M. NAKAI AND S. SEGAWA
KODAI MATH. J.
32 (2009), 109–116

TYPES OF AFFORESTED SURFACES[1]

MITSURU NAKAI AND SHIGEO SEGAWA

Abstract

We form, what we call, an afforested surface R over a plantation P by foresting with trees T_n ($n \in \mathbf{N}$: the set of positive integers). If all of P and T_n ($n \in \mathbf{N}$) belong to the class $\mathcal{O}_s$ of hyperbolic Riemann surfaces W carrying no singular harmonic functions on W, then we will show that, under a certain diminishing condition on roots of trees T_n ($n \in \mathbf{N}$), the afforested surface R also belongs to $\mathcal{O}_s$.

1. Introduction

In 1961 Parreau introduced two terms, quasibounded and singular, for positive harmonic functions on hyperbolic Riemann surfaces and showed the so called Parreau decomposition that any positive harmonic function can be uniquely expressed as the sum of quasibounded and singular positive harmonic functions ([6], cf. e.g. [1]). It is well known that there exists a Riemann surface of any given finite harmonic dimension (i.e. the dimension of the linear space generated by positive harmonic functions) carrying no singular positive harmonic functions (cf. e.g. [7]). In view of this it has been asked whether there exists a Riemann surface of infinite harmonic dimension carrying no singular positive harmonic functions (cf. e.g. [5]). The purpose of this paper is to give a result which implies an affirmative answer to the above question.

2. Afforested surface

We take an open Riemann surface P and a sequence $(T_n)_{n \in \mathbf{N}}$ of open Riemann surfaces T_n ($n \in \mathbf{N}$). We fix a sequence $(V_n)_{n \in \mathbf{N}}$ of simply connected Jordan regions V_n in P such that $(V_n)_{n \in \mathbf{N}}$ does not accumulate in P and $\overline{V}_i \cap \overline{V}_j = \emptyset$ ($i \neq j$); we also choose a simply connected Jordan region U_n in T_n for each $n \in \mathbf{N}$. We identify $\overline{V}_n$ and $\overline{U}_n$ as a parametric disc $\{|z| \leq 1\}$ in P and T_n for every $n \in \mathbf{N}$. Let $s_n \in (0, 1/2)$ and put

[1] 2000 *Mathematics Subject Classification*. Primary 30F20; Secondary 30F25, 31A05.

Key words and phrases. essentially positive; Green function; Parreau decomposition; quasibounded; singular; Wiener compactification; Wiener (harmonic) boundary.

Received April 1, 2008; revised July 1, 2008.

$$\sigma_n := [-s_n, s_n] = \{z \in V_n = U_n : |\mathrm{Re}\, z| \le s_n, \mathrm{Im}\, z = 0\} \tag{2.1}$$

for every $n \in \mathbf{N}$ so that $\sigma_n \subset P$ and at the same time $\sigma_n \subset T_n$ $(n \in \mathbf{N})$. For each $n \in \mathbf{N}$ we attach $T_n \backslash \sigma_n$ to $P \backslash \bigcup_{i \in \mathbf{N}} \sigma_j$ by connecting them crosswise along σ_n. By the standard procedure we can make the resulting surface $R = \langle P, (T_n)_{n \in \mathbf{N}} \rangle$ as a Riemann surface. To make the situation impressive we call R as an *afforested Riemann surface* over a *plantation* P by foresting *trees* T_n $(n \in \mathbf{N})$. The slit σ_n will be called the *root* of the tree T_n for each $n \in \mathbf{N}$. In R, the slit σ_n is also understood in the Carathéodory topology so that it is viewed as a Jordan curve $\sigma_n^+ \cup \sigma_n^-$ by considering both sides σ_n^+ and σ_n^- of the cut σ_n.

