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ON COMPLEX WEYL-HLAVATÝ CONNECTIONS

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§ 0. Introduction.

To generalize results in conformal Riemannian geometry to those in Kaehlerian geometry, one of the present authors introduced in [4] what he calls a complex conformal connection in a Kaehlerian manifold. In this study, a curvature tensor introduced by Bochner [1] plays the rôle of the conformal curvature tensor of Weyl.

It is well known that the so-called Weyl-Hlavatý connection, that is, a linear connection D without torsion such that $\nabla_k g_{ji}=-2p_k g_{ji}$, p_k being a covector field, plays an important rôle in conformal Riemannian geometry, [5].

The main purpose of the present paper is to introduce a complex analogue of Weyl-Hlavatý connection in a Kaehlerian manifold and study its properties.

In § 1, we state some preliminaries on Kaehlerian geometry and on the Bochner curvature tensor and in § 2 we introduce what we call a complex Weyl-Hlavatý connection. § 3 is devoted to the study of the curvature tensor of a complex Weyl-Hlavatý connection. Using the results obtained in § 3, we prove our main theorem in § 4.

§ 1. Preliminaries.

We consider a Kaehlerian manifold M of real n dimensions ($n \geqq 4$) covered by a system of coordinate neighborhoods $\{U;\, x^h\}$ and denote by g_{ji} and $F_i{}^h$ components of the Hermitian metric tensor and those of the almost complex structure tensor of M respectively, where and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

Then we have

$$F_i{}^t F_t{}^h=-\delta_i^h, \qquad F_j{}^t F_i{}^s g_{ts}=g_{ji} \tag{1.1}$$

and

$$\nabla_k g_{ji}=0, \qquad \nabla_k F_i{}^h=0, \qquad \nabla_k F_{ji}=0, \tag{1.2}$$

where ∇_k denotes the operator of covariant differentiation with respect to the

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Christoffel symbols $\left\{ {h \atop j\ i} \right\}$ formed with g_{ji} and $F_{ji}=F_j{}^t g_{ti}$, and consequently $F_{ji} = -F_{ij}$.

We denote by $K_{kji}{}^h$, K_{ji} and K the curvature tensor, the Ricci tensor and the scalar curvature of M respectively. It is well known that these tensors satisfy

$$K_{kjt}{}^h F_i{}^t - K_{kji}{}^t F_t{}^h = 0\,, \qquad K_{kji}{}^h + K_{kjt}{}^s F_i{}^t F_s{}^h = 0\,, \tag{1.3}$$

$$K_{kjit} F_h{}^t - K_{kjht} F_i{}^t = 0\,, \qquad K_{kjih} - K_{kjts} F_i{}^t F_h{}^s = 0\,, \tag{1.4}$$

$$K_i{}^t F_t{}^h - F_i{}^t K_t{}^h = 0\,, \qquad K_i{}^h + K_t{}^s F_i{}^t F_s{}^h = 0 \tag{1.5}$$

and

$$K_{jt} F_i{}^t + K_{it} F_j{}^t = 0\,, \qquad K_{ji} - K_{ts} F_j{}^t F_i{}^s = 0\,, \tag{1.6}$$

where $K_{kjih} = K_{kji}{}^t g_{th}$ and $K_i{}^h = K_{it} g^{th}$, g^{th} being contravariant components of g_{ji}.

We define $H_i{}^h$ by

$$2H_i{}^h = -K_{kji}{}^h F^{kj}\,, \tag{1.7}$$

where $F^{kj} = g^{kt} F_t{}^j$. We then have

$$2H_{ih} = -K_{tsih} F^{ts} = -K_{ihts} F^{ts}\,, \tag{1.8}$$

where $H_{ih} = H_i{}^t g_{th}$, H_{ih} being skew-symmetric.

The relations between K_{ji} and H_{ji} are given by

$$K_{ji} = H_{jt} F_i{}^t\,, \qquad H_{ji} = -K_{jt} F_i{}^t\,. \tag{1.9}$$

The Bochner curvature tensor (Bochner [1], Tachibana [2], Yano and Bochner [3]) is given by

$$\begin{aligned} B_{kji}{}^h = {} & K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} \\ & + F_k{}^h M_{ji} - F_j{}^h M_{ki} + M_k{}^h F_{ji} - M_j{}^h F_{ki} - 2(M_{kj} F_i{}^h + F_{kj} M_i{}^h)\,, \end{aligned} \tag{1.10}$$

where

$$L_{ji} = -\frac{1}{n+4} K_{ji} + \frac{1}{2(n+2)(n+4)} K g_{ji}\,, \tag{1.11}$$

$$M_{ji} = -L_{jt} F_i{}^t\,, \tag{1.12}$$

that is,

$$M_{ji} = -\frac{1}{n+4} H_{ji} + \frac{1}{2(n+2)(n+4)} K F_{ji} \tag{1.13}$$

and

$$L_k{}^h = L_{kt} g^{th}\,, \qquad M_k{}^h = M_{kt} g^{th}\,. \tag{1.14}$$

Since H_{ji} and F_{ji} are both skew-symmetric, so is also M_{ji}.

