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ON INTEGRAL MEANS OF THE DERIVATIVES OF BLASCHKE PRODUCTS

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Abstract

Kutbi and Protas showed that if the zeros $\{a_k\}$ of a given Blaschke product B satisfy $\sum_k (1-|a_k|)^\alpha < \infty$ for some $\alpha \in (0,1)$, then the integral means of its derivative $B^{(n)}$ satisfies certain estimate. In this paper, we extend their result.

1. Introduction and results

A function

$$B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \bar{a}_k z} = \prod_{k=1}^{\infty} b_k(z)$$

is called a Blaschke product, where $\{a_k\}_{k=1}^{\infty}$ is a sequence in the unit disk D of the complex plane satisfying $\sum_{k=1}^{\infty}(1-|a_k|) < \infty$. The Blaschke product B is a holomorphic function on D satisfying $|B(z)| < 1$. In this paper, we investigate the mean value of the n'th derivative $B^{(n)}$ of B under the condition

$$\sum_{k=1}^{\infty} (1-|a_k|)^\alpha < \infty, \tag{1.1}$$

where $0 < \alpha < 1$. We recall the known results. For $n \in \mathbf{N}$, set

$$\Omega_n = \left\{ (\alpha, p) \mid 0 < \alpha < 1, p \geq \alpha, p > \frac{1-\alpha}{n} \right\},$$

$$G_n = \left\{ (\alpha, p) \mid 0 < \alpha < \frac{1}{n+1}, p > \frac{1-\alpha}{n} \right\},$$

$$\Gamma_n = \left\{ (\alpha, p) \mid 0 < \alpha < \frac{1}{n+1}, p \leq \frac{1-\alpha}{n} \right\}.$$

Then $G_n \subset \Omega_n$ and $\Omega_n \subset \Omega_m$ $(n \leq m)$. Assume that (1.1) holds.

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i) If $(\alpha, p) \in \Gamma_n$, then $B^{(n)}$ belongs to the standard Hardy space H^p, (Protas [5] $(n = 1)$ and Linden [4] $(n \geq 2)$).

ii) If $(\alpha, p) \in G_n$, then

$$\int_0^{2\pi} |B^{(n)}(re^{i\theta})|^p \, d\theta = o\left(\frac{1}{(1-r)^{np+\alpha-1}}\right), \quad r \to 1, \tag{1.2}$$

and this estimate is best possible (Kutbi [2], [3]).

iii) If $n = 1$ we can replace the set G_1 with Ω_1 in the statement ii) above (Kutbi [2] and Protas [6]). Protas showed that if $(\alpha, p) \in \Omega_1 \cap \{\alpha > 1/2\}$ then (1.2) holds. Moreover, we can easily verify that (1.2) holds if $(\alpha, p) \in \Omega_1 \cap \{\alpha \geq 1/2\}$ by repeating the argument in Kutbi [2], because [2] Lemma 3 holds for $0 < \alpha < 1$, $p \geq \alpha$, $p + \alpha > 1$.

Note that if $p \geq 1$, then the estimate (1.2) holds for $n = 1$ if and only if the integral modulus of continuity

$$\omega_p(t) = \sup_{0<h<t} \left(\int_0^{2\pi} |B(e^{i(\theta+h)}) - B(e^{i\theta})|^p \, d\theta\right)^{1/p}$$

of B satisfies $\omega_p(t) = o(t^{(1-\alpha)/p})$, $t \to 0$ (cf. [1]).

The purpose of the present paper is to extend the results ii) and iii) above.

