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ON PROPERTY OF COMPLEMENTS OF AN ALGEBRAIC CURVE WITH AT LEAST 4 IRREDUCIBLE COMPONENTS IN $\mathbf{P}^2$

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Abstract

For the manifold $M := \mathbf{P}^2 - A(l)$ $(l \geq 4)$ where $A(l)$ is an algebraic curve with l irreducible components, the notion that M is of log general type, measure hyperbolic and Δ_M is a curve or empty set, where Δ_M is the degeneracy locus of the Kobayashi pseudodistance d_M on M, coincide with each other.

0. Introduction

In [7] and [10], the little Picard theorem, that is, the holomorphic map of $\mathbf{C}^k$ $(k \geq 1)$ to above M is algebrically degenerate always, was proved as a special case.

In Theorems 2 and 3 in [4], the Montel theorem was generalized for M, that is, there are only two cases such as (a): M is tautly imbedded modulo some curve S in $\mathbf{P}^2$ (then M is hyperbolically imbedded modulo S in $\mathbf{P}^2$) or (b): there exists a holomorphic rational function f on M such that all irreducible components of every lebel curve of f are holomorphically isomorphic to either $\mathbf{C}$ or $\mathbf{C}^*$.

In [5] and [3], for an arbitrary complex manifold N, Δ_N (for its definition, see Proposition 1.2) is a pseudoconcave set of order 1 and the same for $S_N(X)$ where $S_N(X)$ is the degeneracy locus of limiting d_N to $\overline{N}$, which is compact in the manifold X (precisely, see Definition 1.1 and 1.3).

So, in the case $M = \mathbf{P}^2 - A(l)$ $(l \geq 4)$, if $S_M(\mathbf{P}^2)$ or Δ_M is contained in a curve, it is a curve or empty set because it is a pseudoconcave set (compliment of the set is a pseudoconvex set) in the two dimensional case. Then we can make clear the above case (a) to (a)$'$, that is, M is hyperbolically imbedded modulo $S_M(\mathbf{P}^2)$ which is a curve or empty set.

In [2], the notion that M is tautly imbeded modulo $S_M(\mathbf{P}^2)$ in $\mathbf{P}^2$ and M is hyperbolically imbedded modulo $S_M(\mathbf{P}^2)$ in $\mathbf{P}^2$ coincide with each other when $S_M(\mathbf{P}^2)$ is a curve or empty set.

In this paper, we prove that the notion that M is of log general type, measure hyperbolic and hyperbolically imbedded modulo $S_M(\mathbf{P}^2)$ in $\mathbf{P}^2$ coincide with each other when $S_M(\mathbf{P}^2)$ is a curve or empty set.

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1. Preliminaries

Let X be a connected complex manifold and M a relatively compact domain in X. We denote by d_M the Kobayashi pseudodistance on M. For the precise definition of d_M, see [9, p. 50]. For $p, q \in \overline{M}$, we define

$$\bar{d}_M(p,q) = \liminf_{p' \to p,\, q' \to q} d_M(p', q'), \quad p', q' \in M.$$

For properties of $\bar{d}_M$, see [2, p. 386].

Definition 1.1 (Definition 1.1 in [2]). Let $S_M(X) = \{p \in \overline{M}$ such that there exists some $q \in \overline{M} - \{p\}$ such as $\bar{d}_M(p,q) = 0\}$. We call $S_M(X)$ the degeneracy locus of $\bar{d}_M$ in X.

It is easy to see the following:

Proposition 1.2. *$S_M(X) \cap M = \Delta_M := \{p \in M$ such that there exists some $q \in M - \{p\}$ such as $d_M(p,q) = 0\}$.*

Definition 1.3 (cf. [13] and [6]). A closed set E of X is called a pseudoconcave set of order 1, if for any coordinate neighborhood

$$U : |z_1| < 1, \ldots, |z_n| < 1$$

of X and psitive numbers r, s with $0 < r < 1$, $0 < s < 1$ such that $U^* \cap E = \emptyset$, one obtains $U \cap E = \emptyset$, where

$$U^* = \{p \in U; |z_1(p)| \le r\} \cup \left\{ p \in U; s \le \max_{2 \le i \le n} |z_i(p)| \right\}.$$

Theorem 1.4 (Theorem 2 in [5] and Theorem 1.12 in [3]). *The sets $S_M(X)$ is a pseudoconcave set of order 1 in X, and Δ_M is the same in M.*