§ 2. Complex Weyl-Hlavatý connections.

We consider an affine connection D with torsion in a Kaehlerian manifold M and denote by Γ_{ji}^{h} the components of the connection D and by D_j the operator of covariant differentiation with respect to Γ_{ji}^{h}.

If the affine connection D satisfies

$$D_k g_{ji} = -2p_k g_{ji}\,, \tag{2.1}$$

$$D_k F_{ji} = -2p_k F_{ji} \qquad (or\ D_k F_j{}^h = 0)\,, \tag{2.2}$$

$$\Gamma_{ji}^{h} - \Gamma_{ij}^{h} = -2F_{ji} q^h \tag{2.3}$$

for a certain non-zero covector field p_k and a vector field q^h, then we call D a complex Weyl-Hlavatý connection.

First of all, solving (2.1) and (2.3) with respect to Γ_{ji}^{h}, we find

$$\Gamma_{ji}^{h} = \left\{ {h \atop j\ i} \right\} + \delta_j^h p_i + \delta_i^h p_j - g_{ji} p^h + F_j{}^h q_i + F_i{}^h q_j - F_{ji} q^h\,, \tag{2.4}$$

where $p^h = p_t g^{th}$ and $q_i = q^t g_{ti}$. Next we compute $D_k F_{ji}$ using (2.4).

We then obtain

$$D_k F_{ji} = -2p_k F_{ji} - g_{kj}(p_t F_i{}^t + q_i) + g_{ki}(p_t F_j{}^t + q_j) + F_{kj}(p_i - q_t F_i{}^t) - F_{ki}(p_j - q_t F_j{}^t)\,,$$

from which, using (2.2),

$$g_{kj}(p_t F_i{}^t + q_i) - g_{ki}(p_t F_j{}^t + q_j) - F_{kj}(p_i - q_t F_i{}^t) + F_{ki}(p_j - q_t F_j{}^t) = 0\,.$$

Transvecting this equation with g^{kj}, we find

$$(n-2)(p_t F_i{}^t + q_i) = 0\,,$$

from which

$$q_i = -p_t F_i{}^t\,, \qquad p_i = q_t F_i{}^t\,. \tag{2.5}$$

Conversely, as is easily seen, the Γ_{ji}^{h} given by (2.4) where p_i and q_i are related by $q_i = -p_t F_i{}^t$ satisfy (2.1), (2.2) and (2.3).

Thus we have

PROPOSITION 2.1. *In a Kaehlerian manifold M with Hermitian metric tensor g_{ji} and the almost complex structure tensor $F_i{}^h$, a complex Weyl-Hlavatý connection is given by* (2.4) *where $q_i = -p_t F_i{}^t$.*

§ 3. Curvature tensor of a complex Weyl-Hlavatý connection.

We consider a complex Weyl-Hlavatý connection Γ^h_{ji} in a Kaehlerian manifold M and compute the curvature tensor of Γ^h_{ji}:

$$R_{kji}{}^h=\partial_k\Gamma^h_{ji}-\partial_j\Gamma^h_{ki}+\Gamma^h_{kt}\Gamma^t_{ji}-\Gamma^h_{jt}\Gamma^t_{ki}, \qquad (\partial_k=\partial/\partial x^k). \tag{3.1}$$

