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KODAI MATH. J.
30 (2007), 41–54

DIVERGENCE THEOREM FOR SYMMETRIC $(0,2)$-TENSOR FIELDS ON A SEMI-RIEMANNIAN MANIFOLD WITH BOUNDARY

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Abstract

We prove in this paper a divergence theorem for symmetric $(0,2)$-tensors on a semi-Riemannian manifold with boundary. We obtain a generalization of results obtained by Ünal in [9, Acta Appl. Math. 40(1995)] and E. García-Río and D. N. Kupeli in [4, Proceeding of the Third World Congress of Nonlinear Analysts, Part 5 (Catania, 2000). Nonlinear Anal. 47 (5) 2995–3004, 2001].

As a tool, we use an induced volume form on the degenerate boundary by introducing a star like operator.

A vanishing theorem for gradient timelike Killing vector fields on Einstein semi-Riemannian manifolds is obtained.

1. Introduction and preliminaries

Our aim in this paper is to establish a divergence theorem for symmetric $(0,2)$-tensor fields on a semi-Riemannian manifold with smooth boundary (theorem 1). More specifically, we use an induced volume form on the degenarate boundary to solve the problem of divergence theorem on a semi-Riemannian manifold with smooth degenerate boundary (theorem 2, corollary 1).

There have been several attempts to extend the divergence theorem. To the best of our knowledge, K. L. Duggal is the first to have attempted in [1, section 3] to extend this theorem. However, in order to use the divergence theorem, he had to restric is study to the so-called regular semi-Riemannian manifold.

In [7], S. E. Stepanov considers the intrinsic geometry defined by the second fundamental form on the boundary without distinguishing the causal character (timelike, spacelike or null) of the tangent vectors at the boundary points.

2000 *Mathematics Subject Classification*. Primary: 53C50.

Key words and phrases. Divergence theorem, semi-Riemannian Manifolds, lightlike hypersurface, Hodge star operator.

The second author was support by ANTSI/UNESCO.

The authors would like to thank the referee for valuable remarks.

Received July 26, 2005; revised July 20, 2006.

The main problem to face is that the outward (or the inward) unit normal vector field which is needed to integrate on the degenerate boundary is not well defined at degenerate points.

1.1. A quick review of the geometry of the degenerate boundary. Let's consider a semi-Riemannian manifold (M, g) with boundary ∂M, possibly empty. A normal vector to ∂M at a point may have one of the three causal characters with respect to $g_{\partial M}$, called the induced metric on ∂M. One denotes ∂M_+, ∂M_-, ∂M_0, the sets of the points where normal vectors are spacelike, timelike, lightlike respectively. The subsets ∂M_+ and ∂M_- are open in ∂M and the subset ∂M_0 is closed in ∂M.

Clearly

$$\partial M = \partial M_+ \cup \partial M_- \cup \partial M_0 \tag{1}$$

and those subsets are pairwise disjoint. Consequently, $\partial M' = \partial M_+ \cup \partial M_-$ is an open submanifold of ∂M and may be considered as the nondegenerate boundary of M and ∂M_0 is referred to as the degenerate boundary of M. Let's suppose that (M, g) is oriented and let v_g denote its volume element. Then ∂M is also oriented and its area element is $\eta = i_N v_g$ where N is the outward unit normal vector field. According to (1) we have the following splitting of N into normal vector fields

$$N = \begin{cases} N_+ & \text{to } \partial M_+ \\ N_- & \text{to } \partial M_- \\ N_0 & \text{to } \partial M_0 \end{cases} \tag{2}$$

where N_0 is the null transverse vector field. The induced volume element on nondegenerate boundary is well known. The one on the degenerate boundary is constructed by using a *Hodge star like operator* defined as follows.

Let's recall that the Hodge star, $\star^H$ defined on an oriented $(n+2)$-semi-Riemannian manifold M, is a linear operator on $\Omega^*(M)$ mapping a p-form into an $(n+2-p)$-form. It may be locally defined, but it does not depend on a particular coordinate system. Its square is given by

$$\star^H \star^H = (-1)^{p(n+1)+s} I_{\Omega^p(M)}$$

where s is the number of minus signs in the metric.

