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ON THE STABILITY OF TWO-DIMENSIONAL LINEAR STOCHATIC SYSTEMS

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1. Introduction and preliminaries.

Consider a two-dimensional linear system of temporally homogeneous stochastic differential equations:

$$dX(t)=B\cdot X(t)dt+C\cdot X(t)dB_1(t)+D\cdot X(t)dB_2(t) \tag{1.1}$$

where B, C, and D are 2×2 constant matrices and $B_i(t)$ ($i=1, 2$) are independent Brownian motions. Our concern is the asymptotic stability with probability 1 of the system (1.1), i. e., we say that $X^{x_0}(t)$ is stable if

$$P_{x0}\{\lim_{t\to\infty}|X(t)|=0\}=1,$$

and that it is divergent if

$$P_{x0}\{\lim_{t\to\infty}|X(t)|=\infty\}=1$$

(here and later on $X^{x_0}(t)$ stands for a solution of (1.1) satisfying $X^{x_0}(0)=x_0$). Applying Ito's formula to $\rho(t)\equiv\log|X(t)|$, Khas'minskii [6] showed that

$$\lim_{T\to\infty}\frac{1}{T}(\rho(T)-\rho(0))=\lim_{T\to\infty}\frac{1}{T}\int_0^T Q(\theta(t))dt \qquad \text{a. s.,} \tag{1.2}$$

where $\theta(t)$ is the angular component of $X(t)$ and

$$Q(\theta)\equiv(B\cdot e(\theta), e(\theta))+\frac{1}{2}\mathrm{Sp}\cdot A(e(\theta))-(A(e(\theta))\cdot e(\theta), e(\theta)), \tag{1.3}$$

in which

$$a(x)_{ij}\equiv\sum_{m,n=1}^{2}(c_{im}c_{jn}+d_{im}d_{jn})x_m x_n \tag{1.4}$$

$$e(\theta)\equiv(\cos\theta, \sin\theta)$$

(we denote by c_{ij} and x_i an (i, j)-element of a matrix C and an i-element of a vector X, respectively, etc.). Then, he has proved: if

$$J\equiv\lim_{T\to\infty}\frac{1}{T}\int_0^T Q(\theta(t))dt$$

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exists and J is a constant independent of samples (which may depend on x^0), then, for an arbitrary x_0 ($\neq 0$),

$X^{x_0}(t)$ is stable in case that $J<0$,

it is divergent in case that $J>0$,

it is neither stable nor divergent in case that $J=0$.

However, to show the existence of the constant J, he needed the non-degenerate condition, i.e.,

$$(A(x)\lambda, \lambda) \geqq a|x|^2|\lambda|^2, \qquad (a \text{ is some positive constant}).$$

Our purpose, in this paper, is to determine J without any assumption, then we shall be able to extend Khas'minskii's result to all equations with the form of (1.1). Approaches in this direction were done by Khas'minskii [6] and [7], Kozin-Prodromou [8], and etc., but their results cannot be applied to all equations with the form of (1.1).

In Section 2, we study asymptotic behaviors of one-dimensional diffusion processes in a finite interval with various singular boundaries. In Sections 3 and 4, we classify the system (1.1) into 18 types according to natures of its singular points and discuss to determine J, for each type. Our results are summarized in Section 5, and several examples are discussed in Section 6.

2. The asymptotic behaviors of a one-dimensional diffusion process.

Consider a one-dimensional diffusion process $\hat{x}(t)$, which is given by

$$d\hat{x}(t)=b(\hat{x}(t))dt+\sigma(\hat{x}(t))d\hat{B}(t)\,, \tag{2.1}$$

where we suppose that $b(x)$ and $\sigma(x)$ satisfy the global Lipschitz condition. An associated generator L of $\hat{x}(t)$ is defined by

$$L=b(x)\frac{d}{dx}+\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}\,.$$

Denote by τ_s the firist hitting time for a point r, i.e.,

$$\tau_s=\begin{cases} \inf\limits_{t>0}\{t\,;\,\hat{x}(t)=r\} \\ \infty\,, \qquad \text{if such } t \text{ does not exist}\,. \end{cases}$$

Denote by $\tau[r_1, r_2]$ the first exit time from an interval $[r_1, r_2]$, i.e.,

$$\tau[r_1, r_2]=\begin{cases} \inf\limits_{t>0}\{t\,;\,x(t)\in[r_1, r_2]^c\} \\ \infty\,, \qquad \text{if such } t \text{ does not exist}\,. \end{cases}$$

The following lemma is due to Khas'minskii [7].

