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A NEW CHARACTERIZATION OF COLLECTIONS OF TWO-POINT SETS WITH THE UNIQUENESS PROPERTY

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Abstract

We give a necessary and sufficient condition for a collection of two-point sets to have the uniqueness property for meromorphic functions.

1. Introduction

In the previous paper ([OS]) the author and his collaborator give a sufficient condition for a collection of two-point sets to have the uniqueness property for meromorphic functions. However it is not seemed to be necessary.

For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a finite set S in $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ (or $\boldsymbol{C}$), we write $f^*(S) = g^*(S)$ if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where $f - \infty$ and $g - \infty$ mean that of $1/f$ and $1/g$, respectively.

Let $\mathscr{A} = \{S_1, \dots, S_q\}$ be a finite collection of pairwise disjoint finite sets in $\hat{\boldsymbol{C}}$. If $f^*(S_j) = g^*(S_j)$ $(1 \le j \le q)$ imply $f = g$ for two nonconstant meromorphic functions f and g on $\boldsymbol{C}$, then the collection $\mathscr{A}$ is said to have the *uniqueness property for meromorphic functions* (abbreviated to UPM). As an example of such collections, we know Nevanlinna's four values theorem ([N]):

Theorem A. *Let f and g be two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ and a_j $(j = 1, \dots, 4)$ four distinct point of $\hat{\boldsymbol{C}}$. If $f^*(\{a_j\}) = g^*(\{a_j\})$ $(j = 1, \dots, 4)$, then f is a Möbius transformation of g and two of a_j (say a_3, a_4) are exceptional values of f and g, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

The author and his collaborator showed in [OS]

Theorem B. *Let $q \ge 6$ be an integer and $\mathscr{A} = \{S_1, \dots, S_q\}$ a collection of pairwise disjoint two-point sets. Assume that there is no Möbius transformation T*

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such that $T(\xi_j) = \eta_j$ *and* $T(\eta_j) = \xi_j$ *for three distinct* j*'s, where* $S_j = \{\xi_j, \eta_j\}$. *Then the collection has UPM.*

Let $\mathscr{A} = \{S_j\}_{j=1}^q$ be a collection of pairwise disjoint sets in $\hat{\boldsymbol{C}}$, and T a Möbius transformation. Put $S_0 := \hat{\boldsymbol{C}} \backslash (\bigcup_{j=1}^q S_j)$. If z_0 and $T(z_0)$ do not belong to the same S_j $(0 \leq j \leq q)$, we call z_0 *a wandering point* of T for $\mathscr{A}$.

The aim of this paper is to give a necessary and sufficient condition for a collection of two-point sets to have the uniqueness property which is expressed by wandering points, and so we prove

Theorem 1.1. *Let* $q \geq 6$ *be an integer and* $\mathscr{A} = \{S_1, \ldots, S_q\}$ *a collection of pairwise disjoint two-point sets in* $\hat{\boldsymbol{C}}$. *Assume that there is no Möbius transformation* T *except the identity with at most two wandering point for* $\mathscr{A}$. *Then* $\mathscr{A}$ *has UPM.*

We assume that the reader is familiar with the standard notations and results of the value distribution theory (see, e.g., [C], [H], [R]). In particular, we use symbols $T(r, f)$ and $T_f(r)$ as characteristic functions of a meromorphic function f on $\boldsymbol{C}$ and a holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$, respectively, and moreover, we express by $S(r, f)$ and $S_f(r)$ quantities such that $\lim_{r\to\infty, r\notin E} S(r, f)/T(r, f) = 0$ and $\lim_{r\to\infty, r\notin E} S_f(r)/T_f(r) = 0$, respectively, where E is a subset of $(0, \infty)$ with finite linear measure and it is variable in each cases.

2. Fundamental properties

Let $\mathscr{A} = \{S_1, \ldots, S_q\}$ be a collection of pairwise disjoint finite sets of $\hat{\boldsymbol{C}}$. We denote fundamental properties of wandering points and UPM, and it is easy to show them.