By straightforward computation, we find

$$\begin{aligned}R_{kji}{}^h=K_{kji}{}^h&-\delta^h_k p_{ji}+\delta^h_j p_{ki}-p_k{}^h g_{ji}+p_j{}^h g_{ki}-F_k{}^h q_{ji}+F_j{}^h q_{ki}\\&-q_k{}^h F_{ji}+q_j{}^h F_{ki}-\alpha_{kj}F_i{}^h-F_{kj}\beta_i{}^h+(\nabla_k p_j-\nabla_j P_k)\delta^h_i,\end{aligned} \tag{3.2}$$

where

$$p_{ji}=\nabla_j p_i-p_j p_i+q_j q_i+\frac{1}{2}\lambda g_{ji}, \tag{3.3}$$

$$q_{ji}=\nabla_j q_i-p_j q_i-q_j p_i+\frac{1}{2}\lambda F_{ji}, \tag{3.4}$$

λ being defined by $\lambda=p_i p^i=q_i q^i$,

$$\alpha_{ji}=-(\nabla_j q_i-\nabla_i q_j), \tag{3.5}$$

$$\beta_{ji}=2(p_j q_i-q_j p_i) \tag{3.6}$$

and $p_k{}^h=p_{kt}g^{th}$, $q_k{}^h=q_{kt}g^{th}$, $\beta_{ih}=\beta_i{}^t g_{th}$.

We can easily check that p_{ji}, q_{ji} and α_{ji} are related by

$$p_{ji}=q_{jt}F_i{}^t, \qquad q_{ji}=-p_{jt}F_i{}^t, \tag{3.7}$$

$$\alpha_{ji}=-(q_{ji}-q_{ij}-\lambda F_{ji}). \tag{3.8}$$

We now assume that the holonomy group of the connection D is that of dilatations, that is, we have equations of the form $R_{kji}{}^h=\nu_{kj}\delta^h_i$, ν_{kj} being a 2-form. Then from (3.2) we find

$$R_{kji}{}^h=(\nabla_k p_j-\nabla_j p_k)\delta^h_i. \tag{3.9}$$

Consequently (3.2) becomes

$$\begin{aligned}K_{kji}{}^h&-\delta^h_k p_{ji}+\delta^h_j p_{ki}-p_k{}^h g_{ji}+p_j{}^h g_{ki}-F_k{}^h q_{ji}+F_j{}^h q_{ki}\\&-q_k{}^h F_{ji}+q_j{}^h F_{ki}-\alpha_{kj}F_i{}^h-F_{kj}\beta_i{}^h=0\end{aligned} \tag{3.10}$$

or, in covariant form,

$$\begin{aligned}K_{kjih}=g_{kh}p_{ji}&-g_{jh}p_{ki}+p_{kh}g_{ji}-p_{jh}g_{ki}+F_{kh}q_{ji}-F_{jh}q_{ki}\\&+q_{kh}F_{ji}-q_{jh}F_{ki}+\alpha_{kj}F_{ih}+F_{kj}\beta_{ih}.\end{aligned} \tag{3.11}$$

Transvecting (3.11) with g^{kh} and using (3.7), we find

(3.12) $$K_{ji}=(n-1)p_{ji}+pg_{ji}-F_j{}^tq_{ti}+qF_{ji}+\alpha_{tj}F_i{}^t+F_{tj}\beta_i{}^t$$

where $p=g^{ji}p_{ji}$ and $q=g^{ji}q_{ji}=g^{ji}\nabla_jq_i$.

Transvecting (3.11) with F^{ih} and making use of (3.7), we obtain

(3.13) $$K_{kjts}F^{ts}=-4(q_{kj}-q_{jk})+n\alpha_{kj}+F_{kj}F^{ts}\beta_{ts},$$

from which, using (1.8), (3.8) and $F^{ts}\beta_{ts}=4\lambda$, we have

(3.14) $$H_{kj}=\frac{n+4}{2}(q_{kj}-q_{jk}-\lambda F_{kj}).$$

Transvecting (3.12) with $-F_h{}^i$ and using the second equation of (1.9) and (3.7), we find

$$H_{jh}=(n-1)q_{jh}+pF_{jh}+F_j{}^tp_{th}-qq_{jh}-\alpha_{jh}-F_{tj}F_h{}^s\beta_s{}^t,$$

from which, making use of (3.8) and the equation

$$F_{tj}F_h{}^s\beta_s{}^t=2F_{tj}F_h{}^s(p_sq^t-q_sp^t)=2(p_jq_h-q_jp_h)=\beta_{jh},$$

we have

(3.15) $$H_{ji}=nq_{ji}-q_{ij}+pF_{ji}+F_j{}^tp_{ti}-qg_{ji}-\lambda F_{ji}-\beta_{ji}.$$

Transvecting (3.12) with g^{ji} and making use of $F^{ts}q_{ts}=p$, $\alpha_{ts}F^{ts}=n\lambda-2p$ and $\beta_{ts}F^{ts}=4\lambda$, we find