3. Classification

An open Riemann surface W is said to be hyperbolic (not parabolic) if there exists the Green function $g(z, \zeta; W)$ with its pole ζ at any point of W, where $g(\cdot, \zeta; W)$ is the minimal positive continuous distributional solution of the Poisson equation

$$-\Delta g(\cdot, \zeta; W) = 2\pi \delta_\zeta$$

on W with δ_ζ the Dirac measure supported at $\zeta \in W$. We say that W is parabolic if W is not hyperbolic and we denote by $\mathcal{O}_G$ the class of parabolic Riemann surfaces. We denote by $H(W)$ the vector space of harmonic functions on W. As usual we denote by $\mathscr{F}^+$ the subclass of $\mathscr{F}$ consisting of nonnegative functions in the function class $\mathscr{F}$. Let $HP(W)$ be the vector subspace of $H(W)$ consisting of essentially positive harmonic functions u on W in the sense of that $|u|$ admits a harmonic majorant on W. Therefore $HP(W)$ forms a vector lattice with lattice operations $\vee$ and $\wedge$, where $u \vee v$ ($u \wedge v$, resp.) is the least harmonic majorant (the greatest harmonic minorant, resp.) of u and v in $HP(W)$. Then u^+ (u^-, resp.) is the positive (negative, resp.) part $u^+ := u \vee 0$ ($u^- := -(u \wedge 0) = (-u)^+ = (-u) \vee 0$, resp.) of the Jordan decomposition $u = u^+ - u^-$ of any $u \in HP(W)$ so that $HP(W)$ is generated by $HP(W)^+ = H(W)^+ : HP(W) = H(W)^+ \ominus H(W)^+$. A function $u \in HP(W)$ is said to be *quasibounded* if $u^\pm = \lim_{\mathbf{R} \ni t \to \infty} u^\pm \wedge t$ and *singular* if $u^\pm \wedge t = 0$ for every $t \in \mathbf{R}^+$, where $\mathbf{R}$ is the set of real numbers. We denote by $HB'(W)$ the vector subspace of $HP(W)$ consisting of quasibounded harmonic functions on W and hence $HB(W) \subset HB'(W) \subset HP(W)$, where $HB(W)$ is the vector space of bounded harmonic functions on W. We denote by $\mathcal{O}_{HP}$ the class of open Riemann surfaces W such that $HP(W) = \mathbf{R}$ (cf. e.g. [7]). We also denote by $\mathcal{O}_s$ the class of hyperbolic Riemann surfaces W such that $HP(W) = HB'(W)$, i.e. $\mathcal{O}_s$ is the class of hyperbolic Riemann surfaces W carrying no nonzero singular essentially positive harmonic functions on W. Hence

$$\mathcal{O}_{HP} \backslash \mathcal{O}_G \subset \mathcal{O}_s. \tag{3.1}$$

Concrete examples in $\mathcal{O}_{HP} \backslash \mathcal{O}_G$ given by Toki and also by Sario are famous (cf. e.g. [7]). From the finite n number of copies of a surface in $\mathcal{O}_{HP} \backslash \mathcal{O}_G$ it is easy to construct a $W \in \mathcal{O}_s$ with harmonic dimension n (i.e. $\dim HP(W) = n$ so that $\dim HB(W) = \dim HB'(W) = n$). The question is whether we can construct a surface $W \in \mathcal{O}_s$ from infinitely many copies W_i ($i \in \mathbf{N}$) of a surface in $\mathcal{O}_{HP} \backslash \mathcal{O}_G$.

4. Sizes of roots of trees

We fix a reference point $o \in P \backslash \bigcup_{n \in \mathbf{N}} \overline{V}_n$ and denote by M_n the Harnack constant of the compact set $\{o\} \cup \partial V_n$ with respect to the class $HP(P \backslash \bigcup_{n \in \mathbf{N}} (1/2) \overline{V}_n)^+$ so that M_n is the smallest of all numbers $c \in [1, +\infty)$ such that

$$c^{-1} h(z_1) \leq h(z_2) \leq c h(z_1)$$

for every pair (z_1, z_2) of points z_1 and z_2 in $\{o\} \cup \partial V_n$ and for every h in $HP(P \backslash \bigcup_{n \in \mathbf{N}} (1/2) \overline{V}_n)^+$, where $t\overline{V}_n = \{|z| \leq t\}$ for every $t \in \mathbf{R}$ with $0 < t \leq 1$ (cf. e.g. [2]).

