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GENERALIZED KINKELIN'S FORMULAS

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1. Introduction

Around 150 years ago, Kinkelin [K] (Crelle J. **57** (1860), which was submitted in 1856 July) defined his generalized gamma function $G(x)$ as an integral of the logarithm of the usual gamma function $\Gamma(x)$:

$$G(x) = \exp\left(\int_0^x \log \Gamma(t)\, dt + \frac{x(x-1)}{2} - \frac{x}{2} \log(2\pi)\right).$$

See the formula (7) in [K, p. 124]. A motivation of Kinkelin seems to be the formula

$$\int_x^{x+1} \log \Gamma(t)\, dt = x \log x - x + \frac{1}{2} \log(2\pi)$$

due to Raabe [R] (Crelle J. **28** (1844)) as indicated in [K, p. 124]. Kinkelin proved basic properties such as

$$G(0) = G(1) = 1,$$
$$G(x+1) = G(x)x^x \quad \text{for } x > 0,$$

and

$$G(n+1) = 1^1 2^2 \cdots n^n \quad \text{for integers } n \geq 1.$$

Note that $G(1) = 1$ is equivalent to

$$\int_0^1 \log \Gamma(t)\, dt = \frac{1}{2} \log(2\pi)$$

coming from Raabe at $x = 0$, and that the formula $G(x+1) = G(x)x^x$ is equivalent to Raabe's formula. Moreover, Kinkelin calculated integrals containing trigonometric functions. Especially he obtained his famous formula

$$\text{(K)} \qquad \int_0^x \log(2 \sin \pi t)\, dt = \log \frac{G(1-x)}{G(x)}$$

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for $0 < x < 1$ (see the formula (24) in [K, p. 135]). We notice that this formula includes

$$\int_0^{1/2} \log(2 \sin \pi t)\, dt = 0$$

by setting $x = 1/2$, which is equivalent to the famous "tricky integral"

$$\int_0^{1/2} \log(\sin \pi t)\, dt = -\frac{1}{2} \log 2$$

or

$$\int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2} \log 2$$

of Euler [E] (p. 130). We notice that Euler's formula is equivalent to Raabe's

$$\int_0^1 \log \Gamma(t)\, dt = \frac{1}{2} \log(2\pi)$$

via the reflection formula

$$\sin(\pi t) = \frac{\pi}{\Gamma(t)\Gamma(1-t)}.$$

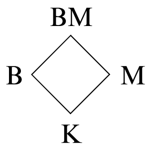
In 1987, Kinkelin's formula (K) is used to calculate the gamma factor of the Selberg zeta function of a Riemann surface by Sarnak [S] and Voros [V]. This is the case of a two-dimensional locally symmetric space, and we refer to [Ku] [KK] for the case of the Selberg zeta function of a general dimensional locally symmetric space.

Kinkelin's function $G(x)$ is the firstly discovered generalized gamma function. Around 1900, it was modified and extended by Barnes [B] to a general order gamma function. Unfortunately, a suitable generalization of Kinkelin's formula (K) seems to be missing still now. Moreover, Kinkelin [K, §1] indicates that his function $G(x)$ would be generalized to a more generalized gamma function $\overset{(k)}{\Gamma}(x)$ (Kinkelin's notation) satisfying

$$\begin{cases} \overset{(k)}{\Gamma}(1) = 1, \\ \overset{(k)}{\Gamma}(x+1) = \overset{(k)}{\Gamma}(x) x^{x^k} & \text{for } x > 0, \\ \overset{(k)}{\Gamma}(n+1) = 1^{1^k} 2^{2^k} \cdots n^{n^k} & \text{for integers } n \geq 1 \end{cases}$$

with $\overset{(0)}{\Gamma}(x) = \Gamma(x)$ and $\overset{(1)}{\Gamma}(x) = G(x)$. Later, Milnor [Mi] also suggested to study $\overset{(k)}{\Gamma}(x)$. In a recent paper [KOW] we described a theory to realize Milnor's suggestion.

The purpose of this paper is to give a generalization for Kinkelin's formula (K) together with a theory of the generalized gamma and the generalized sine function of BM (Barnes-Milnor) type suited to our generalized (K). This is a generalization of the gamma function of Barnes [B] and the gamma function of Milnor [Mi] (see [KOW]) both.



To construct our theory we start from the theory of the multiple Hurwitz zeta function

$$\zeta_r(s, x, (\omega_1, \dots, \omega_r)) = \sum_{n_1, \dots, n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s}$$

introduced by Barnes [B]. In this paper we restrict to the case $\omega_1, \dots, \omega_r > 0$ and $x \geq 0$ for simplicity. We recall that $\zeta_r(s, x, (\omega_1, \dots, \omega_r))$ converges absolutely in $\mathrm{Re}(s) > r$ and it has an analytic continuation to all $s \in \mathbf{C}$ as a meromorphic function. Frequently we omit $(1, \dots, 1)$, so we write

$$\zeta_r(s, x, (1, \dots, 1)) = \zeta_r(s, x)$$

simply. Barnes [B] (1904) defined the multiple gamma function

$$\Gamma_r(x, (\omega_1, \dots, \omega_r)) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, x, (\omega_1, \dots, \omega_r))|_{s=0}\right).$$