Denoting $g_{|\partial M_0}$ the induced degenerate metric on ∂M_0, there exists locally a vector field $\xi \in \Gamma(T\partial M_0)$ such that $g(X, \xi) = 0 \ \forall X \in \Gamma(T\partial M_0)$. Locally, one has

$$TM|_{\partial M_0} = T\partial M_0 \oplus \operatorname{tr}(T\partial M_0), \tag{3}$$

$$T\partial M_0 = S(T\partial M_0) \perp \operatorname{Rad}(T\partial M_0) \tag{4}$$

where

$$\operatorname{Rad}(T\partial M_0) := T\partial M_0 \cap T\partial M_0^{\perp}$$

is the radical vector bundle, $\mathrm{tr}(T\partial M_0)$ is the transverse vector bundle and $S(T\partial M_0)$ is a screen distribution on ∂M_0. For any screen distribution $S(T\partial M_0)$, $\mathrm{tr}(T\partial M_0)$ exists and is unique. Using relation (4), one defines locally on ∂M_0 a smooth 1-form θ^0 by setting $\theta^0(X) = g(N_0, X)$ such that

$$\theta^0(\xi) = 1, \quad \theta^0(X) = 0, \quad \text{for any other } X \in \Gamma(T\partial M_0)$$

and moreover θ^0 is a section of $\mathrm{Rad}(T^*\partial M_0)$.

Let $h = (g_{|\partial M_0})_{S(T\partial M_0)}$ be the restriction of $g_{|\partial M_0}$, as a $(0,2)$-tensor, on the screen distribution $S(T\partial M_0)$.

We also denote the extension of g by g to the space of smooth forms on (M, g). Then, one generalizes h, on the differential forms of the screen distribution as

$$h(\alpha, \beta) = g(\alpha, \beta) \quad \alpha, \beta \in \Omega^k(S(T\partial M_0)) \tag{5}$$

where $\Omega^k(S(T\partial M_0))$ is the space of smooth k-forms of $S(T\partial M_0)$. It is characterized by

$$\Omega^k(S(T\partial M_0)) = \{\alpha \in \Omega^k(\partial M_0), i_\xi \alpha = 0\}.$$

We denote $\Omega^* S(T\partial M_0)$ the graded algebra of forms of $S(T\partial M_0)$ on ∂M_0. The graded algebra $\Omega^*\partial M_0$ splits as

$$\Omega^*\partial M_0 := \Omega^* S(T\partial M_0) \oplus \Omega^* Z \tag{6}$$

with

$$\Omega^* Z = \{\theta^0 \wedge \beta, \beta \in \Omega^* S(T\partial M_0)\} = \{\alpha \in \Omega^*(\partial M_0), \theta^0 \wedge \alpha = 0\}$$

1.2. Hodge star like operator on the boundary. Because h is nondegenerate and that the degenerate boundary ∂M_0 inherits the orientation of (M, g), we can define a star like operator, denoted $\star^s$, on the screen distribution.

According to decompositions (3) and (4) we can choose the field of frames $\{\xi, V_1, \dots, V_{n-2}\}$ on ∂M_0 and $\{N_0, \xi, V_1, \dots, V_{n-2}\}$ on M taking into account the orientation of M, where $n = \dim M$ and $\{V_1, \dots, V_{n-2}\}$ is an orthonormal basis of $\Gamma(S(T\partial M_0))$ [2, p. 9]. Thus, if v_M is a volume element of M and $\tilde{\theta} = g(N_0,)$, then we have

$$v_M = \tilde{\theta} \wedge \theta^0 \wedge \theta^1 \wedge \cdots \wedge \theta^{n-2}$$

where $\theta^i(V_j) = \delta_{ij}$.

Then a *Hodge star like operator* is defined on ∂M_0 as follows

$$\begin{cases} \star\alpha := (-1)^k \theta^0 \wedge \star^s \alpha & \forall \alpha \in \Omega^k(\partial M_0) \text{ and } i_\xi \alpha = 0 \\ \star\alpha := \star^s i_\xi \alpha & \forall \alpha \in \Omega^k(\partial M_0) \text{ and } \theta^0 \wedge \alpha = 0. \end{cases} \tag{7}$$

Thus for each $\alpha \in \Omega^k(S(T\partial M_0))$ defined on ∂M_0, we have

$$\theta^0 \wedge \alpha \wedge \star^s \alpha = h(\alpha, \alpha) v_{\partial M_0} \tag{8}$$

where $v_{\partial M_0}$ is the volume element on $(\partial M_0, g_{\partial M_0})$. It can easily be shown that for each $\alpha \in \Omega^k(\partial M_0)$ there exists a function L on ∂M_0 such that

$$\alpha \wedge \star\alpha = L v_{\partial M_0} \tag{9}$$

and

$$\star\star = (-1)^{kn} I_{\Omega^k(\partial M_0)}. \tag{10}$$

Consequently we define the coderivative on $(\partial M_0, g_{/\partial M_0})$ by

$$\delta = (-1)^{(k+1)(n+1)+1} \star d \star \tag{11}$$

for any k-form on ∂M_0.