Theorem 1.1. *Let* $n \in \mathbf{N}$, $(\alpha, p) \in \Omega_n$, *and* (1.1) *hold. Then*

$$\int_0^{2\pi} |B^{(n)}(re^{i\theta})|^p \, d\theta = o\left(\frac{1}{(1-r)^{np+\alpha-1}}\right), \quad r \to 1.$$

Theorem 1.2. *Let* $n \in \mathbf{N}$, $0 < \alpha < 1$, *and* $p > 0$. *Assume that* $\varepsilon(r)$ *is a positive function on* $(0, 1)$ *satisfying* $\lim_{r\to 1} \varepsilon(r) = 0$. *Then, there exists a Blaschke product B whose zeros satisfy the condition* (1.1) *such that*

$$\int_0^{2\pi} |B^{(n)}(r_k e^{i\theta})|^p \, d\theta > \frac{\varepsilon(r_k)}{(1-r_k)^{np+\alpha-1}}, \quad k = 1, 2, 3, \ldots \tag{1.3}$$

holds for some sequence $\{r_k\}$, $r_k \to 1$.

From Theorems 1.1 and 1.2, the estimate (1.2) is sharp for $(\alpha, p) \in \Omega_n$. Theorem 1.2 is established by Kutbi [3] when $0 < \alpha < 1/(n+1)$. Our proof for Theorem 1.2 simplifies his argument.

Remark 1.1. Since B is bounded, we have

$$|B^{(m)}(z)| = O\left(\frac{1}{(1-|z|)^m}\right), \quad |z| \to 1,$$

(cf. [1]). Hence, if the estimate (1.2) holds for some (p, α), then (1.2) also holds for each (p', α), $p' \geq p$.

Remark 1.2. For a holomorphic function f on D, $p > 0$, $m \in \mathbf{N}$, and $s > 0$,

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta = o\left(\frac{1}{(1-r)^s}\right), \quad r \to 1,$$

holds if and only if

$$\int_0^{2\pi} |f^{(m)}(re^{i\theta})|^p \, d\theta = o\left(\frac{1}{(1-r)^{s+mp}}\right), \quad r \to 1,$$

(cf. [1]). Hence, our result is new only for $(\alpha, p) \in \Omega_n \backslash ((\bigcup_{k=1}^n G_k) \cup \Omega_1)$.

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2. Proof of Theorem 1.1

In the following, $C(a, b, \dots)$ denotes a positive constant depending only on $a, b, \dots$ which may vary from space to space unless stated otherwise, that is, $f \le 2C(p, \alpha)$ implies $f \le C(p, \alpha)$, on the other hand, $f \le 2C_2(p, \alpha)$ does not necessarily imply $f \le C_2(p, \alpha)$. In particular, C denotes an absolute constant.

LEMMA 2.1. *Let f_1, f_2 be bounded holomorphic functions on D, $f = f_1 f_2$, and $n \in \mathbf{N}$. Assume that for each pair of p, m, satisfying $1 \le m \le n$ and $(\alpha, p) \in \Omega_m$, we have*

$$\int_0^{2\pi} |f_j^{(m)}(re^{i\theta})|^p \, d\theta = o\left(\frac{1}{(1-r)^{mp+\alpha-1}}\right), \quad r \to 1, \ (j = 1, 2).$$

Then,

$$\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p \, d\theta = o\left(\frac{1}{(1-r)^{np+\alpha-1}}\right), \quad r \to 1,$$

holds for each p satisfying $(\alpha, p) \in \Omega_n$.

Proof. Assume that f_1 and f_2 satisfy the condition above and $(\alpha, p) \in \Omega_n$. From the Leibnitz formula, it suffices to estimate the mean value of $g = f_1^{(m_1)} f_2^{(m_2)}$ $(0 \le m_j \le n, m_1 + m_2 = n)$. We may assume $1 \le m_j \le n - 1$. Let $X_j = n/m_j$. Then $X_1^{-1} + X_2^{-1} = 1$, and $(\alpha, pX_j) \in \Omega_{m_j}$. Therefore,

$$\begin{aligned}\int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta &\le \prod_{j=1,2} \left(\int_0^{2\pi} |f_j^{(m_j)}(re^{i\theta})|^{pX_j} \, d\theta\right)^{1/X_j} \\ &\le \prod_{j=1,2} \left(o\left(\frac{1}{(1-r)^{m_j p X_j + \alpha - 1}}\right)\right)^{1/X_j} = o\left(\frac{1}{(1-r)^{np+\alpha-1}}\right). \quad \square\end{aligned}$$