LEMMA 2.1. *Assume that there exists a function $V(x)$ such that $V(x)$ is C^2-class and positive in an interval (a_1, a_2), and that*

$$L\,V(x)\leqq -k\,, \qquad \textit{for} \quad x\in[a_1, a_2]\,,$$

where k is a positive constant. Then, for an arbitrary $x_0\in[a_1, a_2]$,

$$E_{x_0}\tau[a_1, a_2]\leqq \frac{1}{k}V(x_0)\,.$$

Let (a_1, a_2) be an open regular interval, i.e., $\sigma^2(x)>0$ for $x\in(a_1, a_2)$. The Feller's canonical scale $\hat{s}(x)$ associated with $\hat{x}(t)$, on (a_1, a_2), is defined by

$$\hat{s}(x)\equiv\int_{b_1}^{x}\exp\left\{-\int_{b_2}^{y}\frac{2b(z)}{\sigma^2(z)}dz\right\}dy\,, \tag{2.2}$$

where b_1 and b_2 are suitably fixed in the interval (a_1, a_2).

Definition. The boundary point a_1+0 (a_2-0) of the interval (a_1, a_2) is repelling if $\hat{s}(a_1+0)=-\infty$ $(\hat{s}(a_2-0)=+\infty)$, and it is attracting otherwise.

Remark. In the Feller's classification of singular points, an exit and a regular boundary are always attracting, and an entrance boundary is always repelling, but we cannot state anything about a natural boundary.

We see asymptotic behaviors of $\hat{x}(t)$ in (a_1, a_2) with some singular boundaries;

(A) $$\sigma(a_1)=\sigma(a_2)=0\,,$$
$$b(a_1)\geqq 0, \quad \text{and} \quad b(a_2)\leqq 0\,.$$

(B) $$\sigma(a_1)=\sigma(a_2)=0\,,$$
$$b(a_1)=0, \quad \text{and} \quad b(a_2)>0\,.$$

By virtue of the assumption that $b(x)$ and $\sigma(x)$ satisfy the global Lipschitz condition, it follows that a_1 and a_2 are, respectively, either the entrance or the natural boundary from (A), and that a_1 and a_2 are, respectively, the natural and the exit boundary from (B). The following lemmas can be proved by a modification of the method of Gikhman-Skorokhod [3].

LEMMA 2.2. *Assume that* (A) *holds. If a_1 and a_2 are both repelling, then $\hat{x}^{x_0}(t)$ is recurrent in (a_1, a_2) for an arbitrary $x_0\in(a_1, a_2)$.*

LEMMA 2.3. *Assume that* (A) *holds. If a_1 is attracting and a_2 is repelling, then for an arbitrary $x_0\in(a_1, a_2)$*

$$P_{x_0}\{\lim_{t\to\infty}\hat{x}(t)=a_1\}=1\,.$$

LEMMA 2.4. *Assume that* (A) *holds. If a_1 and a_2 are both attracting, then for an arbitrary $x_0\in(a_1, a_2)$*

$$P_{x_0}\{\lim_{t\to\infty}\hat{x}(t)=a_1\}=\frac{\hat{s}(a_2)-\hat{s}(x)}{\hat{s}(a_2)-\hat{s}(a_1)}\,,$$

$$P_{x0}\{\lim_{t\to\infty} \hat{x}(t)=a_2\}=\frac{\hat{s}(x)-\hat{s}(a_1)}{\hat{s}(a_2)-\hat{s}(a_1)}.$$