For $\alpha \in \Omega^{k-1}(\partial M_0)$ and $\beta \in \Omega^k(\partial M_0)$, let's define

$$(d\alpha, \beta) := \int_{\partial M_0} d\alpha \wedge \star\beta.$$

Then clearly

$$(d\alpha, \beta) = (\alpha, \delta\beta).$$

The determination of the coderivative δ on ∂M_0 allows one to define the Laplace Beltrami operator, $\Delta = d\delta + \delta d$ on a lightlike hypersurface. Then we may obtain harmonic forms and de Rham decomposition on ∂M_0 if M is Lorentzian manifolds.

As an example, let M be a Monge hypersuface of $\mathbf{R}^4_1$ given by an equation $x^0 = F(x^1, x^2, x^3)$ with immersion:

$$(u^1, u^2, u^3) \mapsto (F(u^1, u^2, u^3), u^1, u^2, u^3)$$

such that

$$(F_1')^2 + (F_2')^2 + (F_3')^2 = 1$$

Then $TM^\perp = \mathrm{Rad}(TM)$ is spanned by

$$\xi = \frac{\partial}{\partial x^0} + \sum_{\alpha=1}^{3} F_\alpha' \frac{\partial}{\partial x^\alpha} \quad \text{where } F_\alpha' = \frac{\partial F}{\partial x^\alpha}.$$

Let's assume that $1 - (F_2')^2 \neq 0$ then $S(TM)$ is spanned by the orthonormal system $\{V_1, V_2\}$ with

$$V_1 = \frac{1}{(1 - (F_2')^2)^{1/2}} \left\{ F_3' \frac{\partial}{\partial x^1} - F_1' \frac{\partial}{\partial x^3} \right\}$$

$$V_2 = \frac{1}{(1 - (F_2')^2)^{1/2}} \left\{ -F_1' F_2' \frac{\partial}{\partial x^1} + (1 - (F_2')^2) \frac{\partial}{\partial x^2} - F_3' F_2' \frac{\partial}{\partial x^3} \right\}.$$

If we set

$$\theta^0 = F_1'\,du^1 + F_2'\,du^2 + F_3'\,du^3,$$

$$\theta^1 = \frac{1}{(1-(F_2')^2)^{1/2}}\{F_3'\,du^1 - F_1'\,du^3\},$$

and

$$\theta^2 = \frac{1}{(1-(F_2')^2)^{1/2}}\{-F_1'F_2'\,du^1 + (1-(F_2')^2)\,du^2 - F_3'F_2'\,du^3\},$$

then the volume element of ∂M_0 is

$$\theta^0 \wedge \theta^1 \wedge \theta^2 = du^1 \wedge du^2 \wedge du^3$$

and

$$\star\theta^0 = \star^s 1 = \theta^1 \wedge \theta^2, \quad \star\theta^1 = -\theta^0 \wedge \star^s\theta^1 = -\theta^0 \wedge \theta^2,$$
$$\star\theta^2 = -\theta^0 \wedge \star^s\theta^2 = \theta^0 \wedge \theta^1$$

So, this volume element is intrinsic (see also [6], p. 148).

Thus if X is a smooth vector field on (M, g) with compact support, we prove in section 2, corollary 1 that

$$\int_M \operatorname{div}(X) v_g = \int_{\partial M_0} g(X, \xi)\eta \tag{12}$$

whenever the boundary $\partial M = \partial M_0$ is degenerate. Note that here η is the area element on the boundary and ξ is an isotropic vector field.

Formula (12) is a new result on divergence theorem for vector fields.

