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KŌDAI MATH. SEM. REP.
27 (1976), 432—435

THE AXIOM OF COHOLOMORPHIC 3-SPHERES IN AN ALMOST TACHIBANA MANIFOLD

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§1. Introduction. Let M be an almost Hermitian manifold with metric tensor $\langle , \rangle$, Riemannian connection ∇, and almost complex structure J. A 2-plane φ is called holomorphic (resp. totally real (or called antiholomorphic)) if $J\varphi=\varphi$ (resp. if $J\varphi$ is prependicular to φ), where we mean an r-dimensional linear subspace of tangent space by r-plane. K. Yano and I. Mogi [9] (resp. B. Y. Chen and K. Ogiue [1]) proved that a Kaehlerian manifold with the axiom of holomorphic 2-planes (resp. the axiom of totally real 2-planes) is a complex space form. A 3-plane is called coholomorphic if it contains a holomorphic 2-planes a holomorphic 2-plane φ. It is clear that a coholomorphic 3-plane also contains a totally real 2-plane. Recently, B. Y. Chen and K. Ogiue [2] have considered the axiom of coholomorphic 3-spheres as follows: *For each point of $x\in M$ and each coholomorph 3-plane π, there exists a 3-dimensional, totally umbilical submanifold N such that $x\in N$ and $T_x(N)=\pi$.*

The purpose of this is to study an almost Tachibana manifold satisfying the axiom of coholomorphic 3-spheres and to prove the following:

THEOREM. *Let M be an n-dimensional non-Kaehlerian almost Tachibana manifold satisfying $\|(\nabla_X J)(Y)\|^2=$constant for all orthonormal vectors X and Y that span a totally real 2-plane. If M admits the axiom of coholomorphic 3-spheres, then M is 6-dimensional manifold of constant curvature $C>0$.*

§2. Submanifold. Let N be a submanifold of M, and ∇ and ∇' be the covariant differentiations on M and N respectively. Then the second fundamental forms B of the immersion is defined by $B(X, Y)=\nabla_X Y-\nabla'_X Y$, where X and Y are vector fields tangent to N. B is a normal bundle valuded symmetric 2-form on N. For a vector field ξ normal to N we write $\nabla_X\xi=-A_\xi(X)+D_X\xi$, where $-A_\xi(X)$ (resp. $D_X\xi$) denotes the tangential (resp. normal) component of $\nabla_X\xi$. The submanifold N is said to be totally umbilical if $B(X, Y)=\langle X, Y\rangle H$, where H is the mean curvature vector of N.

For the second fundamental form B of N in M we define the covariant derivative, denoted by $\bar{\nabla}_X B$, to be

Received Jan. 10, 1975.

$$(\bar{\nabla}_X B)(Y, Z) = D_X(B(Y, Z)) - B(\nabla'_X Y, Z) - B(Y, \nabla'_X Z) \tag{2.1}$$

Then, for all vector fields X, Y, Z, W tangent to N the equations of Gauss and Codazzi take the form

$$\langle R(X, Y)Z, W\rangle = \langle R'(X, Y)Z, W\rangle + \langle B(X, Z), B(Y, W)\rangle - \langle B(Y, Z), B(X, W)\rangle\,, \tag{2.2}$$

$$(R(X, Y)Z)^{\perp} = (\bar{\nabla}_X B)(Y, Z) - (\bar{\nabla}_Y B)(X, Z)\,, \tag{2.3}$$

where $\perp$ in (2.3) means the normal component.

§ 3. **Almost Tachibana manifold.** Let M be an n dimensional almost Hermitian manifold. Then M is said to be an almost Tachibana manifold (K-space or nearly Kähler manifold) provided $(\nabla_X J)(Y) + (\nabla_Y J)(X) = 0$ for any vectors X and Y of M. It is well known that $n \geqq 6$ for a non-Kählerian almost Tachibana manifold M.

Let $R(X, Y)$ be the curvature tensor of M given by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. We denote $R(X, Y; Z, W)$ by $R(X, Y; Z, W) = \langle R(X, Y)Z, W\rangle$. The sectional curvature of M determined by orthonormal vectors X and Y is given by $K(X, Y) = R(X, Y : Y, X)$. The holomorphic sectional curvature $H(X)$ for unit tangent vector X is the sectional curvature $K(X, JX)$. Let x be a point of M. If $H(X)$ is constant for every x and every unit tangent vector X at x, then M is said to be of constant holomorphic sectional curvature.

