $e^{P_1(z)}$ and $e^{P_2(z)}$ are linearly independent, K. Ishizaki and K. Tohge have studied the exponent of convergence of the zero-sequence of solutions of (1.2) and obtained the following results.

Theorem A ([9]). *Suppose that $n = m$, and that $\zeta_1 \neq \zeta_2$ in (1.2). If $\dfrac{\zeta_1}{\zeta_2}$ is non-real, then for any solution $f \not\equiv 0$ of (1.2), we have $\lambda(f) = \infty$.*

Theorem B ([8]). *Suppose that $n = m$, and that $\dfrac{\zeta_1}{\zeta_2} = \rho > 0$ in (1.2). If $0 < \rho < \dfrac{1}{2}$ or $Q_0(z) \equiv 0$, $\dfrac{3}{4} < \rho < 1$, then for any solution $f \not\equiv 0$ of (1.2), we have $\lambda(f) \geq n$.*

When $A(z) = Q_1(z)e^{P_1(z)} + Q_2(z)e^{P_2(z)} + Q_3(z)e^{P_3(z)}$, for the following second order linear differential equation

$$f'' + (Q_1(z)e^{P_1(z)} + Q_2(z)e^{P_2(z)} + Q_3(z)e^{P_3(z)})f = 0, \tag{1.3}$$

in 2007, J. Tu and Z. X. Chen studied the exponent of convergence of the zero-sequence of solutions of (1.3) and obtain the following results.

Theorem C ([13]). *Let $Q_1(z)$, $Q_2(z)$, $Q_3(z)$ be entire functions of order less than n, and $P_1(z)$, $P_2(z)$, $P_3(z)$ be polynomials of degree $n \geq 1$,*

$$P_1(z) = \zeta_1 z^n + \cdots, \quad P_2(z) = \zeta_2 z^n + \cdots, \quad P_3(z) = \zeta_3 z^n + \cdots,$$

where ζ_1, ζ_2, ζ_3 are complex numbers.

(i) *If $\dfrac{\zeta_1}{\zeta_2}$ is non-real, $0 < \lambda = \dfrac{\zeta_3}{\zeta_2} < \dfrac{1}{2}$, then for any solution $f \not\equiv 0$ of (1.3), we have $\lambda(f) = \infty$.*

(ii) *If $0 < \dfrac{\zeta_2}{\zeta_1} < \dfrac{1}{4}$, $0 < \lambda = \dfrac{\zeta_3}{\zeta_2} < 1$, then for any solution $f \not\equiv 0$ of (1.3), we have $\lambda(f) \geq n$.*

Then a natural question is: what is the case if $A(z) = \sum_{j=1}^{l} Q_j e^{P_j(z)}$ $(l \geq 3)$? Can we get the same results as Theorem C?

In this paper, we investigate the exponent of convergence of the zero-sequence of solutions of the following equation

$$f'' + \left(\sum_{j=1}^{l} Q_j(z)e^{P_j(z)}\right)f = 0, \tag{1.4}$$

and obtain the following results which improve and generalize the results in [8, 9, 13].

Theorem 1. *Let* $Q_1(z), Q_2(z), \dots, Q_l(z)$ $(l \geq 3)$ *be entire functions of order less than n, and* $P_1(z), P_2(z), \dots, P_l(z)$ $(l \geq 3)$ *be polynomials of degree* $n \geq 1$,

$$P_1(z) = \zeta_1 z^n + \cdots, \quad P_2(z) = \zeta_2 z^n + \cdots, \dots, \quad P_l(z) = \zeta_l z^n + \cdots,$$

where $\zeta_1, \zeta_2, \dots, \zeta_l$ *are complex numbers.*

(i) *If* $\frac{\zeta_1}{\zeta_2}$ *is non-real,* $0 < \lambda_j = \frac{\zeta_j}{\zeta_2} < \frac{1}{2}$ $(j = 3, \dots, l)$, *then any solution* $f \not\equiv 0$ *of* (1.4) *satisfies* $\lambda(f) = \infty$.

(ii) *If* $0 < \rho = \frac{\zeta_2}{\zeta_1} < \frac{1}{4}$, $\lambda_j = \frac{\zeta_j}{\zeta_2} > 0$ *and* $\sum_{j=3}^{l} \lambda_j < 1$, *then any solution* $f \not\equiv 0$ *of* (1.4) *satisfies* $\lambda(f) \geq n$.

2. Notations and lemmas

To prove the theorem, we need some notations and a series of lemmas. Let $P_j(z)$ $(j = 1, \dots, l)$ be polynomials of degree $n \geq 1$, where $P_j(z) = (\alpha_j + i\beta_j)z^n + \cdots$, $\alpha_j, \beta_j \in \mathbf{R}$.

Define

$$\delta(P_j, \theta) = \delta_j(\theta) = \alpha_j \cos n\theta - \beta_j \sin n\theta, \quad \theta \in [0, 2\pi) \ (j = 1, \dots, l),$$
$$S_j^+ = \{\theta \,|\, \delta_j(\theta) > 0\}, \quad S_j^- = \{\theta \,|\, \delta_j(\theta) < 0\} \quad (j = 1, \dots, l).$$

Let $f(z)$ be a meromorphic function in the complex plane, throughout the paper, $S(r, f)$ will be used to denoted any quantity that satisfies $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, outside possibly an exceptional set of r values of finite linear measure. We will use M to denote a positive constant throughout this paper, not always the same at each occurrence. We call a meromorphic function $a(z)$ a small function of $f(z)$ if $T(r, a(z)) = S(r, f)$. A differential polynomial $P(f)$ in f is a polynomial in f and its derivatives with small functions of f as the coefficients (see [7]).