We refer Manin [Ma] for a survery. The case $r = 1$ reduces to the usual gamma function

$$\Gamma_1(x, \omega) = \frac{\Gamma\left(\frac{x}{\omega}\right)}{\sqrt{2\pi}}\omega^{x/\omega - 1/2}$$

by the formula of Lerch [L] (1894) since

$$\zeta_1(s, x, \omega) = \omega^{-s}\zeta\left(s, \frac{x}{\omega}\right)$$

for the usual Hurwitz zeta function

$$\zeta(s, x) = \sum_{n=0}^{\infty}(n + x)^{-s} = \zeta_1(s, x, 1).$$

We notice that Lerch's formula says

$$\frac{\partial}{\partial s}\zeta(s, x)|_{s=0} = \log\frac{\Gamma(x)}{\sqrt{2\pi}} = \log\Gamma_1(x).$$

Combining this formula with

$$\zeta(0,x)=\frac{1}{2}-x,$$

we get the above formula for $\Gamma_1(x,\omega)$. There exists an associated multiple sine function

$$S_r(x,(\omega_1,\dots,\omega_r))=\Gamma_r(x,(\omega_1,\dots,\omega_r))^{-1}\Gamma_r(\omega_1+\cdots+\omega_r-x,(\omega_1,\dots,\omega_r))^{(-1)^r}.$$

See [KK] for a general theory of the multiple sine function.

As noted before, Milnor [Mi] suggested to study the generalized gamma function

$$\exp\left(\frac{\partial}{\partial s}\zeta(s,x)|_{s=-k}\right)$$

for an integer $k\geq 0$ (see [KOW] for details). From our point of view, these generalized gamma functions of Barnes and of Milnor seem to be insufficient to describe fully the generalization of Kinkelin's formula (K). Consequently we investigate the further generalized gamma function

$$\Gamma_{r,k}(x,(\omega_1,\dots,\omega_r))=\exp\left(\frac{\partial}{\partial s}\zeta_r(s,x,(\omega_1,\dots,\omega_r))|_{s=-k}\right)$$

and sine function

$$\begin{aligned}&S_{r,k}(x,(\omega_1,\dots,\omega_r))\\&\quad=\Gamma_{r,k}(x,(\omega_1,\dots,\omega_r))^{-1}\Gamma_{r,k}(\omega_1+\cdots+\omega_r-x,(\omega_1,\dots,\omega_r))^{(-1)^r}\end{aligned}$$

of BM type. It will turn out that the generalized cotangent function

$$\mathrm{Cot}_{r,k}(x,(\omega_1,\dots,\omega_r))=(\log S_{r,k}(x,(\omega_1,\dots,\omega_r)))'.$$

is crucial.

We first reconstruct Kinkelin's theory from our view point:

Theorem 1. (1) $G(x)=\exp\left(\int_0^x\left(\log\Gamma_1(t)+\left(t-\frac{1}{2}\right)\right)dt\right)$.

(2) *$G(x)$ is characterized as*

$$\begin{cases}G(0)=1,\\ \dfrac{G'}{G}(x)=\log\Gamma_1(x)+x-\dfrac{1}{2}.\end{cases}$$

(3) $$\begin{aligned}G(x)&=\exp(\zeta'(-1,x)-\zeta'(-1))\\&=e^{-\zeta'(-1)}\Gamma_{1,1}(x).\end{aligned}$$

(4)

$$\begin{cases}G(0)=G(1)=1,\\ G(x+1)=G(x)x^x\quad \textit{for } x>0,\\ G(n+1)=1^1 2^2\cdots n^n\quad \textit{for an integer } n\geq 1.\end{cases}$$

(5) $G(x) = e^{-\zeta'(-1)}\Gamma_2(x)\Gamma_1(x)^{x-1}$.

(6) $\dfrac{G(1-x)}{G(x)} = S_2(x)S_1(x)^{x-1} = S_{1,1}(x)$.

(7)
$$\int_0^x \log(2\sin \pi t)\, dt = \log\left(\frac{G(1-x)}{G(x)}\right)$$

or
$$\int_0^x \log S_1(t)\, dt = \log S_{1,1}(x).$$

The following result shows that $\tilde{\Gamma}_{1,k}(x) = \Gamma_{1,k}(x)e^{-\zeta'(-k)}$ realizes the generalized gamma function $\overset{(k)}{\Gamma}(x)$ suggested by Kinkelin:

Theorem 2. (1) $\tilde{\Gamma}_{1,k}(1) = 1$.
(2) $\tilde{\Gamma}_{1,k}(x+1) = \tilde{\Gamma}_{1,k}(x)x^{x^k}$ *for* $x > 0$.
(3) $\tilde{\Gamma}_{1,k}(n+1) = 1^{1^k}2^{2^k}\cdots n^{n^k}$ *for an integer* $n \geq 1$.
(4) $\tilde{\Gamma}_{1,0}(x) = \Gamma(x)$ *and* $\tilde{\Gamma}_{1,1}(x) = G(x)$.