Set $r_k = |a_k|$ and

$$\varphi_k(z) = \frac{b_k'(z)}{b_k(z)} = \frac{1 - r_k^2}{(1 - \bar{a}_k z)(z - a_k)} = \frac{1}{z - a_k} - \frac{1}{z - (\bar{a}_k)^{-1}}.$$

Lemma 2.2. *Let* $0 < p < \infty$, $m \geq 0$, *and* $0 < r < 1$. *Assume* $(m+2)p - 1 > 0$ *and* $|r_k - r| \geq (1 - r)/3$. *Then*

$$\int_0^{2\pi} |\varphi_k^{(m)}(re^{i\theta})|^p \, d\theta \leq C(m, p) \frac{(1 - r_k)^p}{(1 - r_k r)^{(m+2)p-1}}.$$

Proof. Assume $|z| = r$. Let D_k be the disk with center a_k and radius $\dfrac{1 - r_k}{4}$. Then, $z \notin D_k$, $\{|\zeta| < 1/9\} \subset b_k(D_k)$, and $|b_k(z)| < 1$, and so $|z - a_k| \leq |1 - \bar{a}_k z| \leq 9|z - a_k|$. Hence,

$$\begin{aligned}
|\varphi_k^{(m)}(z)| &= m! \left| \frac{1}{(z - a_k)^{m+1}} - \frac{1}{(z - (\bar{a}_k)^{-1})^{m+1}} \right| \\
&\leq m! \left| \frac{1}{z - a_k} - \frac{1}{z - (\bar{a}_k)^{-1}} \right| \sum_{k=0}^{m} \frac{1}{|z - a_k|^k |z - (\bar{a}_k)^{-1}|^{m-k}} \\
&\leq C(m) \frac{1 - r_k}{|1 - \bar{a}_k z|^{m+2}}.
\end{aligned}$$

Therefore, the assertion follows from the estimate (cf. [1])

$$\int_0^{2\pi} \frac{d\theta}{|1 - \bar{a}_k r e^{i\theta}|^{(m+2)p}} \leq \frac{C(m, p)}{(1 - r_k r)^{(m+2)p-1}}. \qquad \square$$

Lemma 2.3 (Kutbi [3]). *Let* $n \in \mathbf{N}$ *and* $(\alpha, p) \in \Omega_n$. *Assume* $\{u_k\}_k$ *is a sequence of numbers satisfying* $0 < u_k < 1$ *and* $\sum_k (1 - u_k)^\alpha < \infty$. *Then*

$$\sum_{k=1}^{\infty} \frac{(1 - u_k)^p}{(1 - u_k r)^{(n+1)p-1}} = o\left(\frac{1}{(1 - r)^{np+\alpha-1}} \right), \quad r \to 1.$$

Proof of Theorem 1.1. We show the assertion by induction for n. Let $n \in \mathbf{N}$. Assume that the assertion is true for $1, 2, \ldots, n-1$. Let $(\alpha, p) \in \Omega_n$ and (1.1) hold. Set

$$S_1 = \bigcup_{j=0,2,4,6,\ldots} \{k \mid 2^{-j-1} < 1 - r_k \leq 2^{-j}\},$$

$$S_2 = \bigcup_{j=1,3,5,7,\ldots} \{k \mid 2^{-j-1} < 1 - r_k \leq 2^{-j}\},$$

and $B_j = \prod_{k \in S_j} b_k$ $(j = 1, 2)$. Then $B = B_1 B_2$ and B_j satisfies the condition (1.1). We show that both B_1 and B_2 satisfy the estimate (1.2), which implies that B also satisfies (1.2) because of Lemma 2.1 with the assumption of induction. We show this only for B_1, because the same argument holds for B_2.