- **(P1)** Any Möbius tranformation does not have only one wandering point for $\mathscr{A}$.
- **(P2)** If a Möbius transformation T not the identity has no wandering point for $\mathscr{A}$, then $\mathscr{A}$ does not have UPM by considering g and $f = T(g)$, where g is an arbitrary nonconstant meromorphic function.
- **(P3)** If a Möbius transformation T has only two wandering points w_1 and w_2 for $\mathscr{A}$, then $\mathscr{A}$ does not have UPM by considering g and $f = T(g)$, where g is a nonconstant meromorphic function with two exceptional values w_1 and w_2.

By (P1), (P2) and (P3), it is necessary for $\mathscr{A}$ to have UPM that there exists no Möbius transformation except the identity with at most two wandering points for $\mathscr{A}$.

(P4) If a subcollection of $\mathscr{A}$ has UPM, then $\mathscr{A}$ has UPM.

Let $\mathscr{B} = \{S_1', \ldots, S_p'\}$ be another collection. If there exists partition $I_1 \cup \cdots \cup I_k = \{1, \ldots, p\}$ such that $\bigcup_{j\in I_t} S_j' = S_t$, $1 \leq t \leq k$, then we call $\mathscr{B}$ a refinement of $\mathscr{A}$.

(P5) If a refinement of $\mathscr{A}$ does not have UPM, then neither does $\mathscr{A}$.
We also prove

Lemma 2.1. *Let $\mathscr{A} = \{S_1, \dots, S_q\}$ be a collection of pairwise disjoint finite sets in $\hat{\boldsymbol{C}}$. Let g a nonconstant meromorphic function on $\boldsymbol{C}$ and T a Möbius transformation not the identity. Put $f = T(g)$. If $f^*(S_j) = g^*(S_j)$, $1 \le j \le q$, then T has at most two wandering points for $\mathscr{A}$.*

Proof. Take any $w_0 \in \hat{\boldsymbol{C}}$. If there exists a point z_0 such that $g(z_0) = w_0$, then $f(z_0) = T(w_0)$. Since $g(z_0)$ and $f(z_0)$ belong to the same S_j $(0 \le j \le q)$ by assumption, where $S_0 = \hat{\boldsymbol{C}} \backslash (\bigcup_{j=1}^q S_j)$, w_0 is not a wandering point of T for $\mathscr{A}$. Hence by the little Picard theorem, T has at most two wandering point for $\mathscr{A}$. $\square$

3. Preliminaries from the value distribution theory

In this section we denote the results from the value distribution theory which are used in the next section.

Lemma 3.1 (for the proof, see [S]). *Let f be a nonconstant meromorphic function on $\boldsymbol{C}$ and a_j $(1 \le j \le q)$ distinct points in $\hat{\boldsymbol{C}}$. If all the zeros of $f - a_j$ have the multiplicities at least m_j for each j, where m_j are arbitrarily fixed positive integers, then*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) \le 2.$$

For a nonconstant meromorphic function f and a point $a \in \hat{\boldsymbol{C}}$ we call a a completely ramified value of f if all zeros of $f - a$ have multiplicities greater than one.

Corollary 3.2. (i) *Each nonconstant entire function has at most two finite completely ramified values.*

(ii) *Each nonconstant entire function without zero has no completely ramified values except zero and ∞.*

Lemma 3.3 ([LY, Lemma 7]). *Let f_1 and f_2 be two nonconstant meromorphic functions on $\boldsymbol{C}$ satisfying*

$$\overline{N}(r, f_j) + \overline{N}(r, 1/f_j) = S(r), \quad j = 1, 2.$$

If $f_1^m f_2^n \not\equiv 1$ for all nonzero integers m and n, then for any $\varepsilon > 0$, we have

$$\overline{N}(r, 1; f_1, f_2) \le \varepsilon T(r) + S(r).$$

Here $\overline{N}(r, 1; f_1, f_2)$ denotes the counting function of common zeros $f_1 - 1$ and $f_2 - 1$ counted once and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r) = o(T(r))$ as $r \to \infty$ outside some set of r with finite linear measure.

The following is a degeneracy theorem of holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{P}^2(\boldsymbol{C})$.