Lemma 1 [5]. *Suppose that* $f(z)$ *is meromorphic and transcendental in the plane and that*

$$f^n(z)P(f) = Q(f), \tag{2.1}$$

where $P(f)$, $Q(f)$ *are differential polynomials in* f *with small functions of* f *as the coefficients and the degree of* $Q(f)$ *is at most n. Then*

$$m(r, P(f)) = S(r, f). \tag{2.2}$$

Lemma 2 [6]. *Let* $f(z)$ *be a transcendental meromorphic function with* $\rho(f) = \rho < \infty$, $\Gamma = \{(k_1, j_1), \dots, (k_m, j_m)\}$ *be a finite set of distinct pairs of integers which satisfy* $k_i > j_i \geq 0$ *for* $i = 1, \dots, m$. *And let* $\varepsilon > 0$ *be a given constant, then there exists a set* $E \subset [0, 2\pi)$ *which has linear measure zero, such that if* $\varphi \in [0, 2\pi) \backslash E$, *there is a constant* $R_1 = R_1(\varphi) > 1$, *such that for all* z *satisfying* $\arg z = \varphi$ *and* $|z| = r > R_1$ *and for all* $(k, j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.3}$$

LEMMA 3 [12]. *Suppose that* $P(z) = (\alpha + \beta i)z^n + \cdots$ (α, β *are real numbers*, $|\alpha| + |\beta| \neq 0$) *is a polynomial with degree* $n \ge 1$, *that* $A(z) (\not\equiv 0)$ *is an entire function with* $\rho(A) < n$. *Set* $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $H_1 \subset [0, 2\pi)$ *that has the linear measure zero, such that for any* $\theta \in [0, 2\pi) \backslash (H_1 \cup H_2)$, *there is a constant* $R_2 > 0$ *such that for* $|z| = r > R_2$, *we have*

(i) *If* $\delta(P, \theta) > 0$, *then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1+\varepsilon)\delta(P,\theta)r^n\}; \tag{2.4}$$

(ii) *If* $\delta(P, \theta) < 0$, *then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1-\varepsilon)\delta(P,\theta)r^n\}, \tag{2.5}$$

where $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$ *is a finite set.*

Remark. *Lemma* 3 *also holds when* $A(z)$ *is a meromorphic function with* $\rho(A) < n$.

LEMMA 4 [4]. *Let* $f(z)$ *be an entire function of order* $\rho(f) = \alpha < +\infty$. *Then for any given* $\varepsilon > 0$, *there is a set* $E \subset [1, \infty)$ *that has finite linear measure such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E$, *we have*

$$\exp\{-r^{\alpha+\varepsilon}\} \le |f(z)| \le \exp\{r^{\alpha+\varepsilon}\}. \tag{2.6}$$

LEMMA 5. *Let* $P_j(z)$ $(j = 1, \ldots, l)$ *be polynomials of degree* $n \ge 1$,

$$P_1(z) = \zeta z^n + B_1(z), \quad P_2(z) = \rho_2 \zeta z^n + B_2(z), \quad \ldots, \quad P_l(z) = \rho_l \zeta z^n + B_l(z),$$

where $\zeta = \alpha + \beta i$, $\alpha, \beta \in \mathbf{R}$, $|\alpha| + |\beta| \neq 0$, $0 < \rho_j < 1$, $j = 2, \ldots, l$, $B_1(z), \ldots, B_l(z)$ *are polynomials of degree at most* $n - 1$. *Let* $Q_1(z) \not\equiv 0$, $Q_2(z), \ldots, Q_l(z)$ *be entire functions of order less than* n, *then for any given* $\varepsilon > 0$, *there exist a set* E *with finite linear measure and a constant* $\xi (n - 1 < \xi < n)$ *such that*

$$\begin{aligned} &m(r, Q_1 e^{P_1} + Q_2 e^{P_2} + \cdots + Q_l e^{P_l}) \\ &\quad \ge (1-\varepsilon) m(r, e^{P_1}) + O(r^{\xi}), \quad r \to \infty, \quad (r \notin E). \end{aligned} \tag{2.7}$$

Proof. By definition, for sufficiently large r, we have

$$\begin{aligned} m(r, e^{P_1}) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+|e^{P_1(re^{i\theta})}|\, d\theta = \frac{1}{2\pi}\int_{S_1^+} \log^+|e^{P_1(re^{i\theta})}|\, d\theta \\ &= \frac{|\zeta| r^n}{\pi} + O(r^{n-1}). \end{aligned} \tag{2.8}$$

If $\theta \in S_1^-$, then $\delta(P_j, \theta) < 0$ $(j = 2, \dots, l)$, by Lemma 3 and Lemma 4, for any given $\varepsilon > 0$ and for sufficiently large r, we have

$$(2.9)\quad |Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + \cdots + Q_l e^{P_l(re^{i\theta})}| \le \sum_{j=1}^{l} \exp\{(1-2\varepsilon)\delta(P_j,\theta) r^n\} \le 1.$$

If $\theta \in S_1^+$, since $0 < \rho_j < 1$ $(j = 2, \dots, l)$, by Lemma 3 and Lemma 4, there exist a set E with finite linear measure, for any given $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}(2.10)\qquad & |Q_1 + Q_2 e^{P_2(re^{i\theta}) - P_1(re^{i\theta})} + \cdots + Q_l e^{P_l(re^{i\theta}) - P_1(re^{i\theta})}| \\ & \ge |Q_1| - |Q_2 e^{P_2(re^{i\theta}) - P_1(re^{i\theta})}| - \cdots - |Q_l e^{P_l(re^{i\theta}) - P_1(re^{i\theta})}| \\ & \ge \frac{1}{2} \exp\{-r^{\sigma(Q_1)+\varepsilon}\} \ge \exp\{-r^{\xi}\}, \quad (r \notin E),\end{aligned}$$

where $\rho(Q_1) < \xi < n$. By (2.8)–(2.10), we have

$$\begin{aligned}(2.11)\qquad & m(r, Q_1 e^{P_1} + Q_2 e^{P_2} + \cdots + Q_l e^{P_l}) \\ & = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + \cdots + Q_k e^{P_k(re^{i\theta})}|\, d\theta \\ & = \frac{1}{2\pi} \int_{S_1^+} \log^+ (|e^{P_1(re^{i\theta})}|\, |Q_1 + Q_2 e^{P_2(re^{i\theta}) - P_1(re^{i\theta})} \\ & \quad + \cdots + Q_l e^{P_l(re^{i\theta}) - P_1(re^{i\theta})}|)\, d\theta \\ & = \frac{(1-\varepsilon)|\zeta| r^n}{\pi} - O(r^{\xi}), \quad (r \notin E).\end{aligned}$$

By (2.8) and (2.11), we obtain (2.7).