Assume that the estimate (1.2) holds for $r = r^{(j)} = 1 - 3/(2 \cdot 4^j)$, $j = 1, 2, 3, \ldots$. For general r, if we take $r^{(j)}$ so that $r^{(j)} \le r < r^{(j+1)}$, then

$$\int_0^{2\pi} |B^{(n)}(re^{i\theta})|^p \, d\theta \le \int_0^{2\pi} |B^{(n)}(r^{(j+1)}e^{i\theta})|^p \, d\theta = o\left(\frac{1}{(1-r)^{np+\alpha-1}}\right),$$

because $1 - r \le 4(1 - r^{(j+1)})$ and the integral mean is non-decreasing. Therefore, it suffices to show (1.2) only for $r = r^{(j)}$. Moreover, we may assume $p < 1$ by Remark 1.1.

Since $B_1' = B_1 \sum_{k \in S_1} \varphi_k$, from the Leibnitz formula, it suffices to estimate the integral mean of $g = |B_1^{(m_1)}| \sum_{k \in S_1} |\varphi_k^{(m_2)}|$, where $0 \le m_j \le n - 1$, $m_1 + m_2 = n - 1$. Let $r = r^{(j)}$. Then $|r_k - r| \ge (1 - r)/3$ holds for $k \in S_1$.

First, assume $m_1 = 0$. Since $(\alpha, p) \in \Omega_n$ and so $p > 1/(n+1)$, appealing to Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}\int_0^{2\pi} g(re^{i\theta})^p &\le \int_0^{2\pi} \left(\sum_{k \in S_1} |\varphi_k^{(n-1)}(re^{i\theta})|\right)^p d\theta \le \int_0^{2\pi} \sum_{k \in S_1} |\varphi_k^{(n-1)}(re^{i\theta})|^p \, d\theta \\ &\le C(n, p) \sum_{k \in S_1} \frac{(1 - r_k)^p}{(1 - r_k r)^{p(n+1)-1}} = o\left(\frac{1}{(1-r)^{np+\alpha-1}}\right).\end{aligned}$$

Next, assume $m_1 \ge 1$. Let $1 < X_1, X_2 < \infty$ satisfy $X_1^{-1} + X_2^{-1} = 1$. Then

$$\begin{aligned}\int_0^{2\pi} g(re^{i\theta})^p \, d\theta &\le \left(\int_0^{2\pi} |B_1^{(m_1)}(re^{i\theta})|^{pX_1} \, d\theta\right)^{1/X_1} \left(\int_0^{2\pi} \left(\sum_{k \in S_1} |\varphi_k^{(m_2)}(re^{i\theta})|\right)^{pX_2} d\theta\right)^{1/X_2} \\ &= I_1^{1/X_1} I_2^{1/X_2}.\end{aligned}$$

If $(\alpha, pX_1) \in \Omega_{m_1}$, then from the assumption of induction we have

$$I_1 = o\left(\frac{1}{(1-r)^{m_1 pX_1 + \alpha - 1}}\right).$$

Moreover, if $pX_2 \le 1$ and $(\alpha, pX_2) \in \Omega_{m_2+1}$, then $(m_2 + 2)pX_2 - 1 > 0$, and so from Lemmas 2.2 and 2.3 we obtain

$$\begin{aligned}I_2 &\le \sum_{k \in S_1} \int_0^{2\pi} |\varphi_k^{(m_2)}(re^{i\theta})|^{pX_2} \, d\theta \\ &\le C(m_2, p, X_2) \sum_{k \in S_1} \frac{(1 - r_k)^{pX_2}}{(1 - r_k r)^{(m_2+2)pX_2 - 1}} = o\left(\frac{1}{(1-r)^{(m_2+1)pX_2 + \alpha - 1}}\right).\end{aligned}$$