(3.16) $$K=2(n+1)p-(n+4)\lambda.$$

Transvecting (3.14) with F^{kj}, we have

(3.17) $$K=(n+4)p-\frac{1}{2}n(n+4)\lambda.$$

From (3.16) and (3.17), we find

(3.18) $$\lambda=p_ip^i=-\frac{K}{(n+2)(n+4)}.$$

Now, transvecting (3.12) with $F_t{}^jF_s{}^i$, we have

$$F_t{}^jF_s{}^iK_{ji}=-(n-1)F_t{}^iq_{is}+pq_{ts}+p_{ts}+qF_{ts}-F_t{}^i\alpha_{si}+F_t{}^i\beta_{is},$$

or, using the second equation of (1.6),

$$K_{ji}=-(n-1)F_j{}^tq_{ti}+pg_{ji}+p_{ji}+qF_{ji}-F_j{}^t\alpha_{it}+F_j{}^t\beta_{ti}.$$

Thus comparing (3.12) with this equation, we find

(3.19) $$0=(n-2)p_{ji}+(n-2)F_j{}^tq_{ti}+\alpha_{tj}F_i{}^t+F_j{}^t\alpha_{it},$$

from which, using (3.8),

(3.20) $$(n-1)(p_{ji}+F_j{}^t q_{ti})-(p_{ij}+F_i{}^t q_{tj})=0.$$

From (3.20) and

$$(n-1)(p_{ij}+F_i{}^t q_{tj})-(p_{ji}+F_j{}^t q_{ti})=0,$$

which is equivalent to (3.20), we obtain

$$p_{ji}+F_j{}^t q_{ti}=0,$$

or

(3.21) $$p_{ji}=-F_j{}^t q_{ti}, \qquad q_{ji}=F_j{}^t p_{ti}.$$

Thus, (3.15) can be written as

(3.22) $$H_{ji}=(n+1)q_{ji}-q_{ij}+pF_{ji}-qg_{ji}-\lambda F_{ji}-\beta_{ji}.$$

But, H_{ji} being skew-symmetric, we have from (3.22)

$$0=(n+1)(q_{ji}+q_{ij})-(q_{ji}+q_{ij})-2qg_{ji},$$

from which

(3.23) $$q_{ji}=\frac{2}{n}qg_{ji}-q_{ij}.$$

Substituting (3.23) into (3.14), we find

$$H_{kj}=\frac{n+4}{2}\left(\frac{2}{n}qg_{kj}-2q_{jk}-\lambda F_{kj}\right),$$

from which

$$q_{ji}=\frac{1}{n}qg_{ji}+\frac{1}{2}\lambda F_{ji}+\frac{1}{n+4}H_{ji},$$

or, using (3.18),

(3.24) $$q_{ji}=\frac{1}{n}qg_{ji}-M_{ji}.$$

From (3.7) and (3.24), we have

(3.25) $$p_{ji}=-\frac{1}{n}qF_{ji}-L_{ji}.$$

Substituting (3.24) into (3.8), we find

(3.26) $$\alpha_{ji}=2M_{ji}+\lambda F_{ji}.$$

On the other hand, we have from (3.22)

$$\beta_{ji}=(n+1)q_{ji}-q_{ij}+pF_{ji}-qg_{ji}-\lambda F_{ji}-H_{ji},$$

from which, substituting (3.24) we find

(3.27) $$\beta_{ji}=-(n+2)M_{ji}+(p-\lambda)F_{ji}-H_{ji}\,.$$

But, (3.17) shows that

$$p-\lambda=\frac{1}{n+4}K+\frac{n-2}{2}\lambda$$

and consequently we can write (3.27) in the form

$$\beta_{ji}=-(n+2)M_{ji}+\Big(\frac{1}{n+4}K+\frac{n-2}{2}\lambda\Big)F_{ji}-H_{ji}\,,$$

or, using (1.13) and (3.18),

(3.28) $$\beta_{ji}=2M_{ji}-\lambda F_{ji}\,.$$

§ 4. Theorems

In this last section, we prove the following two theorems.