3. Proof of Theorem 1 (i)

Since $\zeta_j = \lambda_j \zeta_2$, $\lambda_j > 0$, $j = 3, \dots, l$, we have $S_2^+ = S_3^+ = \cdots = S_l^+$, $S_2^- = S_3^- \cdots = S_l^-$. We see that S_j^+ and S_j^- have n components S_{jq}^+ and S_{iq}^- respectively $(j = 1, \dots, l;\ q = 1, 2, \dots, n)$. Hence we write

$$S_j^+ = \bigcup_{q=1}^{n} S_{jq}^+, \quad S_j^- = \bigcup_{q=1}^{n} S_{jq}^- \quad (j = 1, 2, \dots, l).$$

Let $f \not\equiv 0$ be a solution of (1.4). Suppose that $\lambda(f) < \infty$. Write $f = \pi e^h$, where π is the canonical product from zeros of f, and h is an entire function. From our hypothesis, we have $\sigma(\pi) = \lambda(\pi) < \infty$. From (1.4), we get

$$(3.1)\qquad (h')^2 = -h'' - 2\frac{\pi'}{\pi} h' - \frac{\pi''}{\pi} - Q_1 e^{P_1} - Q_2 e^{P_2} - \cdots - Q_l e^{P_l}.$$

Eliminating e^{P_1} from (3.1) and set $\frac{Q_1'}{Q_1} + P_1' = R$, we have

$$(3.2)\quad 2U_1h' = -h''' + \left(R - 2\frac{\pi'}{\pi}\right)h'' + 2\left(R\frac{\pi'}{\pi} - \left(\frac{\pi'}{\pi}\right)'\right)h' + R\frac{\pi''}{\pi} - \left(\frac{\pi''}{\pi}\right)' + \sum_{j=2}^{l}(RQ_j - Q_j' - Q_jP_j')e^{P_j},$$

$$(3.3)\quad U_1 = h'' - \frac{1}{2}Rh'.$$

Eliminating e^{P_2} from (3.1) and set $\frac{Q_2'}{Q_2} + P_2' = T$, we have

$$(3.4)\quad 2U_2h' = -h''' + \left(T - 2\frac{\pi'}{\pi}\right)h'' + 2\left(T\frac{\pi'}{\pi} - \left(\frac{\pi'}{\pi}\right)'\right)h' + T\frac{\pi''}{\pi} - \left(\frac{\pi''}{\pi}\right)' + (TQ_1 - Q_1' - Q_1P_1')e^{P_1} + \sum_{j=3}^{l}(RQ_j - Q_j' - Q_jP_j')e^{P_j},$$

where

$$(3.5)\quad U_2 = h'' - \frac{1}{2}Th'.$$

We next proceed to prove that $\rho(U_1) \le n$ and $\rho(U_2) \le n$. Since $\max\{\rho(Q_j), j = 1, \dots, l\} < n$, we choose constants ξ_1, ξ_2, ξ_3 satisfying $\max\{\rho(Q_j), j = 1, \dots, l\} < \xi_1 < \xi_2 < \xi_3 < n$, then we have

$$|Q_j(re^{i\theta})| \le \exp\{r^{\xi_1}\}, \quad T(r, Q_j) = m(r, Q_j) \le r^{\xi_1}, \quad (j = 1, \dots, l)$$

for sufficiently large r and for any $\theta \in [0, 2\pi)$. We apply Lemma 1 to (3.1), for any given $\varepsilon > 0$, we have

$$T(r, h') = m(r, h') \le m\left(r, \frac{\pi''}{\pi}\right) + m\left(r, \frac{\pi'}{\pi}\right) + m(r, Q_1e^{P_1(z)} + Q_2e^{P_2(z)} + \cdots + Q_le^{P_l(z)}) + S(r, h') \le O(r^{n+\varepsilon}) + S(r, h'),$$

which implies $\rho(h') \le n$. It follows from (3.3) and (3.5) that $\rho(U_1) \le n$ and $\rho(U_2) \le n$ respectively.

We next show that there exists a set $E_0 \subset [0, 2\pi)$ with $m(E_0) = 0$ such that if $\theta \in S_2^- \backslash E_0$, then

$$(3.6)\quad |U_1(re^{i\theta})| \le O(e^{r^{\xi_2}}), \quad \text{as } r \to \infty, \theta \notin E_0,$$

where E_0 denote a set of linear measure zero, not always the same at each occurrence. If $|h'(re^{i\theta})| < 1$, by Lemma 2 and (3.3), we have

$$|U_1(re^{i\theta})| \le \left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| + \frac{1}{2}|R(re^{i\theta})| \le O(r^M) \quad as\ r \to \infty,\ \theta \notin E_0. \tag{3.7}$$

If $|h'(re^{i\theta})| \ge 1$, then from (3.2), we get

$$\begin{aligned} |2U_1(re^{i\theta})| \le {} & \left|\frac{h'''(re^{i\theta})}{h'(re^{i\theta})}\right| + \left(|R(re^{i\theta})| + 2\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|\right)\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| \\ & + 2\left(|R(re^{i\theta})|\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|^2\right) \\ & + |R(re^{i\theta})|\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})\pi'(re^{i\theta})}{\pi(re^{i\theta})^2}\right| \\ & + \sum_{j=2}^{l}(|R(re^{i\theta})Q_j(re^{i\theta})| + |Q_j'(re^{i\theta})| \\ & + |Q_j(re^{i\theta})P_j'(re^{i\theta})|)|e^{P_j(re^{i\theta})}| \\ \le {} & O(e^{r^{\xi_2}}), \quad as\ r \to \infty,\ \theta \in S_2^- \backslash E_0. \end{aligned} \tag{3.8}$$

Since Q and h' are of finite order, combining (3.7) and (3.8), we obtain (3.6).