THEOREM 3.4. *Let $f = (f_0 : f_1 : f_2)$ be a linearly nondegenerate holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{P}^2(\boldsymbol{C})$. Assume that all f_j are entire functions without zeros and that*

$$N_{(2}(r, 1/F) \geq (1-\varepsilon)T_f(r) + S_f(r)$$

for any $\varepsilon > 0$, where $F := f_0 + f_1 + f_2$ and $N_{(p}(r, 1/F)$ is the counting function of zero of F with multiplicity greater than or equal to p for positive integer p. Then $f_{j_1} f_{j_2}/f_{j_0}^2$ is constant for some j_0, j_1, j_2 with $\{j_0, j_1, j_2\} = \{0, 1, 2\}$.

Proof. By replacing f_j by f_j/f_0 and the conclusion by that one of $f_1 f_2$, f_2^2/f_1, f_1^2/f_2 is constant, we may assume that $f_0 \equiv 1$.

Define meromorphic functions g_1 and g_2 by

$$g_1 = -f_1 \cdot \frac{(f_2'/f_2) - (f_1'/f_1)}{f_2'/f_2}, \quad g_2 = f_2 \cdot \frac{(f_2'/f_2) - (f_1'/f_1)}{f_1'/f_1}$$

which are nonconstant since f is linearly nondegenerate. If $F(z_0) = F'(z_0) = 0$, then $g_1(z_0) = g_2(z_0) = 1$. Hence we get

$$\overline{N}_{(2}(r, 1/F) \leq \overline{N}(r, 1; g_1, g_2), \tag{3.1}$$

where $\overline{N}_{(p}(r, 1/F)$ is the counting function of zero of F with multiplicity greater than or equal to p counted once for positive integer p. Since all f_j have no zeros, we have, by H. Cartan's second main theorem and the first main theorem,

$$T_f(r) = N^2(r, 1/F) + S_f(r),$$

where $N^2(r, 1/F)$ is the counting function of zero of F with multiplicity truncated by 2. On the other hand $N_{(3}^2(r, 1/F) = S_f(r)$ since it counts twice some of zeros of the function

$$\begin{vmatrix} f_1'/f_1 & f_2'/f_2 \\ f_1''/f_1 & f_2''/f_2 \end{vmatrix}$$

and f_1 and f_2 are entire functions without zeros. Hence, by assumption, we have for any $\varepsilon > 0$

$$2\overline{N}_{(2}(r, 1/F) = N_{(2}(r, 1/F) + S_f(r) \geq (1-\varepsilon)T(r, f) \geq \frac{1-\varepsilon}{2}T(r) + S_f(r),$$

where $T(r) = T(r, f_1) + T(r, f_2)$. By this inequality and (3.1) we have

$$\overline{N}(r, 1; g_1, g_2) \geq \frac{1-\varepsilon}{4}T(r) + S_f(r).$$

Since f_1 and f_2 are nonconstant entire functions without zeros, we have

$$\overline{N}(r, g_j) + \overline{N}(r, 1/g_j) = o(T(r)) \quad (j = 1, 2).$$

Note $T(r, g_1) + T(r, g_2) = (1 + o(1))T(r)$ and apply Lemma 3.3, then there exist two nonzero integers m and n such that $g_1^m g_2^n = 1$.

Now put $\Phi = \dfrac{f_1'/f_1}{f_2'/f_2}$, then

$$f_1^m f_2^n = \frac{(-1)^m}{\Phi^m (1 - 1/\Phi)^{m+n}}. \tag{3.2}$$

Since the lefthand side is an entire function without zero, so is the righthand side.

If $m + n = 0$, then $f_1/f_2 = -\omega/\Phi$, where ω is an m-th root of one. We get $f_1 = -\omega f_2 + C$ for some constant C, which contradicts to the linear non-degeneracy of f.