Suppose that $P \in \mathcal{O}_s$ and let $g(z, \zeta; P)$ be the Green function on P with its pole $\zeta \in P$. We denote by ζ_n the center of the parametric disc V_n so that ζ_n is also the center of σ_n corresponding to $z = 0$ of the local parameter on V_n. We choose an arbitrary but then fixed sequence $(\varepsilon_n)_{n \in \mathbf{N}}$ of positive numbers ε_n such that

$$\rho := \sum_{n \in \mathbf{N}} \varepsilon_n < 1. \tag{4.1}$$

Observe that

$$\alpha_n := \sup_{z \in P \backslash V_n} g(z, \zeta_n; P) < \infty \tag{4.2}$$

for every $n \in \mathbf{N}$ and

$$\beta_n := \inf_{z \in \sigma_n} g(z, \zeta_n; P) \uparrow \infty \quad (s_n \downarrow 0) \tag{4.3}$$

for each $n \in \mathbf{N}$, where we recall $\sigma_n = [-s_n, s_n] \subset (-1/2, 1/2)$. Hence we can choose and then fix an $s_n \in (0, 1/2)$ such that

$$\alpha_n / \beta_n \leq \varepsilon_n / (4M_n + 1) \quad (n \in \mathbf{N}). \tag{4.4}$$

We are ready to state our main result in this paper.

THE MAIN THEOREM. *Suppose that $P \in \mathcal{O}_s$ and $T_n \in \mathcal{O}_s$ ($n \in \mathbf{N}$). Then the afforested surface $R = \langle P, (T_n)_{n \in \mathbf{N}} \rangle$ over the plantation P with trees T_n ($n \in \mathbf{N}$) also belongs to $\mathcal{O}_s$ if the roots σ_n of T_n shrink so rapidly as to satisfy the condition* (4.4).

5. Wiener harmonic boundary

We denote by W^* the Wiener compactification of an open Riemann surface $W \notin \mathcal{O}_G$, by $\gamma = \gamma W$ the Wiener boundary $W^* \backslash W$, and by $\delta = \delta W$ the Wiener harmonic boundary of W (cf. e.g. [3], [7]). For any $f \in C(\gamma)$ we denote by H_f^W the Perron-Wiener-Brelot solution of the Dirichlet problem on W with boundary values f on $\gamma = \gamma W$. Then the set of regular points in $\gamma = \gamma W$ is $\delta = \delta W$. We will repeatedly use the following facts (cf. e.g. [7]): a $u \in HB'(W)$ is determined uniquely by $u|\delta$ and $u = H_u^W$; a $u \in HP(W)$ is nonzero singular if and only if $u \,|\, \delta = 0$ and $u \neq 0$ on W so that $W \in \mathcal{O}_s$ if and only if $u \,|\, \delta = 0$ implies $u = 0$ on W for every $u \in HP(W)$. We also need the maximum principle. Let S be a subregion of W and $u \in HB'(W)$. Then

$$\sup_{z \in S} u(z) = \sup_{z \in (\delta W \cap \bar{S}) \cup \partial S} u(z). \tag{5.1}$$

Thus, if $W \in \mathcal{O}_s$, then (5.1) is true for every $u \in HP(W)$.

We now prove a technical lemma which plays an essential role for the proof of the main theorem mentioned above. Let $P \in \mathcal{O}_s$, $T_n \in \mathcal{O}_s$ $(n \in \mathbf{N})$, and $R = \langle P, (T_n)_{n \in \mathbf{N}} \rangle$ be the afforested surface over the plantation P with trees $(T_n)_{n \in \mathbf{N}}$, whose roots $(\sigma_n)_{n \in \mathbf{N}}$ satisfy (4.4). Then we have the following result.

The Unicity Principle. *If an $h \in HP(R)$ vanishes on $(\delta P) \cup (\bigcup_{n \in \mathbf{N}} \delta T_n)$, then h vanishes identically on R.*

It may sound more plausible if $(\delta P) \cup (\bigcup_{n \in \mathbf{N}} \delta T_n)$ is replaced by δR. Thus the significance of the above result lies in the fact that $(\delta P) \cup (\bigcup_{n \in \mathbf{N}} \delta T_n)$ is only a part and not the whole of δR. However it is only used in the case h vanishes on the whole δR. Using the above unicity principle and actually only a weaker version of it as mentioned above, we can prove the main theorem instantaneously as follows. Take any $h \in HP(R)$ satisfying $h \,|\, \delta R = 0$. Then, in particular, $h \,|\, (\delta P) \cup (\bigcup_{n \in \mathbf{N}} \delta T_n) = 0$ and the above unicity principle assures that $h = 0$ identically on R so that there is no singular harmonic function in $HP(R)$ and a fortiori we can conclude that $R \in \mathcal{O}_s$.