Theorem 1.5 (Theorem 8.1 in [1]). *For $M = \mathbf{P}^2 - A(l)$ $(l \ge 4)$, there are only two cases.*
(a) *$S_M(\mathbf{P}^2)$ is a curve of $\mathbf{P}^2$ or empty set.*
(b) *$S_M(\mathbf{P}^2) = \mathbf{P}^2$.*

Definition 1.6 (Definition 6.1 in [1]). We call f a rational holomorphic function of $\mathbf{C}$-type (resp. $\mathbf{C}^*$-type) on $\mathbf{P}^2 - A$ if f is a rational function on $\mathbf{P}^2$ and normalization of every irreducible component of all level curves of f except finite number of them is holomorphically isomorphic to $\mathbf{C}$ (resp. $\mathbf{C}^*$) on $\mathbf{P}^2 - (A \cup I_f)$, where A is a curve of $\mathbf{P}^2$ or empty set and I_f is the set of indeterminacy points of f.

Further, we call f a primitive rational function on $\mathbf{P}^2 - A$ if almost all level curves are irreducible except finite ones.

Proposition 1.7 (see Propositions 6.3, 6.4, 6.5 and 8.2 in [1]). *In the case* (b) *of Theorem* 1.5, *there exists a primitive rational holomorphic function f of* $\mathbf{C}$ *or* $\mathbf{C}^*$*-type on* $\mathbf{P}^2 - A(l)$ $(l \geq 4)$ *with lacunary three points. Namely,* $A(l)$ *is the sum of several irreducible components of level curves of f which is a rational function of* $\mathbf{C}$ *or* $\mathbf{C}^*$*-type on* $\mathbf{P}^2$ *and of* $\mathbf{C}$ *or* $\mathbf{C}^*$*-type on* $\mathbf{P}^2 - (A(l) \cup I_f)$, *or* $A(l)$ *is the sum of several irreducible components of the level curves of f which is a rational function of* $\mathbf{C}$*-type on* $\mathbf{P}^2$ *and an irreducible curve of genus* 0 *of* $\mathbf{P}^2$ *such that f is of* $\mathbf{C}^*$*-type on* $\mathbf{P}^2 - (A(l) \cup I_f)$.

We shall consider a complex manifold X of dimension n which has a compactification. According to Hironaka, there is a smooth compactification $\overline{X}$ of X. Namely, $\overline{X}$ is a compact complex manifold of dimension n and $D = \overline{X} - X$ is a divisor which has at most normal crossings. According to F. Sakai [11, p. 245] we can define logarithmic Kodaira dimension $\bar{\kappa}(X)$ of X.

Definition 1.8. If $\bar{\kappa}(X) = n$, X is called a manifold of log general type.

Proposition 1.9 (Proposition 2.4 in [11]). *Let X be a complex manifold with a smooth compactification $\overline{X}$. Let $f : \overline{X} \to \overline{Y}$ be a surjective holomorphic map, where $\overline{Y}$ is a compact complex manifold. Then for a general point y in $f(X)$, we have*

$$\bar{\kappa}(X) \leq \bar{\kappa}(X \cap f^{-1}(y)) + \dim \overline{Y}.$$

Proposition 1.10 (Proposition 1.1 in [11]). *Let X, Y be complex manifolds of dimension n such that $X \subset Y$. Then $\bar{\kappa}(X) \geq \bar{\kappa}(Y)$.*

Definition 1.11 (cf. [8, p. 117]). Let X be a complex manifold. Given a Borel subset Ξ in X, choose holomorphic maps $f_i : \Delta^n \to X$ where Δ^n is the unit polydisk and Borel subsets Ξ_i in Δ^n, such that $\Xi \subset \bigcup_i f_i(\Xi_i)$. Define

$$\mu_X(\Xi) = \inf \sum_i \int_{\Xi_i} V,$$

where the infimum is taken over all possible choices of f_i, Ξ_i, and V is the Poincaré volume form on Δ^n. We say X is measure hyperbolic if $\mu_X(\Xi) > 0$ for all non empty open subsets Ξ in X.

Proposition 1.12 (Theorem (7.1.4) in [9]). *If there exists a nondegenerate holomorphic map $f : \Delta^{n-1} \times \mathbf{C} \to X$, then X is not measure hyperbolic.*

2. Main results

Throughout this section, the manifold $M = \mathbf{P}^2 - A(l)$, where $A(l)$ is an algebraic curve with l $(l \geq 4)$ irreducible components.