Now $m \neq 0$, $n \neq 0$, $m + n \neq 0$, and Φ omits three values 0, 1 and ∞ by (3.2). Hence Φ is constant. So

$$f_1^m f_2^n = C, \tag{3.3}$$

where C is a nonzero constant and we may assume that m and n are relatively prime and that $m + n > 0$. Take integers s and t such that $ms - nt = 1$ and put $\Psi = f_1^t f_2^s$, which is an entire function without zeros. Then by (3.3)

$$\Psi^{-n} = \frac{1}{C^s} f_1, \quad \Psi^m = C^t f_2$$

which shows that Ψ is not constant, and

$$F = 1 + C^s \Psi^{-n} + \frac{1}{C^t}\Psi^m = \Psi^{-n}\left(C^s + \Psi^n + \frac{1}{C^t}\Psi^{m+n}\right),$$

$$F' = \Psi' \cdot \Psi^{-n-1}\left(-nC^s + \frac{m}{C^t}\Psi^{m+n}\right).$$

The multiple zeros of F except those of Ψ' are those of $C^s + \Psi^n + \dfrac{1}{C^t}\Psi^{m+n}$ and of $-nC^s + \dfrac{m}{C^t}\Psi^{m+n}$ and hence those of $1 + \dfrac{1}{C^t}\left(\dfrac{m}{n} + 1\right)\Psi^m$ and $1 + C^s\left(1 + \dfrac{n}{m}\right)\Psi^{-n}$. Therefore if $F(z_0) = F'(z_0) = 0$ and $\Psi'(z_0) \neq 0$, then $\Psi(z_0)^n = -C^s \dfrac{m+n}{m}$, $\Psi(z_0)^m = -C^t \dfrac{n}{m+n}$ and hence $\Psi(z_0) = (-1)^{s-t} \dfrac{n^s m^t}{(m+n)^{s+t}}$. For any solution w_0 of $C^s + X^n + \dfrac{1}{C^t} X^{m+n} = 0$ there exists a point z_0 such that $\Psi(z_0) = w_0$, $F'(z_0) = 0$ and $\Psi'(z_0) \neq 0$ by assumption. Hence the equation $C^s + X^n + \dfrac{1}{C^t} X^{m+n} = 0$ has only one solution. We can induce that $(m, n) = (1, 1), (2, -1), (-1, 2)$. This implies the conclusion of the theorem. $\square$

The following collorary precises a part of Theorem 7.2 of [NWY].

Corollary 3.5. *Let f_0, f_1, f_2 be entire functions without zeros. Assume that f_0, f_1, f_2 are linearly independent over $\boldsymbol{C}$ and that $F := f_0 + f_1 + f_2$ has no simple zero. Then $f_{j_1} f_{j_2} / f_{j_0}^2$ is constant for some j_0, j_1, j_2 with $\{j_0, j_1, j_2\} = \{0, 1, 2\}$.*

4. Proof of Theorem 1.1

It suffices to treat the case of $q = 6$ by (P4) in §2. Assume that $f^*(S_j) = g^*(S_j)$ for two nonconstant meromorphic functions f and g. We may write $f = f_1/f_0$ by entire functions f_0, f_1 without common zeros, and $g = g_1/g_0$ in a similar manner. Also, we put $S_j = \{\xi_j, \eta_j\}$, $1 \leq j \leq 6$.

By Theorem B, it is enough to consider the case where there exists a Möbius transformation exchanging three pairs of $\mathscr{A}$. By renumbering of S_j and by considering a suitable Möbius transformation, we may assume that $\eta_j = -\xi_j$, $j = 1, 2, 3$, where $\xi_j \neq 0, \infty$.

Then, by the assumption $f^*(S_j) = g^*(S_j)$, there are entire functions α_j without zeros such that

$$f_1^2 + b_j f_0^2 = \alpha_j (g_1^2 + b_j g_0^2), \quad j = 1, 2, 3, \tag{4.1}$$

where $b_j = -\xi_j^2$. Note that b_1, b_2, b_3 are distinct nonzero values.

It follows that

$$\begin{aligned} g^2 &\{(b_3 - b_2)\alpha_1 + (b_1 - b_3)\alpha_2 + (b_2 - b_1)\alpha_3\} \\ &= -\{(b_3 - b_2) b_1 \alpha_1 + (b_1 - b_3) b_2 \alpha_2 + (b_2 - b_1) b_3 \alpha_3\}, \\ f^2 &\{(b_3 - b_2)\alpha_2\alpha_3 + (b_1 - b_3)\alpha_3\alpha_1 + (b_2 - b_1)\alpha_1\alpha_2\} \\ &= -\{(b_3 - b_2) b_1 \alpha_2\alpha_3 + (b_1 - b_3) b_2 \alpha_3\alpha_1 + (b_2 - b_1) b_3 \alpha_1\alpha_2\}. \end{aligned}$$

If among four functions of the lefthand sides and the righthand sides above there is one which is identically zero, then we have $\alpha_1 = \alpha_2 = \alpha_3$. In this case $f^2 = g^2$ is induced from (4.1), and we get $f = g$ by using Lemma 2.1.