In an almost Tachibana manifold M, the following identities are well known [4]:

$$\langle R(X, Y)Y, X\rangle - \langle R(X, Y)JY, JX\rangle = \|(\nabla_X J)(Y)\|^2\,, \tag{3.1}$$

$$\langle R(X, Y)Z, W\rangle = \langle R(JX, JY)JZ, JW\rangle\,, \tag{3.2}$$

$$\langle R(X, JX)JY, Y\rangle = \langle R(X, Y)Y, X\rangle + \langle R(X, JY)JY, X\rangle - 2\|(\nabla_X J)(Y)\|^2 \tag{3.3}$$

for any vector fields X, Y, Z, W of M.

We know [4] that an almost Tachibana manifold M has global constant type if and only if there exists a constant α such that

$$\|(\nabla_X J)(Y)\|^2 = \alpha[\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2 - \langle X, JY\rangle^2]$$

for any vectors X, Y of M. Moreover a non-Kaehlerian almost Hermitian manifold M is said to be a special almost Tachibana manifold if the associated 2-form $\Omega(X, Y) = \langle JX, Y\rangle$ is a special Killing 2-form with constant α $(\neq 0)$. Such M is an almost Tachibana manifold with global constant type and a 6-dimensional Einstein manifold [8]. One of the examples of almost Tachibana manifold is a 6-dimensional sphere [3].

§ 4. **Proof of Theorem.** Let X and Y be any orthonormal tangent vectors of M at $x\in M$ such that X and Y span a totally real 2-plane. Then X, Y and JX span a coholomorphic 3-plave π. By the axiom of coholomorphic 3-spheres, there exists a 3-dimensional totally umbilical submanifold N such that $x\in N$ and $T_x(N)=\pi$. Making use of (2.1) and (2.3), it follows that

$$(R(X, JX)Y)^{\perp}=\langle JX, Y\rangle D_X H-\langle X, Y\rangle D_{JX}H=0\,,$$

from which

$$\langle R(X, JX)Y, JY\rangle=0 \tag{4.1}$$

for all orthonormal vectors X and Y that span a totally real 2-plane. It is clear that $(X+Y)/\sqrt{2}$ and $(JX-JY)/\sqrt{2}$ also span a totally real 2-plane. Therefore we have by virtue of (3.1), (3.3) and (4.1)

$$K(X+Y, JX-JY)=\frac{1}{4}[H(X)+H(Y)+2\|(\nabla_X J)(Y)\|^2]\,. \tag{4.2}$$

On the other hand, regarding to (3.3) and (4.1), we get

$$\begin{aligned} K(X+Y, JX-JY)&=-K(X+Y, X-Y)+2\|(\nabla_X J)(Y)\|^2 \\ &=-K(X, Y)+2\|(\nabla_X J)(Y)\|^2 \end{aligned} \tag{4.3}$$

and therefore, by (4.2) we have

$$K(X, Y)=-\frac{1}{4}[H(X)+H(Y)-6\|(\nabla_X J)(Y)\|^2]\,. \tag{4.4}$$

Taking account of JY in stead of Y in (4.4), it follows that

$$K(X, JY)=-\frac{1}{4}[H(X)+H(Y)-6\|(\nabla_X J)(Y)\|^2]\,. \tag{4.5}$$

Adding side by side of (4.4) and (4.5) and using (3.3), it follows that

$$H(X)+H(Y)=2C\,, \tag{4.6}$$

where we put $\|(\nabla_X J)(Y)\|^2=C$. Since M is a non-Kaehlerian almost Tachibana manifold, we find $n\geqq 6$. Hence we have $H(X)=C$, that is, M is of constant holomorphic sectional curvature C. Our assertion follows the following Theorems:

THEOREM [7]. *If a 6-dimensional almost Tachibana manifold is of constant holomorphic sectional curvature C, then either it is Kaehlerian, or it is of constant curvature $C>0$.*

THEOREM [6]. *There does not exist any dimensional, except 6-dimensional, non-Kaehlerian almost Tachibana manifold of constant holomorphic sectional curvature.*

By virtue of Theorem, we have immediately

COROLLARY 1. *Let M be a non-Kaehlerian almost Tachibana manifold with global constant type. If M admits the axiom of coholomorphic* 3*-spheres, then M is* 6*-dimensional manifold of constant curvature $C>0$.*

COROLLARY 2. *Let M be a special almost Tachibana manifold. If M admits the axiom of coholomorphic* 3*-spheres, then M is* 6*-dimensional manifold of constant curvature $C>0$.*

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