In the following, we prove that for any $\theta \in [0, 2\pi)$,

$$|U_1(re^{i\theta})| \le O(e^{r^{\xi_3}}), \quad as\ r \to \infty. \tag{3.9}$$

We note that there exist $\bar{\theta}_j$ $(j = 1, 2, \dots, l)$ satisfying $\delta_j(\theta) = 0$ on the rays $\arg z = \bar{\theta}_j + \frac{q\pi}{n}$, where $q = 0, \dots, 2n-1$, which form $2n$ sectors of opening $\frac{\pi}{n}$ respectively. Without loss of generality, we may assume that $\bar{\theta}_j \in \left[0, \frac{\pi}{n}\right)$. Since $\lambda_j = \frac{\zeta_j}{\zeta_2} > 0$ $(j = 3, \dots, l)$, we have $\bar{\theta}_j = \bar{\theta}_2$ $(j = 3, \dots, l)$. Set $\bar{\theta}_{jq} = \bar{\theta}_j + \frac{q\pi}{n}$, $j = 1, 2$, if there are some integers q_1 and q_2 such that $\bar{\theta}_{1q_1} = \bar{\theta}_{2q_2}$, then $\bar{\theta}_1 - \bar{\theta}_2 + (q_1 - q_2)\frac{\pi}{n} = 0$, we have that $\tan n\bar{\theta}_j = \frac{\alpha_j}{\beta_j}$, $j = 1, 2$. Which gives

$$0 = \tan(n\bar{\theta}_1 - n\bar{\theta}_2 + (q_1 - q_2)\pi) = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1\alpha_2 + \beta_1\beta_2} = -\Im\mathfrak{m}\frac{\zeta_1}{\zeta_2}.$$

This contradicts the assumption that $\frac{\zeta_1}{\zeta_2}$ is non-real. Hence we see that each component of S_1^+ and S_2^+ contains a component of $S_1^+ \cap S_2^+$. The boundaries of the components of $S_1^+ \cap S_2^+$ are some of the rays $\arg z = \bar{\theta}_{jq}$, we fix a component of $S_1^+ \cap S_2^+$, say S^*. We may write

$$S^* = \{\theta \in S_1^+ \cap S_2^+ : \theta_1^* < \theta < \theta_2^*, \delta_1(\theta_1^*) = \delta_2(\theta_2^*) = 0\}$$

or

$$S^* = \{\theta \in S_1^+ \cap S_2^+ : \theta_2^* < \theta < \theta_1^*, \delta_1(\theta_1^*) = \delta_2(\theta_2^*) = 0\}.$$

Furthermore, we define

$$D_{12} = \{\theta \in S_1^+ \cap S_2^+ : \delta_1(\theta) > (2\lambda + 2)\delta_2(\theta)\},$$
$$D_{21} = \left\{\theta \in S_1^+ \cap S_2^+ : \delta_2(\theta) > \frac{\lambda+1}{\lambda}\delta_1(\theta)\right\},$$

where $\lambda = \max\{\lambda_j : j = 3, \dots, l\} < \frac{1}{2}$. Since each component of S_1^+ and S_2^+ is a sector of opening $\frac{\pi}{n}$, the rays $\arg z = \theta_1^*$ and $\arg z = \theta_2^*$ are contained in S_2^+ and S_1^+ respectively. We prove the first case, the proof of the second case can be obtained similarly. Hence there exist $\eta_1 > 0$, $\eta_2 > 0$ such that

$$\{\theta : \theta_1^* < \theta < \theta_1^* + \eta_1\} \subset D_{21}, \quad \{\theta : \theta_2^* - \eta_2 < \theta < \theta_2^*\} \subset D_{12}.$$

Hence there exists a $\theta \in (S_{2k}^+ \cap D_{12}) \backslash E_0$ for any $k = 1, 2, \dots, n$. Set $0 < (2\lambda + 2)\delta_2 < \rho_2 < \rho_1 < \delta_1$, $0 < \varepsilon_{11} < 1 - \frac{\rho_1}{\delta_1}$, $0 < \varepsilon_{12} < \frac{\rho_2}{2\delta_2} - 1$, $0 < \varepsilon_{1j} < \frac{\rho_2}{2\lambda_j\delta_2} - 1$, $(j = 3, \dots, l)$, by Lemma 3, we have

$$\begin{aligned} (3.10)\qquad & |Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + \cdots + Q_l e^{P_l(re^{i\theta})}| \\ & \geq |Q_1 e^{P_1(re^{i\theta})}| \left| 1 - \left| \frac{Q_2}{Q_1} e^{P_2(re^{i\theta}) - P_1(re^{i\theta})} \right| - \cdots - \left| \frac{Q_l}{Q_1} e^{P_l(re^{i\theta}) - P_1(re^{i\theta})} \right| \right| \\ & \geq (1 - o(1)) e^{(1-\varepsilon_{11})\delta_1 r^n} \geq (1 - o(1)) e^{\rho_1 r^n}, \quad \textit{as } r \to \infty. \end{aligned}$$

We assume that there exists an unbounded sequence $\{r_m\}_{m=1}^{\infty}$ such that $0 < |h'(r_m e^{i\theta})| \leq 1$. From (3.1), (3.10) and Lemma 2, we get for an $N_1 \in \mathbf{N}$

$$\begin{aligned} e^{\rho_1 r_m^n}(1 - o(1)) & \leq 1 + \left| \frac{h''(r_m e^{i\theta})}{h'(r_m e^{i\theta})} \right| + 2\left| \frac{\pi'(r_m e^{i\theta})}{\pi(r_m e^{i\theta})} \right| + \left| \frac{\pi''(r_m e^{i\theta})}{\pi(r_m e^{i\theta})} \right| \\ & \leq r_m^{N_1}, \quad \textit{as } m \to \infty. \end{aligned}$$