Thus, if we can take X_1, X_2, $1 < X_j < \infty$, $X_1^{-1} + X_2^{-1} = 1$, so that

$$pX_2 \le 1, \quad (\alpha, pX_1) \in \Omega_{m_1}, \quad (\alpha, pX_2) \in \Omega_{m_2+1}, \tag{2.1}$$

then we obtain the required estimate

$$\int_0^{2\pi} g(re^{i\theta})^p \, d\theta = o\left(\frac{1}{(1-r)^{np+\alpha-1}}\right).$$

Since $pX_1 \ge \alpha$ and $pX_2 \ge \alpha$, the condition (2.1) is equivalent to

$$\frac{1}{X_1} < \frac{m_1 p}{1-\alpha}, \quad p \le \frac{1}{X_2} < \frac{(m_2+1)p}{1-\alpha}.$$

Now, since $p < (m_2+1)p/(1-\alpha)$ and

$$p < 1 < \frac{np}{1-\alpha} = \frac{m_1 p}{1-\alpha} + \frac{(m_2+1)p}{1-\alpha},$$

we can always take such $X_j = X_j(m_1, p, \alpha)$, which completes the proof.

3. Proof of Theorem 1.2

Lemma 3.1 (cf. Kutbi [3] Lemmas 8 and 9). *Let* $0 < r < 1$, $q \in \mathbf{N}$, $n \in \mathbf{N}$, *and*

$$b(z) = \frac{r^q - z^q}{1 - r^q z^q}.$$

Assume $p > 1/(n+1)$. *Then there exist constants* $N_1 = N_1(n,p) \in \mathbf{N}$, $\varepsilon_1 = \varepsilon_1(n,p) > 0$ *such that if* r, q *satisfy* $1 - r^q < \varepsilon_1$, $q > N_1$, *then*

$$\int_0^{2\pi} |b^{(n)}(re^{i\theta})|^p \, d\theta \quad \text{and} \quad \frac{q}{(1-r)^{np-1}}$$

are comparable with constant factors depending only on n *and* p.

Let $0 < \alpha < 1$ and $p > 0$. We may assume $p > 1$ by Remark 1.1. Let $\varepsilon(r)$ be a given continuous function on $(0,1)$ satisfying $\lim_{r\to 1} \varepsilon(r) = 0$. We can take a positive function $\delta(r)$ on $(0,1)$, so that $\varepsilon(r) \le \delta(r)$, $\lim_{r\to 1} \delta(r) = 0$, and $\lim_{r\to 1} \delta(r)(1-r)^{-\beta} = \infty$ for each $\beta > 0$. Let $\{r_k\}$ be a sequence of numbers increasing to 1 such that $\sum_k \delta(r_k) < \infty$. Let $q(r) = [\delta(r)(1-r)^{-\alpha}]$: the integral part of $\delta(r)(1-r)^{-\alpha}$, and $q_k = q(r_k)$. Since $q_k \to \infty$, we may assume $q_k \ge N_1$ and $\{q_k\}$ is increasing, where $N_1 = N_1(n,p)$ is the constant in Lemma 3.1. Set

$$B(z) = \prod_{k=1}^{\infty} \frac{r^{q_k} - z^{q_k}}{1 - r^{q_k} z^{q_k}} = \prod_{k=1}^{\infty} b_k(z).$$

Since $\sum_k q_k(1-r_k)^\alpha \le \sum_k \delta(r_k) < \infty$, B satisfies the condition (1.1). In particular, $r_k^{q_k} \to 1$. Hence, we may assume $1 - r_k^{q_k} \le \varepsilon_1$ where $\varepsilon_1 = \varepsilon_1(n, p)$ is the constant in Lemma 3.1.