Kinkelin's construction of $G(x)$ is generalized as folllows:

Theorem 3. *For* $k \geq 1$, *we have*
$$\int_0^x \left(\log \Gamma_{r,k-1}(t,(\omega_1,\ldots,\omega_r)) - \frac{1}{k}\zeta_r(1-k,t,(\omega_1,\ldots,\omega_r))\right) dt$$
$$= \frac{1}{k}\log\frac{\Gamma_{r,k}(x,(\omega_1,\ldots,\omega_r))}{\Gamma_{r,k}(0,(\omega_1,\ldots,\omega_r))}.$$

In the simplest case where $r = k = \omega_1 = 1$, this is the starting point of Kinkelin (Theorem 1(1) above)
$$\int_0^x \left(\log \Gamma_1(t) - \left(\frac{1}{2} - t\right)\right) dt = \log G(x)$$

since
$$\Gamma_{1,0}(t,1) = \Gamma_1(t) = \frac{\Gamma(t)}{\sqrt{2\pi}},$$
$$\zeta_1(0,t,1) = \frac{1}{2} - t,$$
$$\Gamma_{1,1}(x,1) = G(x)e^{\zeta'(-1)},$$

and
$$\Gamma_{1,1}(0,1) = e^{\zeta'(-1)}.$$

The generalized cotangent has interesting properties:

THEOREM 4. *Let* $k \geq 1$.
(1) $\mathrm{Cot}_{r,k}(x, (\omega_1, \ldots, \omega_r)) = k \log S_{r,k-1}(x, (\omega_1, \ldots, \omega_r))$.
(2) $\mathrm{Cot}'_{r,k}(x, (\omega_1, \ldots, \omega_r)) = k\, \mathrm{Cot}_{r,k-1}(x, (\omega_1, \ldots, \omega_r))$.
(3) *If* $\mathrm{Cot}_r(x, (\omega_1, \ldots, \omega_r)) = \mathrm{Cot}_{r,0}(x, (\omega_1, \ldots, \omega_r))$ *satisfies an algebraic differential equation, then* $\mathrm{Cot}_r(x, (\omega_1, \ldots, \omega_r))$ *also satisfies an algebraic differential equation. Especially,* $\mathrm{Cot}_r(x, (\omega_1, \ldots, \omega_r))$ *satisfies an algebraic differential equation if ratios of* $\omega_1, \ldots, \omega_r$ *are rational numbers.*

We prove a generalized Kinkelin's formula in the following form:

THEOREM 5. *For* $k \geq 1$, *we have*

$$\int_0^x \log S_{r,k-1}(t, (\omega_1, \ldots, \omega_r))\, dt = \frac{1}{k} \log\left(\frac{S_{r,k}(x, (\omega_1, \ldots, \omega_r))}{S_{r,k}(0, (\omega_1, \ldots, \omega_r))}\right).$$

Especially,

$$\int_0^x \log S_r(t, (\omega_1, \ldots, \omega_r))\, dt = \log\left(\frac{S_{r,1}(x, (\omega_1, \ldots, \omega_r))}{S_{r,1}(0, (\omega_1, \ldots, \omega_r))}\right).$$

We notice that the original Kinkelin's formula (K) is

$$\int_0^x \log S_1(t)\, dt = \log S_{1,1}(x)$$

as in Theorem 1(7) recalling $S_{1,1}(0) = 1$.

As an application of this formula we obtain some integrals as below:

THEOREM 6. (1) *For an odd* r

$$\int_0^r \log S_r(x)\, dx = -2r \log S_{r+1}(1).$$

(2) *For an even* r

$$\int_0^r \log S_r(x)\, dx = 0.$$

THEOREM 7. (1) $\int_0^{1/2} \log S_2(x)\, dx = -\dfrac{7\zeta(3)}{8\pi^2}$.
(2) $\int_0^1 \log S_2(x)\, dx = 0$.
(3) $\int_0^{3/2} \log S_2(x)\, dx = -\dfrac{7\zeta(3)}{8\pi^2}$.
(4) $\int_0^2 \log S_2(x)\, dx = 0$.

THEOREM 8. (1) $\int_0^{1/2} \log S_3(x)\, dx = -\dfrac{9\zeta(3)}{8\pi^2}$.
(2) $\int_0^1 \log S_3(x)\, dx = -\dfrac{\zeta(3)}{2\pi^2}$.

(3) $\int_0^{3/2} \log S_3(x)\, dx = -\dfrac{3\zeta(3)}{4\pi^2}$.

(4) $\int_0^{2} \log S_3(x)\, dx = -\dfrac{\zeta(3)}{\pi^2}$.

(5) $\int_0^{5/2} \log S_3(x)\, dx = -\dfrac{3\zeta(3)}{8\pi^2}$.

(6) $\int_0^{3} \log S_3(x)\, dx = -\dfrac{3\zeta(3)}{2\pi^2}$.