6. Proof of the unicity principle

Choose an arbitrary $h \in HP(R)$ such that $h = 0$ on $\alpha := (\delta P) \cup (\bigcup_{n \in \mathbf{N}} \delta T_n)$. We are to show that $h = 0$ identically on R. Let $h = h_q + h_s$ be the Parreau decomposition of h into the sum of the quasibounded part h_q and the singular part h_s on R. First of all, h_s^+ and h_s^- are singular along with h_s and therefore $h_s^+ \,|\, \delta R = h_s^- \,|\, \delta R = 0$ and in particular $h_s^+ \,|\, \alpha = h_s^- \,|\, \alpha = 0$. Secondary, $h_q \,|\, \alpha = h \,|\, \alpha - h_s \,|\, \alpha = 0$. Since

$$h_q^{\pm} = H^R_{\max(\pm h_q, 0)}$$

and $\max(\pm h_q, 0) \,|\, \alpha = 0$, we see that $h_q^+ \,|\, \alpha = h_q^- \,|\, \alpha = 0$. Hence every summand in the right hand term of $h = h_q^+ - h_q^- + h_s^+ - h_s^-$ vanishes on α. Therefore we can assume that $h \geq 0$ on R, i.e. $h \in HP(R)^+$, in proving $h = 0$ identically on R under the assumption $h \,|\, \alpha = 0$. Moreover we may assume that $h(o) \leq 1/2$.

The first step: We construct auxiliary function u_n on R for each $n \in \mathbf{N}$ as follows. First on $P \backslash \bigcup_{m \in \mathbf{N}} \sigma_m$, u_n is determined by the following conditions: $u_n \in HB(P \backslash \bigcup_{m \in \mathbf{N}} \sigma_m) \cap C(P^* \backslash \sigma_n)$; $u_n = 0$ on $(\bigcup_{m \in \mathbf{N} \backslash \{n\}} \sigma_m) \cup \delta P$; $u_n = h$ continuously on $\sigma_n = \sigma_n^+ \cup \sigma_n^-$ considered as the Carathéodory boundary. Second on the rest of $P \backslash \bigcup_{m \in \mathbf{N}} \sigma_m$ in R, i.e. on $\bigcup_{m \in \mathbf{N}} T_m$, u_n is given by $u_n = h$ on $T_n \backslash \sigma_n$ and $u_n = 0$ on each T_m for each $m \in \mathbf{N} \backslash \{n\}$.

Let $(P_i)_{i \in \mathbf{N}}$ be an exhaustion of P such that $o \in P_1$, $\bigcup_{j \le i} \overline{V}_j \subset P_i$, and $\bigcup_{j > i} \overline{V}_j \subset P \backslash P_i$ for each $i \in \mathbf{N}$; let u_{ni} $(i > n)$ be in $C(\overline{P}_i \backslash \sigma_n) \cap HB(P_i \backslash \bigcup_{j \le i} \sigma_j)$ with $u_{ni} = h$ continuously on $\sigma_n = \sigma_n^+ \cup \sigma_n^-$ in the Carathéodory sense and $u_{ni} = 0$ on $(\bigcup_{j \le i, j \ne n} \sigma_j) \cup \partial P_i$. We extend u_{ni} to R by setting $u_{ni} = 0$ on $(P \backslash P_i) \cup (\bigcup_{j \in \mathbf{N} \backslash \{n\}} T_j)$ and $u_{ni} = h$ on $T_n \backslash \sigma_n$. Since $(u_{ni})_{i > n}$ is uniformly bounded on P and $u_n = 0$ on δP, it is easily seen that

$$u_{ni} \uparrow u_n \tag{6.1}$$

on R. Again, since $h \ge 0$ on R, we see that $\sum_{m \le n} u_{mi} \le h$ on R and therefore, by letting $i \uparrow \infty$, $\sum_{m \le n} u_m \le h$ on R for each $n \in \mathbf{N}$. We can thus conclude that the function

$$u := h - \sum_{n \in \mathbf{N}} u_n \ge 0 \tag{6.2}$$

is well defined on R.