For instance, consider the cylinder $M = S^1 \times [-1, 1]$ with Lorentzian metric

$$g = \frac{1}{2}[d\theta \otimes dt + dt \otimes d\theta] + (1-t)\,d\theta \otimes d\theta$$

where $t \in [-1, 1]$ and θ is the polar coordinate of S^1.

The boundary $\partial M = (S^1 \times \{1\}) \cup (S^1 \times \{-1\})$ is degenerate, i.e $\partial M = \partial M_0$

Let $X = t\dfrac{\partial}{\partial t}$ and $v_g = -\dfrac{1}{2}\,d\theta \wedge dt$, then $\int_M \operatorname{div} X v_g = 2\pi$.

We have $\eta_0 = \dfrac{1}{2}\,d\theta$ and $\xi = \dfrac{\partial}{\partial\theta}$ so that

$$\int_{\partial M_0} g(X, \xi)\eta_0 = \int_{S^1 \times \{1\}} g(X, \xi)\eta_0 - \int_{S^1 \times \{-1\}} g(X, \xi)\eta_0 = 2\pi$$

Thus our formula (12) still remains valid (compare to the counterexample in [9]).

2. Divergence theorem on manifolds with degenerate boundary

Consider a symmetric $(0,2)$-tensor field T on a n-dimensional semi-Riemannian manifold (M,g). The divergence of T is defined as the 1-form $\mathrm{div}(T)$ given by

$$\mathrm{div}\ T(X) = g^{kj}\nabla_{e_k} T(X,e_j), \quad \forall X \in \Gamma(TM)$$

where $B = \{e_i, i = 1,\dots,n\}$ is an orthonormal frame of parallel vector fields and ∇ is the Levi-Civita connection on (M,g).

Let's denote $\#$ the index uppering operator for g and let $T(e_i)^{\#}$ be the vector field associated, by duality, to the 1-form $T(e_i)$ defined by $T(e_i)(X) = T(e_i,X)$

Lemma 1. *Let T be a symmetric $(0,2)$-tensor on (M,g). Then*

$$\mathrm{div}\ T(X) = \mathrm{div}[(TX)^{\#}] - \frac{1}{2} g(L_X g, T) \tag{13}$$

Proof. Let $\mathbf{B} = \{e_j, j = 1,\dots,n\}$ be an orthonormal frame of parallel vector fields and g the metric on M.

$$\begin{aligned}\mathrm{div}\ T(X) &= g^{kj}(\nabla_{e_k} T)(X,e_j)\\ &= g^{kj}[\nabla_{e_k}(T(X,e_j)) - T(\nabla_{e_k}X, e_j) - T(X, \nabla_{e_k} e_j)\\ &= g^{kj}\nabla_{e_k}(T(X,e_j)) - g^{kj}g^{lm}g(\nabla_{e_k}X, e_l)T(e_j,e_m)\\ &\quad i,j,k = 1,\dots,n\\ \mathrm{div}\ T(X) &= g^{kj}\nabla_{e_k}(T(X,e_j)) - \frac{1}{2}g(T, L_X g)\\ &= g^{kj}\nabla_{e_k} g((TX)^{\#}, e_j) - \frac{1}{2}g(T, L_X g)\end{aligned}$$

so $\mathrm{div}[(TX)^{\#}] = g^{kj}\nabla_{e_k} g((TX)^{\#}, e_j)$. □

Theorem 1. *Let (M,g) be an oriented n-dimensional semi-Riemannian manifold with boundary ∂M, and T be a symmetric $(0,2)$-tensor field. Then*

$$\begin{aligned}\int_M \mathrm{div}\ T(X) v_g &= -\frac{1}{2}\int_M g(T, L_X g) v_g + \int_{\partial M_+} T(X, N_+)\eta_+\\ &\quad - \int_{\partial M_-} T(X, N_-)\eta_- + \int_{\partial M_0} T(X,\xi)\eta_0\end{aligned} \tag{14}$$

where $\eta_{\pm} = i_{N_{\pm}} v_g$, $\eta_0 = i_{N_0} v_g$ and ξ is such that $g(\xi, N_0) = 1$ and $g(\xi,\xi) = 0$ and X is a smooth vector field with compact support on M.

Two intermediary results are necessary before we can prove Theorem 1.