Since the "fundamental domain" of $S_r(x)$ is $0 \le x < r$, $\int_0^2 \log S_2(x)\, dx = 0$ and $\int_0^3 \log S_3(x)\, dx = -\dfrac{3\zeta(3)}{2\pi^2}$ are considered to be analogues to Euler-Raabe formula $\int_0^1 \log S_1(x)\, dx = 0$.

2. A reconstruction of Kinkelin's theory

We prove Theorem 1. (1) follows from the definition of $G(x)$ and the fact

$$\Gamma_1(t) = \frac{\Gamma(t)}{\sqrt{2\pi}}.$$

(2) is obvious from (1).

(3) Let $H(x) = \exp(\zeta'(-1,x) - \zeta'(-1))$. Then

$$\zeta(s, x+1) = \zeta(s, x) - x^{-s}$$

shows that

$$\zeta'(-1, x+1) = \zeta'(-1, x) + x \log x.$$

Hence

$$H(x+1) = H(x)x^x.$$

In particular, $H(0) = H(1)$. Thus

$$H(0) = H(1) = \exp(\zeta'(-1) - \zeta'(-1)) = 1.$$

Consequently, from (2), it is sufficient to show that

$$\frac{H'}{H}(x) = \log \Gamma_1(x) + x - \frac{1}{2}.$$

Now, using

$$\frac{\partial}{\partial s}\zeta(s, x) = -s\zeta(s+1, x),$$

we have

$$
\begin{aligned}
\frac{H'}{H}(x) &= (\log H(x))' \\
&= \frac{\partial}{\partial x}\zeta'(-1,x) \\
&= \frac{\partial^2}{\partial s \partial x}\zeta(s,x)|_{s=-1} \\
&= \frac{\partial}{\partial s}(-s\zeta(s+1,x))|_{s=-1} \\
&= \zeta'(0,x) - \zeta(0,x) \\
&= \log\frac{\Gamma(x)}{\sqrt{2\pi}} - \left(\frac{1}{2} - x\right) \\
&= \log \Gamma_1(x) + \left(x - \frac{1}{2}\right)
\end{aligned}
$$

by Lerch's formula. Hence $H(x) = G(x)$.

(4) As seen in the proof of (3), $H(x)$ satisfies

$$H(0) = H(1) = 1$$

and

$$H(x+1) = H(x)x^x \quad \text{for } x > 0.$$

Hence

$$H(n+1) = 1^1 2^2 \cdots n^n \quad \text{for an integer } n \geq 1.$$

Thus $G(x) = H(x)$ satisfies the formula of (4).

(5) We show that

$$\zeta(s-1,x) = \zeta_2(s,x) + (x-1)\zeta_1(s,x).$$

Then, differentiation at $s = 0$ implies

$$\Gamma_{1,1}(x) = \Gamma_2(x)\Gamma(x)^{x-1}.$$

Hence, (5) follows from (3). The needed identity is shown as

$$
\begin{aligned}
\zeta(s-1,x) &= \sum_{n=0}^{\infty}(n+x)(n+x)^{-s} \\
&= \sum_{n=0}^{\infty}((n+1)+(x-1))(n+x)^{-s} \\
&= \sum_{n=0}^{\infty}(n+1)(n+x)^{-s} + (x-1)\sum_{n=0}^{\infty}(n+x)^{-s} \\
&= \zeta_2(s,x) + (x-1)\zeta_1(s,x)
\end{aligned}
$$

since

$$\zeta_2(s,x) = \sum_{n_1,n_2\geq 0}(n_1+n_2+x)^{-s} = \sum_{n=0}^{\infty}(n+1)(n+x)^{-s}.$$

(6) From (5),

$$\frac{G(1-x)}{G(x)} = \frac{\Gamma_2(1-x)\Gamma_1(1-x)^{-x}}{\Gamma_2(x)\Gamma_1(x)^{x-1}}.$$

Here we use the periodicity

$$\Gamma_2(1-x) = \Gamma_2(2-x)\Gamma_1(1-x).$$

We notice that the general periodicity of $\Gamma_r(x,(\omega_1,\dots,\omega_r))$ is

$$\Gamma_r(x+\omega_i,(\omega_1,\dots,\omega_r)) = \Gamma_r(x,(\omega_1,\dots,\omega_r))\Gamma_{r-1}(x,(\omega_1,\dots,\omega_{i-1},\omega_{i+1},\dots,\omega_r))^{-1};$$

see [KK]. Hence

$$\frac{G(1-x)}{G(x)} = \frac{\Gamma_2(2-x)\Gamma_1(1-x)^{1-x}}{\Gamma_2(x)\Gamma_1(x)^{x-1}} = S_2(x)S_1(x)^{x-1}.$$

We have also

$$\frac{G(1-x)}{G(x)} = \frac{\Gamma_{1,1}(1-x)}{\Gamma_{1,1}(x)} = S_{1,1}(x)$$

from (3).