Now we may assume that any of these are not identically zero. Hence, we have

$$g^2 = -\frac{(b_3 - b_2) b_1 \alpha_1 + (b_1 - b_3) b_2 \alpha_2 + (b_2 - b_1) b_3 \alpha_3}{(b_3 - b_2)\alpha_1 + (b_1 - b_3)\alpha_2 + (b_2 - b_1)\alpha_3}, \tag{4.2}$$

$$f^2 = -\frac{(b_3 - b_2) b_1 \alpha_2\alpha_3 + (b_1 - b_3) b_2 \alpha_3\alpha_1 + (b_2 - b_1) b_3 \alpha_1\alpha_2}{(b_3 - b_2)\alpha_2\alpha_3 + (b_1 - b_3)\alpha_3\alpha_1 + (b_2 - b_1)\alpha_1\alpha_2}. \tag{4.3}$$

For α_1, α_2, α_3, we consider three cases: (I) they are linearly dependent over $\boldsymbol{C}$; (II) they are linearly independent over $\boldsymbol{C}$ and $(\alpha_3/\alpha_1)^m (\alpha_2/\alpha_1)^n = 1$ for some nonzero constants m and n; (III) they are linearly independent over $\boldsymbol{C}$ and $(\alpha_3/\alpha_1)^m (\alpha_2/\alpha_1)^n \neq 1$ for any nonzero constants m and n.

The case (I).

If $\alpha_3 = c_1\alpha_1 + c_2\alpha_2$ with constants c_1 and c_2 either nonzero, then by substituting it into (4.2) we have

$$g^2 = -\frac{\{(b_3-b_2)b_1 + (b_2-b_1)b_3c_1\}\alpha_1 + \{(b_1-b_3)b_2 + (b_2-b_1)b_3c_2\}\alpha_2}{\{(b_3-b_2) + (b_2-b_1)c_1\}\alpha_1 + \{(b_1-b_3) + (b_2-b_1)c_2\}\alpha_2}. \tag{4.4}$$

If α_2/α_1 is constant, so is g. Hence it is nonconstant and by Corollary 3.2 it has no more completely ramified finite values except zero. It follows from (4.4) that

$$(b_3-b_2)b_1 + (b_2-b_1)b_3c_1 = (b_1-b_3) + (b_2-b_1)c_2 = 0 \tag{4.5}$$

or

$$(b_1-b_3)b_2 + (b_2-b_1)b_3c_2 = (b_3-b_2) + (b_2-b_1)c_1 = 0, \tag{4.6}$$

and

$$g^2 = -b_3\frac{\alpha_1}{\alpha_2} \tag{4.7}$$

or

$$g^2 = -b_3\frac{\alpha_2}{\alpha_1} \tag{4.8}$$

if (4.5) or (4.6) holds, respectively. By the same way from (4.3), we have

$$f^2 = -\frac{(b_1-b_3)b_2c_1\alpha_1^2 + \{(b_3-b_2)b_1c_1 + (b_1-b_3)b_2c_2 + (b_2-b_1)b_3\}\alpha_1\alpha_2 + (b_3-b_2)b_1c_2\alpha_2^2}{(b_1-b_3)c_1\alpha_1^2 + \{(b_3-b_2)c_1 + (b_1-b_3)c_2 + (b_2-b_1)\}\alpha_1\alpha_2 + (b_3-b_2)c_2\alpha_2^2},$$

and hence, by noting that none of four coefficients of α_1^2 and α_2^2 is zero,

$$\{(b_3-b_2)b_1c_1 + (b_1-b_3)b_2c_2 + (b_2-b_1)b_3\}^2 = 4(b_1-b_3)b_2c_1\cdot(b_3-b_2)b_1c_2,$$
$$\{(b_3-b_2)c_1 + (b_1-b_3)c_2 + (b_2-b_1)\}^2 = 4(b_1-b_3)c_1\cdot(b_3-b_2)c_2.$$

and

$$f = \pm\frac{1}{\xi_2}\frac{2(b_1-b_3)b_2c_1(\alpha_1/\alpha_2) + \{(b_3-b_2)b_1c_1 + (b_1-b_3)b_2c_2 + (b_2-b_1)b_3\}}{2(b_1-b_3)c_1(\alpha_1/\alpha_2) + \{(b_3-b_2)c_1 + (b_1-b_3)c_2 + (b_2-b_1)\}}.$$