LEMMA 2. *Let (M, g) be an oriented n-dimensional semi-Riemannian manifold with boundary ∂M. If X is a smooth vector field with compact support on M, then*

$$\int_M \operatorname{div}(X) v_g = \int_M d(\operatorname{trace}(X \otimes v_g)) = \int_{\partial M} \operatorname{trace}(X \otimes v_g) \tag{15}$$

Proof. Let $\mathbf{B} = \{e_i, i = 1, \dots, n\}$ be an orthonormal frame which can possibly have isotropic vector field on M [2]. Then we have

$$\begin{aligned} \operatorname{div}(X) v_g &= L_X v_g \\ &= d i_X v_g \\ &= d(g^{kj} g(X, e_k) i_{e_j} v_g) \end{aligned}$$

where $k, j = 1, \dots, n$, $X = g^{kj} g(X, e_k) e_j$ and $g^{k,j}$ is the (k, j)-entry of the inverse g^{-1} of g. Thus

$$\operatorname{div}(X) v_g = d(\operatorname{trace}(X \otimes v_g)).$$

Using Stokes' theorem yields the proof. $\square$

LEMMA 3. *Let (M, g) be an n-dimensional oriented semi-Riemannian manifold with degenerate boundary ∂M_0. Let v_M be the volume element of (M, g), Then $i_{N_0} v_M$ is the area element of ∂M_0 at each point of ∂M_0.*

Proof. Let's consider the pseudo-orthonormal coordinate system $\{N_0, \xi, V_1, \dots, V_{n-2}\}$ on M and its dual $\{\tilde{\theta}, \theta^0, \theta^1, \dots, \theta^{n-2}\}$, where $\tilde{\theta}(N_0) = 1$ and $\theta^k(V_j) = \delta_{kj}$. We choose $\{V_1, \dots, V_{n-2}\}$ as an orthonormal coordinate system of $S(T\partial M_0)$. We have the volume element on (M, g):

$$v'_M = \tilde{\theta} \wedge \theta^0 \wedge \theta^1 \wedge \cdots \wedge \theta^{n-2}$$

since

$$\star^s \theta^1 = \theta^2 \wedge \cdots \wedge \theta^{n-2}.$$

Thus

$$i_{N_0} v'_M = \theta^0 \wedge \theta^1 \wedge \cdots \wedge \theta^{n-2} = \theta^0 \wedge \theta^1 \wedge \star^s \theta^1.$$

This equality is independent of the choice of $S(T\partial M_0)$. $\square$

Proof of Theorem 1. When integrating relation (13), we have

$$\int_M \operatorname{div}(T) X v_g = -\frac{1}{2} \int_M g(T, L_X g) v_g + \int_{\partial M} \operatorname{trace}[(TX)^{\#} \otimes v_g].$$

$$\begin{aligned}\operatorname{trace}((TX)^{\#}\otimes v_g)_{|\partial M} &= \operatorname{trace}((TX)^{\#}\otimes v_g)_{|\partial M_+} + \operatorname{trace}((TX)^{\#}\otimes v_g)_{|\partial M_-}\\ &\quad + \operatorname{trace}((TX)^{\#}\otimes v_g)_{|\partial M_0}\\ &= g((TX)^{\#}, N_+)\eta_+ - g((TX)^{\#}, N_+)\eta_+ + g((TX)^{\#}, \xi)\eta_0\\ &= T(X, N_+)\eta_+ - T(X, N_-)\eta_- + T(X, \xi)\eta_0.\end{aligned}$$

We can then use the relation (15), to conclude the proof. □

The trace operator is independent of the choice of coordinates. Therefore in the case of degenerate boundary, the divergence theorem is independent of the choice of screen distribution.

Theorem 1 can be used to generalize Ünal's results in [9] as follows

Theorem 2. *Let (M, g) be an oriented semi-Riemannian manifold with the boundary ∂M. If X is a smooth vector field with compact support on M, then*

$$\int_M \operatorname{div} X v_g = \int_{\partial M_+} g(X, N_+)\eta_+ - \int_{\partial M_-} g(X, N_-)\eta_- + \int_{\partial M_0} g(X, \xi)\eta_0$$

Moreover, if one of the following conditions holds.