(7) Since the equality holds at $x=0$, it is sufficient to show that

$$\left(\log\left(\frac{G(1-x)}{G(x)}\right)\right)' = \log(2\sin\pi x).$$

From (2), the left-hand side above is calculated as

$$-\frac{G'}{G}(1-x) - \frac{G'}{G}(x) = -\left(\log\Gamma_1(1-x)+\frac{1}{2}-x\right) - \left(\log\Gamma_1(x)+x-\frac{1}{2}\right) = \log\left(\frac{1}{\Gamma_1(x)\Gamma_1(1-x)}\right) = \log(2\sin\pi x).$$

This proves Theorem 1. □

3. A generalized gamma function

We prove Theorem 2.

(1) We have

$$\begin{aligned}\tilde{\Gamma}_{1,k}(1) &= \exp(\zeta'(-k,1) - \zeta'(-k)) \\ &= \exp(\zeta'(-k) - \zeta'(-k)) \\ &= 1.\end{aligned}$$

(2) From $\zeta(s, x+1) = \zeta(s, x) - x^{-s}$ we obtain

$$\zeta'(-k, x+1) = \zeta'(-k, x) + x^k \log x.$$

Hence

$$\tilde{\Gamma}_{1,k}(x+1) = \tilde{\Gamma}_{1,k}(x) x^{x^k}.$$

(3) From (1) and (2) we get

$$\begin{aligned}\tilde{\Gamma}_{1,k}(n+1) &= \tilde{\Gamma}_{1,k}(n) n^{n^k} \\ &= 1^{1^k} 2^{2^k} \cdots n^{n^k}.\end{aligned}$$

(4) $\tilde{\Gamma}_{1,0}(x) = \Gamma(x)$ by $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. The equality $\tilde{\Gamma}_{1,1}(x) = G(x)$ is shown in Theorem 1(3). □

4. Generalized Kinkelin's formulas

We prove Theorems 3, 4 and 5. To show Theorem 3 we calculate

$$\begin{aligned}&(\log \Gamma_{r,k}(x, (\omega_1, \ldots, \omega_r)))' \\ &= \frac{\partial}{\partial x}(\zeta_r'(-k, x, (\omega_1, \ldots, \omega_r))) \\ &= \frac{\partial^2}{\partial s \partial x} \zeta_r(s, x, (\omega_1, \ldots, \omega_r))|_{s=-k} \\ &= \frac{\partial}{\partial s}(-s\zeta_r(s+1, x, (\omega_1, \ldots, \omega_r)))|_{s=-k} \\ &= k\zeta_r'(1-k, x, (\omega_1, \ldots, \omega_r)) - \zeta_r(1-k, x, (\omega_1, \ldots, \omega_r)) \\ &= k \log \Gamma_{r,k-1}(x, (\omega_1, \ldots, \omega_r)) - \zeta_r(1-k, (\omega_1, \ldots, \omega_r)),\end{aligned}$$

where we used

$$\frac{\partial}{\partial x}\zeta_r(s, x, (\omega_1, \ldots, \omega_r)) = -s\zeta_r(s+1, x, (\omega_1, \ldots, \omega_r)).$$

Then we obtain Theorem 3.

Hence

$$
\begin{aligned}
&(\log S_{r,k}(x,(\omega_1,\dots,\omega_r)))' \\
&\quad = -(\log \Gamma_{r,k}(x,(\omega_1,\dots,\omega_r)))' \\
&\qquad + (-1)^{r+k}(\log \Gamma_{r,k}(\omega_1+\dots+\omega_r - x,(\omega_1,\dots,\omega_r)))' \\
&\quad = -(k \log \Gamma_{r,k-1}(x,(\omega_1,\dots,\omega_r)) - \zeta_r(1-k,x,(\omega_1,\dots,\omega_r))) \\
&\qquad + (-1)^{r+k-1}\begin{pmatrix} k \log \Gamma_{r,k-1}(\omega_1+\dots+\omega_r - x,(\omega_1,\dots,\omega_r)) \\ -\zeta_r(1-k,\omega_1+\dots+\omega_r - x,(\omega_1,\dots,\omega_r)) \end{pmatrix} \\
&\quad = k \log(\Gamma_{r,k-1}(x,(\omega_1,\dots,\omega_r))^{-1} \\
&\qquad \times \Gamma_{r,k-1}(\omega_1+\dots+\omega_r - x,(\omega_1,\dots,\omega_r))^{(-1)^{r+k-1}}) \\
&\qquad - (-\zeta_r(1-k,x,(\omega_1,\dots,\omega_r)) + (-1)^{r+k-1} \\
&\qquad \times \zeta_r(1-k,\omega_1+\dots+\omega_r - x,(\omega_1,\dots,\omega_r))) \\
&\quad = k \log S_{r,k-1}(x,(\omega_1,\dots,\omega_r)) \\
&\qquad - (-\zeta_r(1-k,x,(\omega_1,\dots,\omega_r)) + (-1)^{r+k-1} \\
&\qquad \times \zeta_r(1-k,\omega_1+\dots+\omega_r - x,(\omega_1,\dots,\omega_r))).
\end{aligned}
$$

