ON CONFORMALLY FLAT SPACES WITH DEFINITE RICCI CURVATURE II

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1. Introduction. There is a formal similarity between the theory of hypersurfaces and conformally flat \( d \)-dimensional spaces of constant scalar curvature provided \( d \geq 3 \). For, then, the symmetric linear transformation field \( Q \) defined by the Ricci tensor satisfies the "Codazzi equation"

\[
(V_X Q)Y = (V_Y Q)X.
\]

This observation together with the technique and results in [2] and [3] yields the following statement.

**Theorem.** Let \( M \) be a compact conformally flat manifold with definite Ricci curvature. If the scalar curvature \( r \) is constant and \( \text{tr} \, Q^2 \leq r^2/d - 1 \), \( d \geq 3 \), then \( M \) is a space of constant curvature.

The corresponding result for hypersurfaces is due to M. Okumura [3].

**Corollary.** A 3-dimensional compact conformally flat manifold of constant scalar curvature whose sectional curvatures are either all negative or all positive is a space of constant curvature.

Note that, in general \( \text{tr} \, Q^2 \geq r^2/d \) with equality, if and only if, \( M \) is an Einstein space.

Examples of compact negatively curved space forms are given in the paper by A. Borel [1].

2. Definitions and formulas. Let \( (M, g) \) be a Riemannian manifold with metric tensor \( g \). The curvature transformation \( R(X, Y), X, Y \in M \) — the tangent space at \( m \in M \), and \( g \) are related by

\[
R(X, Y) = \nabla_{[X, Y]} - \nabla_{[Y, X]},
\]

where \( \nabla_X \) is the operation of covariant differentiation with respect to \( X \) defined in terms of the Levi-Civita connection. In terms of a basis \( X_1, \ldots, X_d \) of \( M \), we set

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We denote the scalar curvature by \( r \), that is \( r = \text{tr} Q \), where \( Q = (R^i_j) \), \( R^i_j = g^{ik}R_{jk} \). The manifold \((M, g)\) is conformally flat if \( g \) is conformally related to a locally flat metric. The Weyl conformal curvature tensor defined by

\[
C^i_{jkh} = R^i_{jkh} - \frac{1}{d-2} (R_{jk} \delta^i_h - R_{jh} \delta^i_k + g_{jh} R^i_k - g_{jh} R^k_i) \\
+ \frac{r}{(d-1)(d-2)} (g_{jh} \delta^i_k - g_{jh} \delta^k_i)
\]

consequently vanishes, so if \((M, g)\) is conformally flat

\[
R^i_{jkh} = \frac{1}{d-2} (R_{jk} \delta^i_h - R_{jh} \delta^i_k + g_{jh} R^i_k - g_{jh} R^k_i) \\
- \frac{r}{(d-1)(d-2)} (g_{jh} \delta^i_k - g_{jh} \delta^k_i).
\]

From (2.1) and the second Bianchi identity

\[
\nabla_i C^i_{jkh} = (d-3)C_{jkh},
\]

where

\[
C_{jkh} = \frac{1}{d-2} (\nabla_h R_{jk} - \nabla_k R_{jh}) - \frac{1}{2(d-1)(d-2)} (g_{jh} \nabla^i_r - g_{jh} \nabla^r_i).
\]

For \( d=3 \) it can be shown that if \((M, g)\) is conformally flat, then \( C_{i,j,k} = 0 \).

### 3. The Laplacian of the square length of the Ricci tensor.

The following formula may be found in [2]:

\[
\frac{1}{2} \Delta \text{tr} Q^2 = g^{ab} \nabla_a R^i_j \nabla_b R_{ij} + R^{ij} g^{ab} \nabla_a (\nabla_b R_{ij} - \nabla_i R_{bj}) \\
+ \frac{1}{2} R^{ij} \nabla_i \nabla_j + K,
\]

where \( \text{tr} Q^2 = R^{ij} R_{ij} \) and

\[
K = R^{lk} (R_{jk} R_{lk} + R_{kj} R_{lk}) + \frac{1}{2} R^{ij} \nabla_i \nabla_j + K.
\]

If \( r = \text{const.} \), the third term on the right-hand side of (3.1) vanishes. If, furthermore, \( M \) is conformally flat and \( d \geq 3 \), then from (2.3) and (2.4), the second term on the right-hand side of (3.1) also vanishes. Substituting (2.2) into the right-hand side of (3.2), we obtain
\[ \frac{1}{2} \Delta \text{tr } Q^2 = K + g(FQ, FQ), \]

where

\[ (d-1)(d-2)K = d(d-1) \text{tr } Q^2 - r(2d-1) \text{tr } Q^2 + r^3. \]

4. Proof of Theorem. Put

\[ S = Q - \frac{r}{d} I, \]

where \( I \) is the identity. Since \( \text{tr } S^2 \geq 0, \)

\[ \text{tr } Q^2 \geq \frac{r^2}{d}, \]

equality holding if and only if \( M \) is an Einstein space. Since the scalar curvature is constant, the Laplacian \( \Delta f^2 \) of the function \( f^2 = \text{tr } S^2, f \geq 0, \) satisfies

\[ \Delta f^2 = \Delta \text{tr } Q^2, \]

so that

\[ \frac{1}{2} \Delta f^2 = K + g(FQ, FQ). \]

From the definition of \( S, \) we get

\[ \text{tr } S = 0, \]

\[ \text{tr } Q^2 = \text{tr } S^2 + \frac{r^2}{d}, \]

\[ \text{tr } Q^2 = \text{tr } S^2 + \frac{3r}{d} \text{tr } S^2 + \frac{r^3}{d^2}. \]

Substituting (4.3) and (4.4) in (3.3), we obtain

\[ (d-1)(d-2)K = d(d-1) \left( \text{tr } S^2 + \frac{3r}{d} f^2 + \frac{r^3}{d^2} \right) - r(2d-1) \left( f^2 + \frac{r^2}{d} \right) + r^3. \]

**Lemma.** Let \( a_i, i = 1, \ldots, d \) be real numbers such that

\[ \sum_{i=1}^{d} a_i = 0, \quad \sum_{i=1}^{d} a_i^2 = k^2, \quad k = \text{const.} \geq 0. \]

Then,

\[ -\frac{d-2}{\sqrt{d(d-1)}} k^3 \leq \sum_{i=1}^{d} a_i^3 \leq \frac{d-2}{\sqrt{d(d-1)}} k^3. \]

Applying the lemma to the eigenvalues of \( S, \) (4.5) yields the following inequality

\[ (d-1)K \geq f^2 (r - \sqrt{d(d-1)} f). \]
Thus, since $f \leq r/\sqrt{d(d-1)}$, $\Delta f \geq 0$, from which since $M$ is compact, $f^2 = \text{const.}$, so $\text{tr} \ Q^2 = \text{const.}$ It follows from (3.1) that $\nabla Q = 0$. Theorem 1 of [2] then gives the desired result.

In case the sectional curvatures are all positive the corollary is due to M. Tani [4].

The condition that the Ricci tensor is definite is essential. For, if $M = M_1 \times N$ where $M_1$ has constant curvature and $N$ is 1-dimensional, then $M$ is conformally flat, $r$ is constant, and $\text{tr} \ Q^2 = r^2/d-1$.

BIBLIOGRAPHY


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