1. *∂M_0 is a set of null measure in ∂M*
2. *X is tangent to ∂M at any point of ∂M_0,*

then

$$\int_M (\operatorname{div} X) v_g = \int_{\partial M_+} g(X, N_+)\eta_+ - \int_{\partial M_-} g(X, N_-)\eta_- \tag{16}$$

Proof. Let's $T = g$, then $\operatorname{div}(T) = 0$ and $g(g, L_X g) = 2\operatorname{div}(X)$.
One can then use theorem 1 to conclude the proof. □

Corollary 1. *Let (M, g) be an oriented semi-Riemannian manifold, ∂M its boundary and X a smooth vector field with compact support on M. If one of the following conditions holds,*

- *X is tangent to ∂M at the points of ∂M_+ and ∂M_-,*
- *$\partial M_+ = \partial M_- = \emptyset$,*

then

$$\int_M (\operatorname{div} X) v_g = \int_{\partial M_0} g(X, \xi)\eta_0 \tag{17}$$

Example 1. Let (M, g) be an Euclidean 3-dimensional Lorentzian manifold such that the boundary ∂M is defined by $x = F(y, z)$ where F is a smooth function such that

$$(F'_y)^2 + (F'_z)^2 = 1, \quad F'_\alpha = \frac{\partial F}{\partial \alpha}.$$

Then there exists a submersion f such that $f^{-1}(0) = \partial M$, where f is given by

$$(x, y, z) \mapsto f(x, y, z) = x - F(y, z),$$

and we have

$$df = -dx + F_y' \, dy + F_z' \, dz \neq 0.$$

Clearly ∂M is degenerate ($\partial M = \partial M_0$) and we can construct a Duggal-Bejancu basis by

$$N_0 = \frac{1}{2}\left(-\frac{\partial}{\partial x} + F_y' \frac{\partial}{\partial y} + F_z' \frac{\partial}{\partial z}\right); \quad \xi = \frac{\partial}{\partial x} + F_y' \frac{\partial}{\partial y} + F_z' \frac{\partial}{\partial z}$$

The area element on the boundary is given by

$$\eta = i^*\left[-\frac{1}{2}(dy \wedge dz + F_y' \, dx \wedge dz - F_z' \, dx \wedge dy)\right] = du^1 \wedge du^2$$

where i is the immersion

$$i(u^1, u^2) = (F(u^1, u^2), u^1, u^2).$$

If X is a smooth vector field, $X = (X^1, X^2, X^3)$ with compact support on M, we have

$$\begin{aligned}
\int_M \operatorname{div}(X) v_g &= \int_M d(X^1 \, dy \wedge dz) + \int_M d(X^2 \, dz \wedge dx) \int_M d(X^3 \, dx \wedge dy) \\
&= \int_{\partial M_0} i^*(X^1 \, dy \wedge dz + X^2 \, dz \wedge dx + X^3 \, dx \wedge dy) \\
&= \int_{\partial M_0} (X^1 - X^2 F_y - X^3 F_z) \circ i \, du^1 du^2 \\
&= \int_{\partial M_0} g(X, \xi) \eta
\end{aligned}$$

By theorem 2 we can extend a Stepanov result 8 to degenerate boundary case.

Let X be a smooth vector field on M and we put $A_X = -\nabla X$. Then X is said to be special concircular vector field with compact support, if

$$A_X Y = -\nabla_Y X = -\left(\frac{1}{n} \operatorname{div} X\right) Y, \quad \forall Y \in \Gamma(TM).$$

THEOREM 3. *Let (M, g) be an n-dimensional oriented Lorentzian manifold with degenerate boundary. If ξ is a conformal isotropic vector field with compact support such that* $\operatorname{Ric}(\xi, \xi) \leq 0$ *then ξ is parallel.*

Moreover there is no conformal isotropic vector field with compact support, which satisfies the condition $\operatorname{Ric}(\xi, \xi) < 0$.

The following lemma is necessary for the proof of Theorem 3

LEMMA 4. *Let M and ∂M be as in theorem 3. For $X \in \Gamma(TM)$ with compact support and tangent to ∂M, we have*

$$\int_M \{\mathrm{Ric}(X,X) + \mathrm{trace}(A_X)^2 - (\mathrm{trace}\ A_X)^2\} v_g = \int_{\partial M} B(X,X)\eta$$

where B is the second fundamental form of ∂M.

Proof. The proof comes from the following classic relation

$$\mathrm{div}(W) = \mathrm{Ric}(X,X) + \mathrm{trace}(A_X)^2 - (\mathrm{trace}\ A_X)^2 \tag{18}$$

where $W = \mathrm{trace}(A_X) - A_X X$.