The second step: We now show that $u = 0$ on R, or equivalently, we have the representation

$$h = \sum_{n \in \mathbf{N}} u_n \tag{6.3}$$

on R. By the very definition, $u = 0$ on $R \backslash (P \backslash \sum_{i \in \mathbf{N}} \sigma_i)$ and thus we only have to prove that $u = 0$ on P. Take the harmonic measure w_n of σ_n on $P \backslash \sigma_n$ for each $n \in \mathbf{N}$, i.e. $w_n \in C(P^*) \cap HB(P \backslash \sigma_n)$ with $w_n \,|\, \sigma_n = 1$ and $w_n \,|\, \delta P = 0$. In view of (4.4), we see that

$$w_n \le \frac{\varepsilon_n}{4M_n + 1} \tag{6.4}$$

on $P \backslash V_n$ for every $n \in \mathbf{N}$. Then

$$v := \sum_{n \in \mathbf{N}} \frac{2M_n}{\rho} w_n$$

is finitely continuous on P and harmonic on $P \backslash \bigcup_{n \in \mathbf{N}} \sigma_n$. Clearly $\sum_{j \le n} (2M_j/\rho) w_j$ is a potential on P for every $n \in \mathbf{N}$ and $v = \sum_{j \in \mathbf{N}} (2M_j/\rho) w_j$ is superharmonic on P. Hence v is a potential on P (cf. e.g. [4]) so that $v = 0$ on δP. Finally consider the function $u + v \in HP(P \backslash \bigcup_{j \in \mathbf{N}} \sigma_j)^+ \cap C(P)$. Observe that $(u+v)(o) \leq h(o) + \sum_{n \in \mathbf{N}} (2M_n/\rho) \cdot (\varepsilon_n/(4M_n + 1)) \leq 1$ and hence $u + v \leq M_n$ on ∂V_n. On V_n, $u + v$ is the sum of $u + (2M_n/\rho)w_n$ and the function $\sum_{j \in \mathbf{N} \backslash \{n\}} (2M_j/\rho)w_j$. Note that $u + (2M_n/\rho)w_n \leq u + v \leq M_n$ on ∂V_n and $u + (2M_n/\rho)w_n = 2M_n/\rho > 2M_n$ on σ_n and therefore $u + (2M_n/\rho)w_n < 2M_n/\rho$ on $\overline{V}_n \backslash \sigma_n$. Hence the local supermean value property is valid at each point of σ_n and we can conclude that $u + v$ is superharmonic on P. Clearly u is subharmonic on P and therefore there exists a $q \in H(P)$ with $0 \leq u \leq q \leq u + v$ on P. By the assumption that $P \in \mathcal{O}_s$ we see that $q \in HB'(P)^+$. By the fact that $0 \leq u \leq h = 0$ on δP, we see that $0 \leq q \leq u + v = 0$ on δP so that $q = 0$ on δP. By the maximum principle we can conclude that $q = 0$ on P so that $u = 0$ on P.

The third step: We next prove that h is bounded on each $T_n \backslash \sigma_n$ $(n \in \mathbf{N})$. Take the $h_n \in HB(T_n \backslash \sigma_n) \cap C(T_n^* \backslash \sigma_n)$ determined by $h_n = h$ continuously on $\sigma_n = \sigma_n^+ \cup \sigma_n^-$ in the Carathéodory sense and $h_n \,|\, \delta T_n = 0$. Let $(T_{ni})_{i \in \mathbf{N}}$ be the exhaustion of T_n and $h_{ni} \in HB(T_{ni} \backslash \sigma_n) \cap C(\overline{T}_{ni} \backslash \sigma_n)$ given by $h_{ni} = h$ continuously on $\sigma_n = \sigma_n^+ \cup \sigma_n^-$ in the Carathéodory sense and $h_{ni} \,|\, \partial T_{ni} = 0$. Clearly $h_{ni} \uparrow h_n$ $(n \uparrow \infty)$ on $T_n \backslash \sigma_n$ and $h_{ni} \leq h$. Hence $h_n \leq h$ on $T_n \backslash \sigma_n$. Set $t_n := h - h_n \in H(T_n \backslash \sigma_n)^+ \cap C(T_n^*)$ and consider $f_n \in HB(T_n \backslash \sigma_n)^+ \cap C(T_n^*)$ with $f_n \,|\, \sigma_n = 1$ and $f_n \,|\, \delta T_n = 0$. Recall that U_n is a parametric disc in T_n which was identified with V_n in P. Let $b := \sup_{\partial U_n} f_n < 1$, $c := \sup_{\partial U_n} t_n$, and $a := 1 + c/(1-b)$ so that $c < a(1-b)$. Observe that $t_n + af_n \in HP(T_n \backslash \sigma_n)^+ \cap C(T_n^*)$ with $(t_n + af_n) \,|\, \sigma_n = a$. Then $t_n + af_n \leq c + ab < a$ on ∂U_n and $t_n + af_n = a$ on σ_n. Hence $t_n + af_n < a$ on $U_n \backslash \sigma_n$ so that the supermean value property is fulfiled at each point of σ_n and thus $t_n + af_n$ is superharmonic on T_n. Since t_n is subharmonic on T_n and $t_n \leq t_n + af_n$, there exists a $p \in HB'(T_n)$ such that $0 \leq t_n \leq p \leq t_n + af_n$. Then $0 \leq p \,|\, \delta T_n \leq (t_n + af_n) \,|\, \delta T_n = 0$, or $p \,|\, \delta T_n = 0$. The maximum principle yields that $p = 0$ on T_n so that $t_n = 0$ on T_n. Hence $h = h_n \in HB(T_n \backslash \sigma_n)$, i.e. h is bounded on $T_n \backslash \sigma_n$ for each $n \in \mathbf{N}$.