Which is absurd. Hence we may assume that $|h'(re^{i\theta})| \geq 1$ for sufficiently large r. It follows from (3.1) and Lemma 2, for an $N_2 \in \mathbf{N}$

$$\begin{aligned} (3.11)\qquad & |Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + \cdots + Q_l e^{P_l(re^{i\theta})}| \\ & \leq |h'(re^{i\theta})|^2 \left(1 + \left| \frac{h''(re^{i\theta})}{h'(re^{i\theta})} \right| + 2\left| \frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})} \right| + \left| \frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})} \right| \right) \\ & \leq |h'(re^{i\theta})|^2 (1 + O(r^{N_2})), \quad \textit{as } r \to \infty. \end{aligned}$$

Thus, by (3.10) and (3.11) and for sufficiently large r, we have

$$|h'(re^{i\theta})| \geq e^{(1/2)\rho_2 r^n}. \tag{3.12}$$

From Lemma 2, (3.2) and (3.12), we get

$$\begin{aligned}
|2U_1(re^{i\theta})| \leq{}& \left|\frac{h'''(re^{i\theta})}{h'(re^{i\theta})}\right| + \left(|R(re^{i\theta})| + 2\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|\right)\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| \\
&+ 2\left(|R(re^{i\theta})|\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|^2\right) \\
&+ |R(re^{i\theta})|\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})\pi'(re^{i\theta})}{\pi(re^{i\theta})^2}\right| \\
&+ \sum_{j=2}^{l}(|R(re^{i\theta})Q_j(re^{i\theta})| + |Q_j'(re^{i\theta})| \\
&+ |Q_j(re^{i\theta})P_j'(re^{i\theta})|)\left|\frac{e^{P_j(re^{i\theta})}}{h'(re^{i\theta})}\right| \\
\leq{}& O(r^{N_2}) + (1+o(1))\exp\left\{\left(\delta_2(1+\varepsilon_{12}) - \frac{\rho_2}{2}\right)r^n\right\} \\
&+ \sum_{j=3}^{l}(1+o(1))\exp\left\{\left(\lambda_j\delta_2(1+\varepsilon_{1j}) - \frac{\rho_2}{2}\right)r^n\right\}, \quad \textit{as } r \to \infty.
\end{aligned} \tag{3.13}$$

Since $\delta_2(1+\varepsilon_{12}) - \frac{\rho_2}{2} < 0, \lambda_j\delta_2(1+\varepsilon_{1j}) - \frac{\rho_2}{2} < 0$ $(j = 3, \ldots, l)$, it gives that for an $N_3 \in \mathbf{N}$ and for sufficiently large r, we have

$$|U_1(re^{i\theta})| \leq r^{N_3}. \tag{3.14}$$

Now we fix a $\gamma (= \gamma_{2k}) \in (S_{2k}^+ \cap D_{21}) \backslash E_0$, $k = 1, 2, \ldots, n$. Then we find $\gamma_1, \gamma_2 \in S_2^- \backslash E_0$, $\gamma_1 < \gamma < \gamma_2$ such that $\gamma - \gamma_1 < \frac{\pi}{n}$, $\gamma_2 - \gamma < \frac{\pi}{n}$. We first show that (3.9) holds for any $\theta \in [\gamma_1, \gamma]$. Write $\gamma - \gamma_1 = \frac{\pi}{n + \tau_1}$, $\tau_1 > 0$, since $\rho(U_1) \leq n$, we have that $|U_1(re^{i\theta})| \leq e^{r^{n+\tau_2}}$, $0 < \tau_2 < \tau_1$ for sufficiently large r. Set $g(z) = U_1(z)/\exp((ze^{-((\gamma+\gamma_1)/2)i})^{\xi_3})$, then $g(z)$ is analytic in the region $\{z : \gamma_1 \leq \arg z \leq \gamma\}$. Since $\gamma_1 \leq \arg z = \theta \leq \gamma$, $\gamma - \gamma_1 < \frac{\pi}{n}$, we infer that $\cos(\arg((ze^{-((\gamma+\gamma_1)/2)i})^{\xi_3})) \geq K$ for some $K > 0$. In fact,

$$-\frac{\pi}{2} < -\frac{\pi\xi_3}{2n} \leq -\xi_3\frac{\gamma - \gamma_1}{2} \leq \arg((ze^{-((\gamma+\gamma_1)/2)i})^{\xi_3}) \leq \xi_3\frac{\gamma - \gamma_1}{2} \leq \frac{\pi\xi_3}{2n} < \frac{\pi}{2}.$$

Hence for $\gamma_1 < \theta < \gamma$,

$$|g(re^{i\theta})| \le \left|\frac{U_1(re^{i\theta})}{e^{Kr^{\xi_3}}}\right| \le O(e^{r^{n+\tau_2}}), \quad as \ r \to \infty.$$

It follows from (3.6) and (3.14) that for some $M > 0$, as $r \to \infty$

$$|g(re^{i\gamma_1})| \le \frac{O(e^{r^{\xi_2}})}{e^{Kr^{\xi_3}}} \le M$$

and

$$|g(re^{i\gamma})| \le \frac{O(r^{N_3})}{e^{Kr^{\xi_3}}} \le M.$$

By the Phragmen-Lindelöf theorem, we obtain (3.9). Similarly we see that (3.9) holds for any $\theta \in [\gamma, \gamma_2]$. Hence we conclude that (3.9) holds for any $\theta \in [0, 2\pi)$.