Here we use that

$$
\begin{aligned}
&-\zeta_r(1-k,x,(\omega_1,\dots,\omega_r)) \\
&\quad + (-1)^{r+k-1}\zeta_r(1-k,\omega_1+\dots+\omega_r - x,(\omega_1,\dots,\omega_r)) = 0.
\end{aligned}
$$

This vanishing result can be seen by expressing these special values via generalized Bernoulli polynomials. We show a shorter way below. Put

$$
f_{r,k}(t,x,(\omega_1,\dots,\omega_r)) = \frac{-e^{-tx} + (-1)^{r+k-1}e^{-t(\omega_1+\dots+\omega_r - x)}}{(1-e^{-t\omega_1})\cdots(1-e^{-t\omega_r})},
$$

and let

$$
f_{r,k}(t,x,(\omega_1,\dots,\omega_r)) = \sum_{m \ge -r} c_m(x,(\omega_1,\dots,\omega_r))t^m
$$

be the Laurent expansion around $t = 0$. Then

$$
\begin{aligned}
&-\zeta_r(s,x,(\omega_1,\dots,\omega_r)) + (-1)^{r+k-1}\zeta_r(s,\omega_1+\dots+\omega_r - x,(\omega_1,\dots,\omega_r)) \\
&\quad = \frac{1}{\Gamma(s)}\int_0^\infty f_{r,k}(t,x,(\omega_1,\dots,\omega_r))t^{s-1}\,dt
\end{aligned}
$$

for $\mathrm{Re}(s) > r$. The usual method of the analytic continuation implies that

$$
\begin{aligned}
&-\zeta_r(1-k, x, (\omega_1, \dots, \omega_r)) + (-1)^{r+k-1}\zeta_r(1-k, \omega_1 + \cdots + \omega_r - x, (\omega_1, \dots, \omega_r)) \\
&\quad = \frac{1}{\Gamma(s)} \int_0^1 f_{r,k}(t, x, (\omega_1, \dots, \omega_r)) t^{s-1}\, dt|_{s=1-k} \\
&\quad = (-1)^{k-1}(k-1)! c_{k-1}(x, (\omega_1, \dots, \omega_r)).
\end{aligned}
$$

Now, the equality

$$f_{r,k}(-t, x, (\omega_1, \dots, \omega_r)) = (-1)^k f_{r,k}(t, x, (\omega_1, \dots, \omega_r))$$

shows that

$$(-1)^m c_m(x, (\omega_1, \dots, \omega_r)) = (-1)^k c_m(x, (\omega_1, \dots, \omega_r)).$$

Hence

$$c_{k-1}(x, (\omega_1, \dots, \omega_r)) = 0.$$

Thus we have

$$
\begin{aligned}
&-\zeta_r(1-k, x, (\omega_1, \dots, \omega_r)) \\
&\qquad + (-1)^{r+k-1}\zeta_r(1-k, \omega_1 + \cdots + \omega_r - x, (\omega_1, \dots, \omega_r)) = 0.
\end{aligned}
$$

Thus we have proved Theorem 4(1) and Theorem 5. Now, Theorem 4(2) follows from Theorem 4(1), and the former half of Thorem 4(3) comes from Thoerem 4(2). Lastly, we have the latter half of Theorem 4(3) by applying the differential algebraicity result on the multiple sine function proved in [KW]. Thus we have proved Theorems 3, 4 and 5. □

Remark 1. From the calculation above we have

$$
\begin{aligned}
\int_0^x t\, \mathrm{Cot}_{r,k-1}(t, \boldsymbol{\omega})\, dt &= x \log S_{r,k-1}(x, \boldsymbol{\omega}) - \int_0^x \log S_{r,k-1}(t, \boldsymbol{\omega})\, dt \\
&= \frac{1}{k} \log\left(\frac{S_{r,k-1}(x, \boldsymbol{\omega})^{kx} S_{r,k}(0, \boldsymbol{\omega})}{S_{r,k}(x, \boldsymbol{\omega})}\right).
\end{aligned}
$$

This is a generalization of the case $r = 1$ and $k = 1$ due to Kinkelin [K] (p. 135):

$$
\begin{aligned}
\int_0^x t\pi \cot(\pi t)\, dt &= x \log(2 \sin \pi x) - \int_0^x \log(2 \sin \pi t)\, dt \\
&= \log\left(\frac{(2 \sin \pi x)^x G(x)}{G(1-x)}\right)
\end{aligned}
$$

which was used by Sarnak [S] and Voros [V]. Kinkelin's cotangent integral gave the origin of the theory of the multiple sine functions of Hölder [H] (1886) and [KK].