Proposition 2.1 (cf. Conjecture 4 in [9, p. 80]). *If* $\bar{\kappa}(M) = 2$, *then* M *is in the case* (a) *of Theorem* 1.5.

Proof. We shall prove that if M is in the case (b) of Theorem 1.5, then $\bar{\kappa}(M) \leq 1$. By blowing up in finite times, there is a smooth compactification of $\overline{M}$ and f of Theorem 1.5 is extended to $\bar{f} : \overline{M} \to \mathbf{P}^1$ which is a surjective holomorphic map. For a general point y in $\mathbf{P}^1$, $X \cap \bar{f}^{-1}(y)$ is an irreducible curve which is holomorphically isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$. By Proposition 1.9, $\bar{\kappa}(M) \leq 1$. □

Proposition 2.2. *If* M *is in the case* (a) *of Theorem* 1.5, *then* $\bar{\kappa}(M) = 2$.

Proof. By blowing up in finite times, there is a smooth compactification $\overline{M}$ and $\overline{M} - M = C$ is a semi-stable curve. For the definition of semi-stable curve, see [12, p. 90]. By checking the tables 1 and 2 in [12, p. 90] and more pricisely, Theorem (2.7) and (3.15) in [12], in the case where $\overline{M}$ is a rational surface, we see $\bar{\kappa}(M) \leq 1$ only when M is in the case (b) of Theorem 1.5. Therefore, the above proposition is proved. □

By Corollary (7.2.12) in [9], we have the following proposition.

Proposition 2.3. *If* M *is in the case* (a) *of Theorem* 1.5, M *is measure hyperbolic.*

Proposition 2.4. *If* M *is measure hyperbolic*, M *is in the case* (a) *of Theorem* 1.5.

Proof. We shall prove that if M is in the case (b) of Theorem 1.5, then M is not mesaure hyperbolic. By blowing up in finite times, there is a smooth compactification $\overline{M}$ and there is a primitive rational holomorphic function f of $\mathbf{C}$ or $\mathbf{C}^*$-type on M. If f is of $\mathbf{C}$-type, we can take a neighborhood δ in $f(M)$ such that $f^{-1}(y)$ is holomorphically isomorphic to $\mathbf{C}$, where $y \in \delta$. By well known Nishino's theorem, $f^{-1}(\delta)$ is biholomorphic to $\delta \times \mathbf{C}$. So there exists a nondegenerate holomorphic map $g : \Delta \times \mathbf{C} \to M$. By Proposition 1.12, M is not measure hyperbolic.

If f is of $\mathbf{C}^*$-type, there are two cases. The one is that f is a rational function of $\mathbf{C}^*$-type on $\mathbf{P}^2$ and A is the sum of level curves of f. The other is that f is a rational function of $\mathbf{C}$-type on $\mathbf{P}^2$ and except A_1 such as one of the irreducible components of A, A is level curves of f and A_1 is a curve of genus 0 of $\mathbf{P}^2$ such that f is of $\mathbf{C}^*$-type on $\mathbf{P}^2 - A$.

In the former case, we can take a neighborhood δ in $f(M)$ such that $f^{-1}(y)$ is holomorphically isomorphic to $\mathbf{C}^*$ for every $y \in \delta$. By Theorem 3 in [14], $f^{-1}(\delta)$ is biholomorphic to $\delta \times \mathbf{C}^*$. Since there exists a nondegenerate holomorphic map of $\delta \times \mathbf{C}$ to $\delta \times \mathbf{C}^*$, there exsits a nondegenerate holomorphic map $h : \Delta \times \mathbf{C} \to M$. By Proposition 1.12, M is not measure hyperbolic.

In the latter case, it is easy to see that there is a neighborhood δ in A_1 such that every $f^{-1}(y)$ is holomorphicaly isomorphic mutually to $\mathbf{C}^*$ for every $y \in \delta$. By the same reason of the former case, M is not measure hyperbolic. □

From Propositions 2.1 through 2.4 and Remark on Theorem 4.4 in [2], we conclude the following

THEOREM 2.5. *For* $M = \mathbf{P}^2 - A(l)$ $(l \geq 4)$ *such that* $S_M(\mathbf{P}^2)$ *is a curve or empty set, the following notions coincide with each other*:
(1) M *is hyperbolically imbedded modulo* $S_M(\mathbf{P}^2)$ *in* $\mathbf{P}^2$.
(2) M *is tautly imbedded modulo* $S_M(\mathbf{P}^2)$ *in* $\mathbf{P}^2$.
(3) M *is of log general type.*
(4) M *is measure hyperbolic.*

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