The fourth and the final step: We are ready to show that $h = 0$ on R, which was the desired conclusion to assure $R \in \mathcal{O}_s$. For each $n \in \mathbf{N}$, let $S_n := (T_n \backslash \sigma_n) \bowtie_{\sigma_n} (V_n \backslash \sigma_n)$, where $(X \backslash \sigma) \bowtie_{\sigma} (Y \backslash \sigma)$ for a common slit σ in Riemann surfaces X and Y denotes the Riemann surface obtained by connecting $X \backslash \sigma$ and $Y \backslash \sigma$ crosswise along σ. Observe that $h \in HB(S_n)^+ \cap C(\overline{S}_n) \cap C(S_n^*)$. As the nonnegative function on $P \backslash \bigcup_{i \in \mathbf{N}} \sigma_i$ with the value at most $1/2 \leq 1$ at o, $h \leq M_n$ on ∂V_n. Hence, as the function in $HB(S_n)^+$, the maximum principle with $h \,|\, \delta T_n = 0$ implies that $h \leq M_n$ on S_n. As the function in $HB(P \backslash \bigcup_{i \in \mathbf{N}} \sigma_i) \cap C(P^* \backslash \sigma_n)$ with vanishing values on $(\bigcup_{i \in \mathbf{N} \backslash \{n\}} \sigma_i) \cup \delta P$, $u_n \leq M_n w_n$ on $P \backslash \sigma_n$ and hence, by (6.4), $u_n \leq \varepsilon_n$ on $P \backslash V_n$. A fortiori we see that

$$h = \sum_{n \in \mathbf{N}} u_n \leq \sum_{n \in \mathbf{N}} \varepsilon_n = \rho$$

on $P \backslash \bigcup_{n \in \mathbf{N}} V_n$. In particular, $h \,|\, \partial V_n \le \rho$ with $h \,|\, \delta T_n = 0$ implies $h \le \rho$ on S_n for every $n \in \mathbf{N}$. This proves that $0 \le h \le \rho$ on R.

Starting from the fact just established that $0 \le h \le \rho$ on R, we proceed as follows. Since $u_n = w_n = 0$ on δP and $u_n \le \rho w_n$ on $\bigcup_{i \in \mathbf{N}} \sigma_i$, we see that $u_n \le \rho w_n$ on $P \backslash \sigma_n$. On the other hand, since $w_n \le \varepsilon_n/(4M_n + 1) \le \varepsilon_n$ on ∂V_n and thus

$$h = \sum_{n \in \mathbf{N}} u_n \le \sum_{n \in \mathbf{N}} \varepsilon_n \rho = \rho^2$$

on $\partial(P \backslash \bigcup_{i \in \mathbf{N}} V_i)$ and also on $\partial S_n = \partial V_n$ for every $n \in \mathbf{N}$. In view of $h = 0$ on δP and also on the ideal harmonic boundary part of S_n, i.e. δT_n, for every $n \in \mathbf{N}$, we see that $h \le \rho^2$ on $P \backslash \bigcup_{i \in \mathbf{N}} V_i$ and also on S_n for every $n \in \mathbf{N}$. This finally assures that $h \le \rho^2$ on R. By the same method, starting from $h \le \rho^2$ on R, we can deduce that $h \le \rho^3$ on R. Repeating the same procedure we can show that

$$0 \le h \le \rho^k \quad (k \in \mathbf{N})$$

on R. On letting $k \uparrow \infty$ in the above inequality and recalling $\rho \in (0, 1)$, we deduce $h = 0$ identically on R as desired. $\square$