We next need to prove that for any $\theta \in [0, 2\pi)$,

$$|U_2(re^{i\theta})| \le O(e^{r^{\xi_3}}), \quad as \ r \to \infty. \tag{3.15}$$

By recalling the previous reasoning in (3.6) and (3.8), we can also obtain that there exists a set $E_0 \subset [0, 2\pi)$ with $m(E_0) = 0$ such that if $\theta \in S_1^- \cap S_2^- \backslash E_0$, then

$$|U_2(re^{i\theta})| \le O(e^{r^{\xi_2}}), \quad as \ r \to \infty. \tag{3.16}$$

By the similar proof in (3.9), there exists a $\theta \in (S_{1k}^+ \cap D_{21}) \backslash E_0$ for any $k = 1, 2, \dots, n$. Set $0 < (2\lambda + 2)\delta_1 < 2\lambda\delta_2 < \rho_4 < \rho_3 < \delta_2$, $0 < \varepsilon_{21} < 1 - \frac{\rho_3}{\delta_2}$, $0 < \varepsilon_{22} < \frac{\rho_4}{2\delta_1} - 1$, $0 < \varepsilon_{2j} < \frac{\rho_4}{2\lambda_j\delta_2} - 1$, $(j = 3, \dots, l)$. By Lemma 3, we have

$$\begin{aligned} &|Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + \cdots + Q_l e^{P_l(re^{i\theta})}| \\ &\quad \ge |Q_2 e^{P_2(re^{i\theta})}| \left|1 - \left|\frac{Q_1}{Q_2} e^{P_1(re^{i\theta}) - P_2(re^{i\theta})}\right| - \cdots - \left|\frac{Q_l}{Q_2} e^{P_l(re^{i\theta}) - P_2(re^{i\theta})}\right|\right| \\ &\quad \ge (1 - o(1)) e^{(1-\varepsilon_{21})\delta_2 r^n} \ge (1 - o(1)) e^{\rho_3 r^n}, \quad as \ r \to \infty. \end{aligned} \tag{3.17}$$

We assume that there exists an unbounded sequence $\{r_m\}_{m=1}^{\infty}$ such that $0 < |h'(r_m e^{i\theta})| \le 1$. From (3.1), (3.17) and Lemma 2, we get for an $N_4 \in \mathbf{N}$

$$\begin{aligned} e^{\rho_3 r_m^n}(1 - o(1)) &\le 1 + \left|\frac{h''(r_m e^{i\theta})}{h'(r_m e^{i\theta})}\right| + 2\left|\frac{\pi'(r_m e^{i\theta})}{\pi(r_m e^{i\theta})}\right| + \left|\frac{\pi''(r_m e^{i\theta})}{\pi(r_m e^{i\theta})}\right| \\ &\le r_m^{N_4}, \quad as \ m \to \infty. \end{aligned}$$

This is absurd. Hence we may assume that $|h'(re^{i\theta})| \ge 1$ for sufficiently large r. It follows from (3.1) and Lemma 2, for an $N_5 \in \mathbf{N}$

$$(3.18)\quad \begin{aligned}&|Q_1e^{P_1(re^{i\theta})}+Q_2e^{P_2(re^{i\theta})}+\cdots+Q_le^{P_l(re^{i\theta})}|\\&\le |h'(re^{i\theta})|^2\left(1+\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right|+2\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|+\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right|\right)\\&\le |h'(re^{i\theta})|^2(1+O(r^{N_5})),\quad \text{as } r\to\infty.\end{aligned}$$

Combining (3.17) and (3.18), we obtain for sufficiently large r

$$(3.19)\qquad |h'(re^{i\theta})|\ge e^{(1/2)\rho_4r^n}.$$

It follows from (3.4) and (3.19) that

$$(3.20)\quad \begin{aligned}&|2U_2(re^{i\theta})|\\&\le \left|\frac{h'''(re^{i\theta})}{h'(re^{i\theta})}\right|+\left(|T(re^{i\theta})|+2\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|\right)\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right|\\&+2\left(|T(re^{i\theta})|\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|+\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right|+\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|^2\right)\\&+|T(re^{i\theta})|\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right|+\left|\frac{\pi'''(re^{i\theta})}{\pi(re^{i\theta})}\right|+\left|\frac{\pi''(re^{i\theta})\pi'(re^{i\theta})}{\pi(re^{i\theta})^2}\right|\\&+(|T(re^{i\theta})Q_1(re^{i\theta})|+|Q_1'(re^{i\theta})|+|Q_1(re^{i\theta})P_1'(re^{i\theta})|)\left|\frac{e^{P_1(re^{i\theta})}}{h'(re^{i\theta})}\right|\\&+\sum_{j=3}^{l}(|T(re^{i\theta})Q_j(re^{i\theta})|+|Q_j'(re^{i\theta})|+|Q_j(re^{i\theta})P_j'(re^{i\theta})|)\left|\frac{e^{P_j(re^{i\theta})}}{h'(re^{i\theta})}\right|\\&\le O(r^{N_5})+(1+o(1))\exp\left\{\left(\delta_1(1+\varepsilon_{22})-\frac{\rho_4}{2}\right)r^n\right\}\\&+\sum_{j=3}^{l}(1+o(1))\exp\left\{\left(\lambda_j\delta_2(1+\varepsilon_{2j})-\frac{\rho_4}{2}\right)r^n\right\},\quad \text{as } r\to\infty.\end{aligned}$$

Since $\delta_1(1+\varepsilon_{22})-\frac{\rho_4}{2}<0$, $\lambda_j\delta_2(1+\varepsilon_{2j})-\frac{\rho_4}{2}<0$ $(j=3,\dots,l)$, it gives that for an $N_6\in\mathbf{N}$ and for sufficiently large r,

$$(3.21)\qquad |U_2(re^{i\theta})|\le r^{N_6}.$$

Now we fix a $\gamma'(=\gamma_{2k}')\in(S_{2k}^+\cap D_{12})\backslash E_0$, $k=1,2,\dots,n$. Then we find $\gamma_3,\gamma_4\in S_1^-\cap S_2^-\backslash E_0$, $\gamma_3<\gamma'<\gamma_4$ such that $\gamma'-\gamma_3<\frac{\pi}{n}$, $\gamma_4-\gamma'<\frac{\pi}{n}$. By the same reasoning in (3.14), for any $\gamma_3\le\theta\le\gamma_4$, we have

$$(3.22)\qquad |U_2(re^{i\theta})|\le O(e^{r^{\xi_3}}),\quad \text{as } r\to\infty.$$

Hence we conclude that (3.15) holds for any $\theta\in[0,2\pi)$.