5. Generalized Euler-Raabe integrals

We prove Theorems 6, 7 and 8 at the same time. These calculations are special cases of Theorem 5. In fact, since

$$\int_0^x \log S_r(t)\, dt = \log\left(\frac{S_{r,1}(x)}{S_{r,1}(0)}\right),$$

it remains to obtain $S_{r,1}(x)$. We show that

$$S_{r,1}(x) = S_{r+1}(x)^r S_r(x)^{x-r}.$$

First we notice that

$$\zeta_r(s-1,x) = r\zeta_{r+1}(s,x) + (x-r)\zeta_r(s,x).$$

This follows from

$$\begin{aligned}\zeta_r(s,x) &= \sum_{n_1,\dots,n_r \geq 0} (n_1 + \cdots + n_r + x)^{-s} \\ &= \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} (n+x)^{-s}\end{aligned}$$

as

$$\begin{aligned}\zeta_r(s-1,x) &= \sum_{n=0}^{\infty} (n+x)\binom{n+r-1}{r-1}(n+x)^{-s} \\ &= \sum_{n=0}^{\infty}\left\{r\binom{n+r}{r} + (x-r)\binom{n+r-1}{r-1}\right\}(n+x)^{-s} \\ &= r\zeta_{r+1}(s,x) + (x-r)\zeta_r(s,x).\end{aligned}$$

Hence, differentiating at $s=0$ we get

$$\log \Gamma_{r,1}(x) = r \log \Gamma_{r+1}(x) + (x-r)\log \Gamma_r(x).$$

Thus

$$\Gamma_{r,1}(x) = \Gamma_{r+1}(x)^r \Gamma_r(x)^{x-r}.$$

This gives

$$\begin{aligned}S_{r,1}(x) &= \Gamma_{r,1}(x)^{-1}\Gamma_{r,1}(r-x)^{(-1)^{r+1}} \\ &= (\Gamma_{r+1}(x)^r\Gamma_r(x)^{x-r})^{-1} \times (\Gamma_{r+1}(r-x)^r\Gamma_r(r-x)^{-x})^{(-1)^{r+1}}.\end{aligned}$$

Hence using the periodicity

$$\Gamma_{r+1}(r-x) = \Gamma_{r+1}(r+1-x)\Gamma_r(r-x)$$

we get

$$\begin{aligned} S_{r,1}(x) &= (\Gamma_{r+1}(x)^r \Gamma_r(x)^{x-r})^{-1} \times (\Gamma_{r+1}(r+1-x)^r \Gamma_r(r-x)^{r-x})^{(-1)^{r+1}} \\ &= (\Gamma_{r+1}(x)^{-1} \Gamma_{r+1}(r+1-x)^{(-1)^{r+1}})^r \times (\Gamma_r(x)^{-1} \Gamma_r(r-x)^{(-1)^r})^{x-r} \\ &= S_{r+1}(x)^r S_r(x)^{x-r}. \end{aligned}$$

Moreover, the periodicity

$$S_{r+1}(x+1) = S_{r+1}(x) S_r(x)^{-1}$$

(see [KK]) gives

$$S_{r,1}(x) = S_{r+1}(x+1)^r S_r(x)^x.$$

Hence

$$S_{r,1}(0) = S_{r+1}(1)^r$$

since $S_r(x)$ has a simple zero at $x = 0$ (see [KK]). Thus we get the formula

$$\int_0^x \log S_r(t)\, dt = \log\left(\frac{S_{r+1}(x)^r S_r(x)^{x-r}}{S_{r+1}(1)^r}\right).$$

When r is odd,

$$\begin{aligned} \int_0^r \log S_r(x)\, dx &= \log\left(\frac{S_{r+1}(r)^r}{S_{r+1}(1)^r}\right) \\ &= -2r \log S_{r+1}(1) \end{aligned}$$

since $S_r(x)$ has a simple zero at $x = r$ and $S_{r+1}(r) = S_{r+1}(1)^{-1}$ for odd r. If r is even,

$$\int_0^r \log S_r(x)\, dx = 0$$

trivially from the relation

$$S_r(r-x) = S_r(x)^{-1}$$

for even r. Thus we have Theorem 6. In particular

$$\int_0^1 \log S_1(x)\, dx = -2 \log S_2(1),$$

$$\int_0^2 \log S_2(x)\, dx = 0$$

and

$$\int_0^3 \log S_3(x)\, dx = -6 \log S_4(1).$$

Hence we have Theorem 7(4). We notice that

$$S_2(1) = \frac{\Gamma_2(1)}{\Gamma_2(1)} = 1$$

gives the Euler-Raabe formula

$$\int_0^1 \log S_1(x)\,dx = 0.$$

Now, we see that

$$\begin{aligned} S_4(1) &= S_4(2)S_3(1) \\ &= S_3(1) \\ &= \exp\left(\frac{\zeta(3)}{4\pi^2}\right), \end{aligned}$$

where the last equality was proved in [KK, Theorem 3.8(c)]. Hence

$$\int_0^3 \log S_3(x)\,dx = -\frac{3\zeta(3)}{2\pi^2}.$$

Thus we have Theorem 8(6). We have proved also that

$$\int_0^x \log S_2(t)\,dt = \log(S_3(x)^2 S_2(x)^{x-2}) - \frac{\zeta(3)}{2\pi^2}$$

and

$$\int_0^x \log S_3(t)\,dt = \log(S_4(x)^3 S_3(x)^{x-3}) - \frac{3\zeta(3)}{4\pi^2}$$