Applying divergence theorem 2 leads to

$$\int_M \mathrm{div}\ W = \int_M \mathrm{Ric}(X,X) + \mathrm{trace}(A_X)^2 - (\mathrm{trace}\ A_X)^2 v_g$$
$$= \int_{\partial M} \mathrm{trace}(W \otimes v_g).$$

But we also have

$$g(W,N) = g(\mathrm{trace}(A_X) - A_X X), N) = B(X,X),$$

so the proof is completed. □

Proof of theorem 3. Let $X = \xi$ be an isotropic vector field, then $B(\xi,\xi) = 0$. Since ξ is a conformal vector field

$$\mathrm{trace}(A_\xi)^2 - (\mathrm{trace}\ A_\xi)^2 \le 0.$$

By lemma 4 we have a contradiction to the condition $\mathrm{Ric}(\xi,\xi) < 0$. □

THEOREM 4. *Let (M,g) be an n-dimensional oriented semi-Riemannian manifold with boundary. If X is special concircular smooth vector field of constant norm on M and transverse to ∂M at every point of ∂M, then*

$$\int_M \mathrm{Ric}(X,X) v_g \ge 0$$

Proof. Let $X \in \Gamma(TM)$ be a special concircular vector field with compact support and transverse to ∂M. Using relation (18), we have

$$\int_M \left\{\mathrm{Ric}(X,X) - \frac{n-1}{n}(\mathrm{div}\ X)^2\right\} v_g = -\frac{n-1}{n}\int_{\partial M} f(\mathrm{div}\ X)\eta \tag{19}$$

where $X = Z' + fN$, $Z' \in \Gamma(\partial M)$ and f is a constant function on ∂M. As $f = \text{constant}$, then

$$\int_{\partial M} f(\text{div } X)\eta = 0$$

concluding the proof. □

We end this part by proving the following theorem:

Theorem 5. *There is no gradient non degenerate Killing vector field in a semi-Riemannian, Einstein Ricci non flat compact manifold without boundary.*

Proof. For the Hessian of a function ϕ, we have [3]

$$\begin{aligned}\int_M \text{div } H^\phi(X) &= \int_M (-d(\Delta\phi)(X) + \text{Ric}(\nabla\phi, X))v_g \\ &= \frac{1}{2}\int_M g(H^\phi, L_X g)v_g + \int_{\partial M_+} H^\phi(X, N_+)\eta_+ \\ &\quad - \int_{\partial M_-} H^\phi(X, N_-)\eta_- + \int_{\partial M_0} H^\phi(X, \xi)\eta_0\end{aligned}$$

and in particular if X is a Killing vector field and $\partial M = \emptyset$, then

$$\int_M \text{div } H^\phi(X) = 0$$

so

$$\int_M (-d(\Delta\phi)(X) + \text{Ric}(\nabla\phi, X))v_g = 0$$

moreover

$$\int_M (-d(\Delta\phi)(X)) = 0.$$

Thus

$$\int_M \text{Ric}(\nabla\phi, X)v_g = 0. \qquad \square$$

As an immediate consequence, we have

Corollary 2. *On a connected semi-Riemannian Einstein Ricci non flat compact manifold M without boundary, harmonic functions are constant or have degenerate gradient vector fields.*

Proof. Let ϕ be a harmonic function and $X = \nabla\phi$. Theorem 5 gives

$$\int_M \text{Ric}(\nabla\phi, \nabla\phi) = 0$$

so

$$\int_M |\nabla\phi|^2 = 0.$$

Since M is Ricci non flat, we conclude $\nabla\phi = 0$ and ϕ is constant by the connectedness of M or $|\nabla\phi|^2 = 0$ and $\nabla\phi$ is degenerate. □

3. Electromagnetic tensor fields on lightlike hypersurface

In this part, we apply the Hodge star operator to a class of induced electromagnetic tensor fields to obtain Maxwell equations which will extend the result earlier obtained by Duggal-Bejancu [2, chapter 8].