To complete the proof of Theorem 1 (i), by (3.2) and (3.5), we have

$$U_1 - U_2 = \frac{1}{2}h'(T - R), \tag{3.23}$$

since $\max\{\rho(Q_j), j = 1, 2, \dots, l\} < \xi_2 < \xi_3$, by the theorem on the logarithmic derivative and by (3.1), (3.9), (3.15), (3.23), we have

$$\begin{aligned} m(r, Q_1 e^{P_1(z)} + Q_2 e^{P_2(z)} + \cdots + Q_l e^{P_l(z)}) \\ \leq 2m(r, h') + O(\log r) \leq 2m(r, U_1 - U_2) + O(\log r) \\ \leq O(r^{\xi_3}), \quad \text{as } r \to \infty. \end{aligned} \tag{3.24}$$

Since $\frac{\zeta_1}{\zeta_2}$ is non-real, $S_1^+ \cap S_2^-$ contains an interval $I = [\varphi_1, \varphi_2]$ satisfying $\min_{\theta \in I} \delta_1(\theta) = s > 0$. By Lemma 3, there exists a constant $R_2(\theta)(> 0)$ such that for any $\theta \in I$ and for any given $\varepsilon > 0$, we have for sufficiently large $r \geq R_2(\theta)$

$$\begin{aligned} |Q_1 e^{P_1(re^{i\theta})}| \geq \exp((1 - \varepsilon)\delta_1 r^n), \\ |Q_2 e^{P_2(re^{i\theta})}| \leq \exp((1 - \varepsilon)\delta_2 r^n), \\ |Q_i e^{P_j(re^{i\theta})}| \leq \exp((1 - \varepsilon)\lambda_j \delta_2 r^n), \quad (j = 3, \dots, l). \end{aligned}$$

Hence,

$$\begin{aligned} m(r, Q_1 e^{P_1(z)} + Q_2 e^{P_2(z)} + \cdots + Q_l e^{P_l(z)}) \\ \geq \int_{\varphi_1}^{\varphi_2} \log^+ |Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + \cdots + Q_l e^{P_l(re^{i\theta})}| \, d\theta \\ \geq \int_{\varphi_1}^{\varphi_2} (1 - o(1)) \log^+ |Q_1 e^{P_1(re^{i\theta})}| \, d\theta \\ \geq \int_{\varphi_1}^{\varphi_2} (1 - o(1))(1 - \varepsilon) s r^n \, d\theta \\ \geq (1 - o(1))(1 - \varepsilon) s r^n (\varphi_2 - \varphi_1), \quad \text{as } r \to \infty. \end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25) and recalling that $\xi_3 < n$, we get a contradiction. Hence, $\lambda(f) = \infty$.

4. Proof of Theorem 1 (ii)

Let $f \not\equiv 0$ be a solution of (1.4). Write $f = \pi e^h$, suppose that $\lambda(f) < n$. From our hypothesis, we have $\rho(\pi) = \lambda(\pi) < n$. Eliminating e^{P_1} from (3.1) and recalling that $R = \frac{Q_1'}{Q_1} + P_1'$, we have

$$(3.26)\qquad 2Uh' = -h''' + \left(R - 2\frac{\pi'}{\pi}\right)h'' + 2\left(R\frac{\pi'}{\pi} - \left(\frac{\pi'}{\pi}\right)'\right)h' + R\frac{\pi''}{\pi} - \left(\frac{\pi''}{\pi}\right)' + \sum_{j=2}^{l}(RQ_j - Q_j' - Q_jP_j')e^{P_j},$$

where

$$(3.27)\qquad U = h'' - \frac{1}{2}Rh'.$$

From (3.26) and (3.27), we get

$$C_1(z)h' = C_0(z),$$

where

$$(3.28)\qquad C_0(z) = -U' + \frac{1}{2}RU - 2\frac{\pi'}{\pi}U + R\frac{\pi''}{\pi} - \frac{\pi'''}{\pi} + \frac{\pi''\pi'}{\pi^2} + \sum_{j=2}^{l}(RQ_j - Q_j' - Q_jP_j')e^{P_j},$$

$$(3.29)\qquad C_1(z) = 2U + \frac{1}{2}R' - \frac{1}{4}R^2 - R\frac{\pi'}{\pi} + 2\frac{\pi''}{\pi} - 2\left(\frac{\pi'}{\pi}\right)^2.$$

We next show that $C_0(z) \equiv 0$ and $C_1(z) \equiv 0$. If $C_0(z) \not\equiv 0$, $C_1(z) \not\equiv 0$, by Nevanlinna's first fundamental theorem, we obtain

$$T(r,h') \le T(r,C_0) + T(r,C_1) + o(1).$$

Set $\max\{\rho(Q_j)\ (j=1,\dots,l), \lambda(f)\} < \xi_2 < \xi_3 < n$, from (3.1), we obtain

$$(3.30)\qquad T(r, Q_1e^{P_1(z)} + Q_2e^{P_2(z)} + \cdots + Q_le^{P_l(z)}) \le 2T(r,h') + O(\log r).$$