Since $b_k(z) \to 1$, $b_k^{(m)}(z) \to 0$ $(1 \le m \le n)$ uniformly on compact subsets of D as $k \to \infty$, and $|b_k(z)| \to 1$ as $|z| \to 1$, taking a subsequence if necessary, we may assume

$$|b_k(z)| \ge 1 - 2^{-k-2} \quad (|z| \le r_{k-1} \text{ or } |z| \ge r_{k+1}), \tag{3.1}$$

$$|b_k^{(m)}(z)| \le \varepsilon_2 2^{-k-2}, \quad (|z| \le r_{k-1}),\ 1 \le m \le n, \tag{3.2}$$

where ε_2, $0 < \varepsilon_2 < 1$, is a given constant. Set

$$B_k = \prod_{l \ne k} b_l, \quad \phi_k = \prod_{l \ge k+1} b_l.$$

Since $\prod_{l=1}^{\infty}(1 - 2^{-l-2}) \ge \frac{1}{2}$, from (3.1), we have

$$|B_k(z)| \ge \frac{1}{2}, \quad (|z| = r_k). \tag{3.3}$$

Next, assume $1 \le m \le n$ and $|z| = r_k$. Then

$$\phi_k^{(m)} = m! \sum_{*} \sum_{**} \left(\prod_{j=1}^{l} \frac{b_{s_j}^{(v_j)}}{v_j!} \prod_{\substack{j \ge k+1 \\ j \ne s_1, \dots, s_l}} b_j \right),$$

where $\sum_*$ is taken over all pairs of positive integers l, $v_1, \dots, v_l$, satisfying $v_1 + \cdots + v_l = m$, and $\sum_{**}$ is taken over all pairs of positive integers $s_1, \dots, s_l$ satisfying $k+1 \le s_1 < s_2 < \cdots < s_l$. Thus,

$$|\phi_k^{(m)}| \le m! \sum_{*} \sum_{**} \prod_{j=1}^{l} \frac{|b_{s_j}^{(v_j)}|}{v_j!} \le m! \sum_{*} \prod_{j=1}^{l} \sum_{i=k+1}^{\infty} |b_i^{(v_j)}|$$

$$\le m! \sum_{*} \prod_{j=1}^{l} \sum_{i=k+1}^{\infty} \frac{\varepsilon_2}{2^{i+2}} = m! \sum_{*} \left(\frac{\varepsilon_2}{2^{k+2}} \right)^l \le C(n)\varepsilon_2.$$

Therefore, if $\varepsilon_2 = \varepsilon_2(n) > 0$ is sufficiently small, then we obtain

$$|\phi_k^{(m)}(z)| \le \frac{1}{2}, \quad (|z| = r_k),\ 1 \le m \le n. \tag{3.4}$$

It is to be noted that both the estimates (3.1) and (3.2) hold for each subproduct of B, and so the estimates (3.3) and (3.4) also hold for each subproduct of B.

We show that there exists a subsequence $\{\hat{r}_j\}$, $\hat{r}_j = r_{k_j}$, of $\{r_k\}$, satisfying

$$|\hat{B}_j^{(m)}(z)| \le \Phi(\hat{r}_j) \quad (|z| = \hat{r}_j),\ 1 \le m \le n, \tag{3.5}$$

where $\hat{b}_j = b_{k_j}$, $\hat{B} = \prod_j \hat{b}_j$, $\hat{B}_k = \prod_{j \neq k} \hat{b}_j$, and $\Phi(r)$ is a given function on $(0,1)$ satisfying $\Phi(r) \to \infty$ $(r \to 1)$. We choose $\{\hat{r}_j\}$ inductively as follows. Set $\hat{\phi}_j = \prod_{l \geq j+1} \hat{b}_l$. First, since $|\hat{\phi}_1^{(m)}| \leq 1/2$ $(|z| \leq \hat{r}_1)$ by (3.4), if k_1 is sufficiently large, then $\hat{B}_1 = \hat{\phi}_1$ satisfies (3.5) with an arbitrary choice of k_l $(l \geq 2)$. Next, assume that we can take $k_1, \ldots, k_{j-1}$, so that (3.5) holds for $1, 2, \ldots, j-1$, with an arbitrary choice of k_l $(l \geq j)$. Since $\hat{b}_1 \cdots \hat{b}_{j-1}$ is a finite Blaschke product, $|(\hat{b}_1 \cdots \hat{b}_{j-1})^{(l)}| \leq C(\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_{j-1})$, $0 \leq l \leq n$, holds on D. Hence, from (3.4) again, if $|z| = \hat{r}_j$, then

$$|\hat{B}_j^{(m)}| \leq \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} |(\hat{b}_1 \cdots \hat{b}_{j-1})^{(l)}| \, |\hat{\phi}_j^{(m-l)}| \leq C(n, \hat{r}_1, \hat{r}_2, \ldots, \hat{r}_{j-1}).$$