Therefore f is a Möbius tranformation of g^2 by (4.7) or (4.8) and we have

$$T(r,f) = 2T(r,g) + O(1). \tag{4.9}$$

On the other hand, since f and g share S_j, $1 \le j \le 6$,

$$\begin{aligned} 10T(r,f) &\le \sum_{j=1}^{6}(N(r,1/(f-\xi_j)) + N(r,1/(f-\eta_j))) + S(r,f) \\ &= \sum_{j=1}^{6}(N(r,1/(g-\xi_j)) + N(r,1/(g-\eta_j))) + S(r,f) \\ &\le 12T(r,g) + S(r,f). \end{aligned}$$

This contradicts to (4.9) since $S(r, f)$ and $S(r, g)$ are not distinguished in this case.

The case (II).

Note that none of m, n and $m+n$ is zero since α_1, α_2, α_3 are linearly independent over $\boldsymbol{C}$. We may assume $m > 0$. For suitable m-th roots β_j of α_j, $j = 1, 2$, we have from (4.2)

$$g^2 = -\frac{(b_3 - b_2)b_1\beta_1^m + (b_1 - b_3)b_2\beta_2^m + (b_2 - b_1)b_3\beta_1^{m+n}\beta_2^{-n}}{(b_3 - b_2)\beta_1^m + (b_1 - b_3)\beta_2^m + (b_2 - b_1)\beta_1^{m+n}\beta_2^{-n}}$$

Let $d > 0$ be the greatest common divisor of m and n, and take integers p and q such that $m = dp$ and $n = dq$. Put $\gamma = (\beta_1/\beta_2)^d$, then

$$g^2 = -\frac{(b_3 - b_2)b_1\gamma^p + (b_1 - b_3)b_2 + (b_2 - b_1)b_3\gamma^{p+q}}{(b_3 - b_2)\gamma^p + (b_1 - b_3) + (b_2 - b_1)\gamma^{p+q}}. \tag{4.10}$$

Recall $p > 0$ and first assume that $p + q > 0$. Consider the denominator and the numerator as polynomials of γ. Obviously they have not the factor γ, and by Corollary 3.2 they must have only factors with even exponents except γ after reduction. Also, they have the common factor $\gamma - 1$ but it is not simultaneously multiple factor of both. Finally, each of them has at most one multiple factor and the exponents of multiple factors are two if they exist. After all, the right-hand side of (4.10) is written as

$$C\frac{(\gamma - 1)(\gamma - \lambda_1)^2}{(\gamma - 1)(\gamma - \lambda_2)^2}$$

before reduction, where λ_1, λ_2 and C are nonzero constants and $\lambda_1 \neq 1$, $\lambda_2 \neq 1$, $\lambda_1 \neq \lambda_2$. However, it is expanded as

$$C\frac{\gamma^3 - (2\lambda_1 + 1)\gamma^2 + \lambda_1(\lambda_1 + 2)\gamma - \lambda_1^2}{\gamma^3 - (2\lambda_2 + 1)\gamma^2 + \lambda_2(\lambda_2 + 2)\gamma - \lambda_2^2},$$

which induces a contradiction $\lambda_1 = \lambda_2$ since the coefficients of γ^2 or γ are zero by comparing this with the righthand side of (4.10). In the case of $p + q < 0$, we can also induce a contradiction by the same way as above.

The case (III).