THEOREM 4.1. *Let M be a real n-dimensional Kaehlerian manifold, $(n\geqq 4)$. If M admits a complex Weyl-Hlavatý connection such that its holonomy group is that of dilatations, then the Bochner curvature tensor of M vanishes.*

Proof. Substituting (3.24), (3.25), (3.26) and (3.28) into (3.10), we obtain

$$\begin{aligned}K_{kji}{}^h=&-\delta_k^h L_{ji}+\delta_j^h L_{ki}-L_k{}^h g_{ji}+L_j{}^h g_{ki}\\&-F_k{}^h M_{ji}+F_j{}^h M_{ki}-M_k{}^h F_{ji}+M_j{}^h F_{ki}+2(M_{kj}F_i{}^h+F_{kj}M_i{}^h)\,,\end{aligned}$$

that is

$$B_{kji}{}^h=0\,.$$

THEOREM 4.2. *Let M be a real n-dimensional Kaehlerian manifold, $(n\geqq 4)$. If one of the following conditions is satisfied, then there does not exist a complex Weyl-Hlavatý connection such that its holonomy group is that of dilatations.*

(1) *M is compact,*

(2) *The scalar curvature K is non-negative,*

(3) *$K_{ji}K^{ji}=$constant.*

Proof. (1) From (3.6) and (3.28), we have

$$p_j q_i-p_i q_j=M_{ji}-\frac{\lambda}{2}F_{ji}\,,$$

or, using (1.12) and (3.18)

(4.1) $$p_j p_i+F_j{}^t F_i{}^s p_t p_s-\frac{1}{(n+2)(n+4)}Kg_{ji}+\frac{1}{n+4}K_{ji}=0\,.$$

On the other hand, from (3.3) and (3.25), we have

$$\nabla_j p_i = p_j p_i - F_j{}^t F_i{}^s p_t p_s - \frac{1}{2}\lambda g_{ji} - \frac{1}{n} q F_{ji} - L_{ji},$$

or taking account of (3.18),

$$\nabla_j p_i = p_j p_i - F_j{}^t F_i{}^s p_t p_s - \frac{1}{n} q F_{ji} + \frac{1}{n+4} K_{ji}. \tag{4.2}$$

Eliminating $p_j p_i$ from (4.1) and (4.2), we obtain

$$\nabla_j p_i = -2F_j{}^t F_i{}^s p_t p_s - \frac{1}{n} q F_{ji} + \frac{1}{(n+2)(n+4)} K g_{ji}. \tag{4.3}$$

By covariant differentiation, we have from (3.18)

$$2p^i \nabla_j p_i + \frac{1}{(n+2)(n+4)} \nabla_j K = 0. \tag{4.4}$$

Thus substituting (4.3) into (4.4), we find

$$2p^i\Big[-2F_j{}^t F_i{}^s p_t p_s - \frac{1}{n} q F_{ji} + \frac{1}{(n+2)(n+4)} K g_{ji}\Big]$$
$$+ \frac{1}{(n+2)(n+4)} \nabla_j K = 0,$$

or

$$-\frac{2}{n} q p^i F_{ji} + \frac{2}{(n+2)(n+4)} K p_j + \frac{1}{(n+2)(n+4)} \nabla_j K = 0, \tag{4.5}$$

from which, transvecting with p^j,

$$2K p_j p^j + p^j \nabla_j K = 0,$$

or, taking account of (3.18),

$$\frac{2}{(n+2)(n+4)} K^2 - p^j \nabla_j K = 0. \tag{4.6}$$

Consequently, by Green's theorem, we have

$$\int_M \Big[\frac{2}{(n+2)(n+4)} K^2 + K \nabla_j p^j\Big] dV = 0, \tag{4.7}$$

dV denoting the volume element of M. But (4.2) shows that

$$\nabla_j p^j = \frac{1}{n+4} K.$$

Thus (4.7) becomes

$$\int_M \Big[\frac{2}{(n+2)(n+4)} K^2 + \frac{1}{n+4} K^2\Big] dV = 0,$$

from which it follows that $K=0$ and consequently, from (3.18), we have $p_i=0$.

(2) (3.18) shows that if $K \geqq 0$, then $p_i=0$.

(3) Transvecting (4.1) with $p^j p^i$, we have

$$p_j p^j p_i p^i - \frac{1}{(n+2)(n+4)} K p_i p^i + \frac{1}{n+4} p^j p^i K_{ji} = 0$$

or, using (3.18)

$$p^j p^i K_{ji} = -\frac{2K^2}{(n+2)^2(n+4)} . \tag{4.8}$$

Transvecting (4.1) with K^{ji} and using the second equation of (1.6), we have

$$2p_j p_i K^{ji} - \frac{K^2}{(n+2)(n+4)} + \frac{1}{n+4} K_{ji} K^{ji} = 0 .$$

Substituting (4.8) into this equation, we find

$$K_{ji} K^{ji} = \frac{n+6}{(n+2)^2} K^2$$

from which it follows that $K=$constant by virtue of the assumption $K_{ji}K^{ji}=$ constant.

Thus, from (4.6), we have $K=0$ and consequently, from (3.18), we find $p_i=0$.

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