7. Conclusion

Take the Sario or Toki surface S already referred to in Section 3, which is a Riemann surface in the class $\mathcal{O}_{HP} \backslash \mathcal{O}_G$. Let

$$P = S, \quad T_n = S \quad (n \in \mathbf{N}). \tag{7.1}$$

We form the afforested surface $\hat{S} = \langle P, (T_n)_{n \in \mathbf{N}} \rangle$ by using the special plantation P and trees $(T_n)_{n \in \mathbf{N}}$ given by (7.1) with roots σ_n of trees T_n satisfying the condition (4.4). By our main theorem, noting (3.1), we can conclude that

$$\hat{S} \in \mathcal{O}_s. \tag{7.2}$$

Concerning the Wiener harmonic boundaries $\delta\hat{S}$, δP, and δT_n $(n \in \mathbf{N})$, respectively, of Riemann surfaces $\hat{S}$, P, and T_n $(n \in \mathbf{N})$, respectively, we have

$$\delta\hat{S} \supset (\delta P) \cup (\textstyle\bigcup_{n \in \mathbf{N}} \delta T_n)$$

with $\delta P = \delta T_n = \delta S$, which consists of a single point. Therefore $\delta\hat{S}$ consists of infinitely many points. In general the linear space $HB(W)$ over a Riemann surface W is of finite dimension m if and only if the Wiener harmonic boundary δW consists of finite m points (cf. [7]). Therefore we have $\dim HB(\hat{S}) = \infty$. By the Parreau decomposition, (7.2) is equivalent to $HP(\hat{S}) = HB'(\hat{S}) \supset HB(\hat{S})$ and a fortiori

$$\dim HP(\hat{S}) = \dim HB'(\hat{S}) = \dim HB(\hat{S}) = \infty. \tag{7.3}$$

Thus we can conclude that $\hat{S}$ is a concrete example of a Riemann surface of infinite harmonic dimension carrying no singular essentially positive harmonic functions on it.

References

[1] L. V. Ahlfors and L. Sario, Riemann surfaces, Princeton Univ. Press, Princeton, 1960.
[2] S. Axler, P. Bourdon and W. Ramey, Harmonic function theory, Graduate texts in mathematics **137**, 2nd ed., Springer-Verlag, 2001.
[3] C. Constantinescu und A. Cornea, Ideale Ränder Riemannscher Flächen, Ergebnisse der Mathematik und ihre Grenzgebite **32**, Springe-Verlag, 1963.
[4] F.-Y. Maeda, Dirichlet integrals on harmonic spaces, Lecture notes in mathematics **803**, Springer-Verlag, 1980.
[5] H. Masaoka and S. Segawa, On several classes of harmonic functions on a hyperbolic Riemann surface, Complex analysis and its applications, proceeding of the 15th ICFIDCAA Osaka 2007, OCAMI studies **2** (2008), 289–294.
[6] M. Parreau, Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann, Ann. Inst. Fourier (Grenoble) **3** (1951/1952), 103–197.
[7] L. Sario and M. Nakai, Classification theory of Riemann surfaces, Grudlehlen der mathematischen Wissenschaften in Einzeldarstellungen **164**, Springer-Verlag, Berlin, 1970.

Mitsuru Nakai
Professor Emeritus at:
Department of Mathematics
Nagoya Institute of Technology
Gokiso, Showa, Nagoya 466-8555
Japan
E-mail: nakai@daido-it.ac.jp

Mailing Address:
52 Eguchi, Hinaga
Chita 478-0041
Japan

Shigeo Segawa
Department of Mathematics
Daido Institute of Technology
Takiharu, Minami, Nagoya 457-8530
Japan
E-mail: segawa@daido-it.ac.jp