In this case by Lemma 3.3, for any $\varepsilon > 0$, we have

$$\overline{N}(r, 1; f_1, f_2) \leq \varepsilon T(r) + S(r), \tag{4.11}$$

where $f_1 = \alpha_2/\alpha_1$, $f_2 = \alpha_3/\alpha_1$ and $T(r)$ and $S(r)$ are as in Lemma 3.3. If z_0 is a zero of $F := (b_3 - b_2) + (b_1 - b_3)f_1 + (b_2 - b_1)f_2$ and not common zero of $f_1 - 1$ and $f_2 - 1$, then it is a zero of the denominator of the righthand side of (4.2) and not a zero of the numerator. Hence z_0 is a multiple zero of F. So we have

$$N_{(2}(r, 1/F) \geq N(r, 1/F) - \overline{N}(r, 1; f_1, f_2). \tag{4.12}$$

By H. Cartan's second main theorem and the first main theorem

$$N(r,1/F)=T_\varphi(r)+S_\varphi(r),$$

and $T(r)=T(r,f_1)+T(r,f_2)\le 2T_\varphi(r)\le 2T(r)$ holds, where $\varphi:=((b_3-b_2):(b_1-b_3)f_1:(b_2-b_1)f_2)$. These and (4.11), (4.12) induce

$$N_{(2}(r,1/F)\ge T_\varphi(r)+S_\varphi(r)-(\varepsilon T(r)+S(r))\ge(1-2\varepsilon)T_\varphi(r)+S_\varphi(r).$$

Hence φ satisfies the assumption of Theorem 3.4. Therefore one of the followings holds: (i) $\alpha_3=c\alpha_2^2/\alpha_1$; (ii) $\alpha_2=c\alpha_1^2/\alpha_3$; (iii) $\alpha_1=c\alpha_3^2/\alpha_2$, where c is a nonzero constant.

The case (i).

The case of $\alpha_3=c\alpha_2^2/\alpha_1$, where $c\ne 0$ is a constant. By substituing this into (4.2) and (4.3),

$$g^2=-\frac{(b_3-b_2)b_1+(b_1-b_3)b_2(\alpha_2/\alpha_1)+(b_2-b_1)b_3c(\alpha_2/\alpha_1)^2}{(b_3-b_2)+(b_1-b_3)(\alpha_2/\alpha_1)+(b_2-b_1)c(\alpha_2/\alpha_1)^2},$$

$$f^2=-\frac{(b_3-b_2)b_1c(\alpha_2/\alpha_1)^2+(b_1-b_3)b_2c(\alpha_2/\alpha_1)+(b_2-b_1)b_3}{(b_3-b_2)c(\alpha_2/\alpha_3)^2+(b_1-b_3)c(\alpha_2/\alpha_1)+(b_2-b_1)}.$$

Since α_2/α_1 is not constant and b_1, b_2, b_3 are distinct nonzero constants, by Corollary 3.2 all denominators and numerators above have double roots as quadratic polynomials of α_2/α_1. So we have $b_2^2=b_1b_3$, $c=\dfrac{(b_1-b_3)^2}{4(b_2-b_1)(b_3-b_2)}=\dfrac{(b_2+b_1)^2}{4b_1b_2}$ and $c=\dfrac{4(b_2-b_1)(b_3-b_2)}{(b_1-b_3)^2}=\dfrac{4b_1b_2}{(b_2+b_1)^2}$. Hence $c=\pm 1$, but $c=1$ implies $b_1=b_2$ which is a contradiction. So $c=-1$, and we have

$$g^2=-\frac{b_1\left(b_2(\alpha_2/\alpha_1)+\dfrac{b_1+b_2}{2}\right)^2}{\left(b_1(\alpha_2/\alpha_1)+\dfrac{b_1+b_2}{2}\right)^2},\quad f^2=-\frac{b_2^2\left(b_1(\alpha_2/\alpha_1)-\dfrac{b_1+b_2}{2}\right)^2}{b_1\left(b_2(\alpha_2/\alpha_1)-\dfrac{b_1+b_2}{2}\right)^2}.$$

Therefore f is a Möbius transformation of g, and the conclusion $f=g$ follows from Lemma 2.1 and the assumption.

We can induce $f=g$ by the same way in the cases (ii) and (iii), and the proof is completed.

Problem. In Theorem 1.1 can we remove the condition $q\ge 6$? However q must be greater than or equal to 3.

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