Let $(\overline{M}, \bar{g}, \bar{F})$ be a time oriented 4-dimensional electromagnetic spacetime manifold with Lorentz metric $\bar{g}$ of signature $(-,+,+,+)$ and an electromagnetic tensor field $\bar{F}$. We define a tensor field $\bar{F} = (\bar{F}_a^b)$, of type $(1,1)$,

$$\bar{F}'^b_a = \bar{g}^{bc}\bar{F}_{ca}$$

where $a, b, c \in \{0, \dots, 3\}$

$$\bar{K} = \frac{1}{2}(\bar{F}_{ab}\bar{F}^{ab} + i\bar{F}_{ab}\bar{F}^{*ab})$$

There are two classes of electromagnetic tensor fields by the Ruse-Synge classification. Whether $\bar{F}$ is non-singular or singular depending on $K \neq 0$ or $K = 0$. It is known that $\bar{K}$ can be expressed in terms of Maxwell scalars [2, p. 238] and

$$\bar{K} = 2(\phi_1^2 - \phi_0\phi_2)$$

Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \bar{g}, \overline{F'})$. Hence relation (3) becomes

$$T\overline{M}_{|_M} = TM \oplus \operatorname{tr}(TM). \tag{20}$$

We say that $(M, g, S(TM))$ is electromagnetic invariant if

$$\bar{F}'(X) \in \Gamma(TM) \quad \forall X \in \Gamma(TM).$$

Now let f be the restriction of $\bar{F}'$ on M as an $(1,1)$-tensor field on M.

Then f is a skew symmetric $(1,1)$-tensor field with respect to the induced degenerate metric g (see [2, p. 241] theorem 21). If

$$F(X, Y) = g(f(X), Y) \quad \forall X, Y \in \Gamma(TM), \tag{21}$$

F is an induced electromagnetic tensor field on $(M, g, S(TM))$ and we call $(M, g, S(TM), F)$ an electromagnetic invariant lightlike hypersurface.

The tensor field F is singular (resp. non singular) if $\bar{F}$ is singular (resp. non singular). In terms of Maxwell scalars, one has $K = 2\phi_1^2$, (see [2, p. 242]).

In this text we only deal with the class of induced non-singular electromagnetic tensor fields. However, the result is similar to that of singular ones.

Let $\{\xi, V_1, V_2\}$ be a pseudo-orthonormal coordinate system of M with its dual $\{\theta^0, \theta^1, \theta^2\}$. Then a non-singular induced electromagnetic tensor field is expressed as follows

$$F = \mathrm{Im}(\phi_1)\theta^1 \wedge \theta^2 \tag{22}$$

where $(\mathrm{Im}(\phi_1))$ is the imaginary part of ϕ_1.

THEOREM 6. *Let $(M, g, S(TM), F)$ be an electromagnetic invariant lightlike hypersurface of 4-dimensional spacetime manifold $(\overline{M}, \bar{g}, \bar{F})$, where $\bar{F}$ is a non-singular electromagnetic tensor field on $\overline{M}$ and F is an induced electromagnetic tensor field on M. Assume that the screen distribution $S(TM)$ is integrable and that an induced connection ∇ on M is a metric connection. Then*

$$d \star F = 0 \quad (\delta_M F = 0) \text{ if and only if } V_1.(\mathrm{Im}(\phi_1)) = 0 \text{ and } V_2.(\mathrm{Im}(\phi_1)) = 0.$$

where Hodge star like operator $\star$ is defined by (7)

Proof. By theorem 3.1 [2, p. 248], for any $X \in \Gamma(TM)$, there exists a smooth function $\kappa(X)$ such that

(1) $\nabla_X \theta^1 = \kappa(X)\theta^2$
(2) $\nabla_X \theta^2 = -\kappa(X)\theta^1$
(3) $\nabla_X \theta^0 = 0$

Using (22) we have

$$\star F = \mathrm{Im}(\phi_1)\theta^0$$

and

$$d \star F = d(\mathrm{Im}(\phi_1))\theta^0.$$

Therefore

$$d \star F = 0 \Leftrightarrow V_1.(\mathrm{Im}(\phi_1)) = 0 \text{ and } V_2.(\mathrm{Im}(\phi_1)) = 0. \qquad \square$$

COROLLARY 3. *For $\mathrm{Im}(\phi_1) = constant$, the Maxwell equations obtained in M for the corresponding induced electromagnetic tensor field are*

$$\begin{aligned} dF &= 0 \\ d \star F &= 0 \quad (\delta F = 0) \end{aligned} \tag{23}$$

The condition $\phi_1 =$ constant is satisfied in the class of homogeneous non singular electromagnetic spacetime.

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