By Lemma 5, we have

$$(3.31)\qquad m(r, Q_1e^{P_1(z)} + Q_2e^{P_2(z)} + \cdots + Q_le^{P_l(z)}) \ge (1-\varepsilon)m(r, e^{P_1}) + O(r^{\xi_3}), \quad r \to \infty, (r \notin E),$$

where E has finite linear measure. From (3.30) and (3.31), we obtain

$$(3.32)\qquad T(r,h') \ge \frac{1-\varepsilon}{2}T(r,e^{P_1}) + O(r^{\xi_3}), \quad r \to \infty, (r \notin E).$$

Since $0 < \rho = \dfrac{\zeta_2}{\zeta_1} < \dfrac{1}{4}$, $\lambda_j = \dfrac{\zeta_j}{\zeta_2} > 0$ $(j = 3,\dots,l)$, $\sum_{j=3}^{l}\lambda_j < 1$, we get

$$\delta(P_2,\theta) = \rho\delta(P_1,\theta), \quad S_{1k}^+ = S_{2k}^+ = \cdots = S_{lk}^+,$$
$$S_{1k}^- = S_{2k}^- = \cdots = S_{lk}^-, \quad (k = 1,\dots,n).$$

By the same reasoning in (3.7) and (3.8), we have

$$(3.33)\qquad |U(re^{i\theta})| \le O(e^{r^{\xi_2}}), \quad as\ r \to \infty$$

for any $\theta \in S_1^- \backslash E_0$, $m(E_0) = 0$. Also by the same reasoning in (3.9)–(3.13), we have

$$(3.34)\qquad |U(re^{i\theta})| \le r^{N_3}, \quad as\ r \to \infty$$

for any $\theta \in S_1^+ \backslash E_0$, $m(E_0) = 0$. Since $\rho(U) \le n$, by the Phragmen-Lindelöf theorem, we have

$$(3.35)\qquad |U(re^{i\theta})| \le O(e^{r^{\xi_3}}), \quad as\ r \to \infty$$

for any $\theta \in [0, 2\pi)$. In the following, we estimate $T(r, C_0)$ and $T(r, C_1)$.

$$T(r, C_0) \le T\left(r, U' - \frac{1}{2}RU + 2\frac{\pi'}{\pi}U\right) + T\left(r, R\frac{\pi''}{\pi} - \frac{\pi'''}{\pi} + \frac{\pi''\pi'}{\pi^2}\right) + \sum_{j=2}^{l} T(r, RQ_j - Q_j' - Q_jP_j') + \sum_{j=2}^{l} T(r, e^{P_j}).$$

Since $\max\{\rho(Q_j)\ (j = 1, \dots, l), \rho(R), \rho(\pi)\} < n$, we have

$$(3.36)\qquad T(r, C_0) \le \sum_{j=2}^{l} T(r, e^{P_j})) + O(r^{\xi_3}) = \left(1 + \sum_{j=3}^{l} \lambda_j\right) T(r, e^{P_2}) + O(r^{\xi_3}) \le \left(1 + \sum_{j=3}^{l} \lambda_j\right) \rho T(r, e^{P_1}) + O(r^{\xi_3}), \quad as\ r \to \infty.$$

From (3.29) and (3.35), we have

$$(3.37)\qquad T(r, C_1) \le O(r^{\xi_3}), \quad as\ r \to \infty.$$

From (3.30), (3.32), (3.36) and (3.37), we get

$$(3.38)\qquad \frac{1-\varepsilon}{2} T(r, e^{P_1}) + O(r^{\xi_3}) \le T(r, h') \le \left(1 + \sum_{j=3}^{l} \lambda_j\right) \rho T(r, e^{P_1}) + O(r^{\xi_3}), \quad r \to \infty, (r \notin E).$$

Thus (3.38) implies

$$\left(\frac{1-\varepsilon}{2} - \left(1 + \sum_{j=3}^{l} \lambda_j\right)\rho - o(1)\right) T(r, e^{P_1}) \le 0, \quad r \to \infty, (r \notin E).$$

Since $0 < \rho = \dfrac{\zeta_2}{\zeta_1} < \dfrac{1}{4}$, $0 < \sum_{j=3}^{l} \lambda_j < 1$, we get a contradiction. Hence $C_0(z) \equiv C_1(z) \equiv 0$. From (3.28), we obtain

$$(3.39) \quad \sum_{j=2}^{l} (RQ_j - Q_j' - Q_j P_j') e^{P_j} = U' - \frac{1}{2} RU + 2\frac{\pi'}{\pi} U - R\frac{\pi''}{\pi} + \frac{\pi'''}{\pi} - \frac{\pi''\pi'}{\pi^2}.$$

We assume that $\sum_{j=2}^{l} (RQ_j - Q_j' - Q_j P_j') e^{P_j} \not\equiv 0$, if $\sum_{j=2}^{l} (RQ_j - Q_j' - Q_j P_j') e^{P_j} \equiv 0$, since $\lambda_j = \dfrac{\zeta_j}{\zeta_2} > 0$ $(j = 3, \dots, l)$ and $0 < \sum_{j=3}^{l} \lambda_j < 1$, by Lemma 3 and by a simple calculation, this is a contradiction. From (3.39), by Lemma 5, we obtain

$$\begin{aligned}
(3.40) \quad (1 - \varepsilon) T(r, e^{P_2}) + O(r^{\xi_3}) &\le \sum_{j=2}^{l} T(r, (RQ_j - Q_j' - Q_j P_j') e^{P_j}) \\
&\le T\left(r, U' - \frac{1}{2} RU\right) + T(r, U) + T(r, R) \\
&\quad + T\left(r, \frac{\pi'}{\pi}\right) + T\left(r, \frac{\pi''}{\pi}\right) + T\left(r, \frac{\pi'''}{\pi}\right) + o(1) \\
&\le O(r^{\xi_3}), \quad r \to \infty, \ (r \notin E).
\end{aligned}$$

From (3.40), we have $\rho(e^{P_2}) < \xi_3 < n$, we get a contradiction. Hence $\lambda(f) \ge n$.
Thus, we complete the proof of Theorem 1.

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