Thus, if k_j is sufficiently large, then (3.5) holds for j with an arbitrary choice of k_l $(l \geq j+1)$, which completes the proof of (3.5). In the following, $\{\hat{r}_j\}$ is denoted by $\{r_k\}$ for the simplicity.

Since $B = B_k b_k$, we have from the Leibnitz formula,

$$\int_0^{2\pi} |B^{(n)}(r_k e^{i\theta})|^p \, d\theta \geq C_1(n,p) \int_0^{2\pi} |B_k(r_k e^{i\theta}) b_k^{(n)}(r_k e^{i\theta})|^p \, d\theta - C_2(n,p) \sum_{m=1}^{n} \int_0^{2\pi} |B_k^{(m)}(r_k e^{i\theta}) b_k^{(n-m)}(r_k e^{i\theta})|^p \, d\theta. \tag{3.6}$$

Now, we set

$$\Phi(r) = \varepsilon_3 \min\left\{ \frac{1}{1-r}, \frac{q(r)^{1/p}}{(1-r)^{n-1/p}} \right\}$$

in (3.5), where $\varepsilon_3 > 0$ is a given constant.

As to the first term of the right side of (3.6), from Lemma 3.1 and (3.3), we have

$$\int_0^{2\pi} |B_k(r_k e^{i\theta}) b_k^{(n)}(r_k e^{i\theta})|^p \, d\theta \geq \frac{1}{2^p} \int_0^{2\pi} |b_k^{(n)}(r_k e^{i\theta})|^p \, d\theta \geq \frac{C(n,p) q_k}{(1-r_k)^{np-1}}.$$

Next, we estimate the second term of (3.6). For $m = n$, from (3.5), we have

$$\int_0^{2\pi} |B_k^{(n)}(r_k e^{i\theta})|^p |b_k(r_k e^{i\theta})|^p \, d\theta \leq 2\pi \Phi(r_k)^p \leq \frac{2\pi \varepsilon_3^p q_k}{(1-r_k)^{np-1}}.$$

For $1 \leq m \leq n-1$, from Lemma 3.1 and (3.5), we have

$$\int_0^{2\pi} |B_k^{(m)}(r_k e^{i\theta}) b_k^{(n-m)}(r_k e^{i\theta})|^p \, d\theta \leq \Phi(r_k)^p \cdot \frac{C(n,p) q_k}{(1-r_k)^{(n-m)p-1}} \leq \frac{C(n,p) \varepsilon_3^p q_k}{(1-r_k)^{np-1}}. \tag{3.7}$$

Thus,

$$\sum_{m=1}^{n}\int_0^{2\pi}|B_k^{(m)}(r_ke^{i\theta})b_k^{(n-m)}(r_ke^{i\theta})|^p\,d\theta \le \frac{C(n,p)\varepsilon_3^p q_k}{(1-r_k)^{np-1}}.$$

It follows from (3.6) that if $\varepsilon_3=\varepsilon_3(p,n)$ is sufficiently small, then

$$\int_0^{2\pi}|B^{(n)}(r_ke^{i\theta})|^p\,d\theta \ge \frac{C(n,p)q_k}{(1-r_k)^{np-1}} \ge \frac{C(n,p)\varepsilon(r_k)}{(1-r_k)^{np+\alpha-1}}.$$

Now, repeating the argument above for a suitable "$\varepsilon(r)$", we obtain Theorem 1.2.

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