since

$$S_3(1) = S_4(1) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right).$$

Thus we reach to

$$\int_0^{1/2} \log S_2(x)\,dx = \log\left(S_3\left(\frac{1}{2}\right)^2 S_2\left(\frac{1}{2}\right)^{-3/2}\right) - \frac{\zeta(3)}{2\pi^2}, \tag{7-1}$$

$$\int_0^1 \log S_2(x)\,dx = \log(S_3(1)^2 S_2(1)^{-1}) - \frac{\zeta(3)}{2\pi^2}, \tag{7-2}$$

$$\int_0^{3/2} \log S_2(x)\,dx = \log\left(S_3\left(\frac{3}{2}\right)^2 S_2\left(\frac{3}{2}\right)^{-1/2}\right) - \frac{\zeta(3)}{2\pi^2}, \tag{7-3}$$

$$\int_0^{1/2} \log S_3(x)\,dx = \log\left(S_4\left(\frac{1}{2}\right)^3 S_3\left(\frac{1}{2}\right)^{-5/2}\right) - \frac{3\zeta(3)}{4\pi^2}, \tag{8-1}$$

$$\int_0^1 \log S_3(x)\,dx = \log(S_4(1)^3 S_3(1)^{-2}) - \frac{3\zeta(3)}{4\pi^2}, \tag{8-2}$$

(8-3) $$\int_0^{3/2} \log S_3(x)\,dx = \log\left(S_4\left(\frac{3}{2}\right)^3 S_3\left(\frac{3}{2}\right)^{-3/2}\right) - \frac{3\zeta(3)}{4\pi^2},$$

(8-4) $$\int_0^{2} \log S_3(x)\,dx = \log(S_4(2)^3 S_3(2)^{-1}) - \frac{3\zeta(3)}{4\pi^2},$$

and

(8-5) $$\int_0^{5/2} \log S_3(x)\,dx = \log\left(S_4\left(\frac{5}{2}\right)^3 S_3\left(\frac{5}{2}\right)^{-1/2}\right) - \frac{3\zeta(3)}{4\pi^2}.$$

Hence we get all the results of Theorems 7 and 8 from the following values:

$$S_2\left(\frac{1}{2}\right) = \sqrt{2},$$

$$S_2(1) = 1,$$

$$S_2\left(\frac{3}{2}\right) = \frac{1}{\sqrt{2}},$$

$$S_3\left(\frac{1}{2}\right) = 2^{3/8}\exp\left(-\frac{3\zeta(3)}{16\pi^2}\right),$$

$$S_3(1) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right),$$

$$S_3\left(\frac{3}{2}\right) = S_3\left(\frac{1}{2}\right)S_2\left(\frac{1}{2}\right)^{-1} = 2^{-1/8}\exp\left(-\frac{3\zeta(3)}{16\pi^2}\right),$$

$$S_3(2) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right),$$

$$S_3\left(\frac{5}{2}\right) = 2^{3/8}\exp\left(-\frac{3\zeta(3)}{16\pi^2}\right),$$

$$S_4\left(\frac{1}{2}\right) = 2^{5/16}\exp\left(-\frac{9\zeta(3)}{32\pi^2}\right),$$

$$S_4(1) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right),$$

$$S_4\left(\frac{3}{2}\right) = 2^{-1/16}\exp\left(-\frac{3\zeta(3)}{32\pi^2}\right),$$

$$S_4(2) = 1,$$

and

$$S_4\left(\frac{5}{2}\right) = 2^{1/16}\exp\left(\frac{3\zeta(3)}{32\pi^2}\right).$$

These special values for $S_2(x)$ and $S_3(x)$ are proved in [KK]. Then, values for $S_4(x)$ are shown as follows:

$$\begin{aligned}
S_4\left(\frac{1}{2}\right) &= \frac{\Gamma_4\left(\frac{7}{2}\right)}{\Gamma_4\left(\frac{1}{2}\right)} = \frac{\Gamma_4\left(\frac{5}{2}\right)\Gamma_3\left(\frac{5}{2}\right)^{-1}}{\Gamma_4\left(\frac{3}{2}\right)\Gamma_3\left(\frac{1}{2}\right)} \\
&= \Gamma_3\left(\frac{3}{2}\right)^{-1} S_3\left(\frac{1}{2}\right) = S_3\left(\frac{3}{2}\right)^{1/2} S_3\left(\frac{1}{2}\right), \\
S_4(1) &= S_3(1), \\
S_4\left(\frac{3}{2}\right) &= S_4\left(\frac{1}{2}\right) S_3\left(\frac{1}{2}\right)^{-1} = S_3\left(\frac{3}{2}\right)^{1/2}, \\
S_4(2) &= \frac{\Gamma_4(2)}{\Gamma_4(2)} = 1,
\end{aligned}$$

and

$$S_4\left(\frac{5}{2}\right) = S_4\left(4 - \frac{3}{2}\right) = S_4\left(\frac{3}{2}\right)^{-1}.$$

Thus we have proved Theorems 6, 7 and 8. □

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