LEMMA 2.5. *Assume that* (B) *holds. If a_1 is repelling, then for an arbitrary* $x_0\in(a_1, a_2)$

$$P_{x0}\{\tau_{a2}<\infty\}=1.$$

LEMMA 2.6. *Assume that* (B) *holds. If a_1 is attracting, then for an arbitrary* $x_0\in(a_1, a_2)$

$$P_{x0}\{\tau_{a2}<\infty\}=\frac{\hat{s}(x)-\hat{s}(a_1)}{\hat{s}(a_2)-\hat{s}(a_1)},$$

$$P_{x0}\{\lim_{t\to\infty} \hat{x}(t)=a_1\}=\frac{\hat{s}(a_2)-\hat{s}(x)}{\hat{s}(a_2)-\hat{s}(a_1)}.$$

3. The determination of $J(\theta_0)$.

Let U_1 be a real constant regular matrix. Then, if $Y\equiv U_1\cdot X$, the system (1.1) is transformed into the following system:

$$dY(t)=(U_1\cdot B\cdot U_1^{-1})\cdot Y(t)\,dt+(U_1\cdot C\cdot U_1^{-1})\cdot Y(t)\,dB_1(t)$$
$$+(U_1\cdot D\cdot U_1^{-1})\cdot Y(t)\,dB_2(t),$$

Then we can make the transformed matrix $(U_1\cdot C\cdot U_1^{-1})$ have one of the cononical forms, i. e.,

$$\text{(I)}\ \begin{pmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{pmatrix}\ c_2\neq 0, \qquad \text{(II)}\ \begin{pmatrix} c_1 & 0 \\ c_2 & c_1 \end{pmatrix}\ c_2\neq 0,$$

$$\text{(III)}\ \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}\ c_1\neq c_2, \qquad \text{(IV)}\ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$

Thus, in order to discuss the stability of the solution of the system (1.1), we may assume that the matrix C has one of the forms (I) through (IV).

Since the system (1.1) has a special, namely linear, form, there is no variable but $\theta(t)$ in the right hand side of the equation (1.2). Thus, in order to determine

$$J(\theta_0)\equiv\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T Q(\theta^{\theta_0}(t))dt,$$

it is sufficient to see only the behavior of $\theta(t)$, which is given by the equation

(3.1) $$d\theta(t)=\Phi(\theta(t))dt+\Psi(\theta(t))d\tilde{B}(t),$$

where $\theta(0)=\theta_0$ and

(3.2) $$\Phi(\theta)=-(B\cdot e(\theta), e^*(\theta))+(A(e(\theta))\cdot e(\theta), e^*(\theta)),$$

(3.3) $$\Psi^2(\theta)=(A(e(\theta))e^*(\theta), e^*(\theta))$$

$$e^*(\theta)=(\sin\theta, -\cos\theta),$$

and $\tilde{B}(t)$ is a Brownian motion on the circumference of the unit circle. Note that, since $\Phi(\theta+\pi)=\Phi(\theta)$ and $\Psi^2(\theta+\pi)=\Psi^2(\theta)$,

$$Q(\theta+\pi)=Q(\theta). \tag{3.4}$$

Note that, if $P\{\lim_{t\to\infty}\theta(t)=\alpha\}=1$, where α is a point in the circumferences of the unit circle, then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T Q(\theta(t))dt=Q(\alpha) \qquad \text{a. s.}$$

The following lemma is due to Maruyama-Tanaka [9].

LEMMA 3.1. *If a one-dimensional diffusion process is recurrent in an interval, then it has an invariant measure.*

We shall determine $J(\theta_0)$ for the forms (I) and (II) of the matrix C.

(I) (1°). In this case $\theta(t)$ is non-singular, because

$$\Psi^2(\theta)=c_2^2+\Psi_D^2(\theta)\geqq c_2^2>0,$$

where

$$\Psi_D^2(\theta)\equiv\{-d_{21}\sin^2\theta+(d_{11}-d_{22})\sin\theta\cos\theta+d_{21}\cos^2\theta\}^2.$$

Since a non-singular diffusion on the circumference is recurrent. $\theta(t)$ has an invariant measure there, by virtue of Lemma 3.1. The density of an invariant measure exists, and it is the solution of Kolmogorov's forward equation, associated with $\theta(t)$, with the normal condition and the periodic condition. Thus, we have 1°) of Summary in Section 5.

(II) Since for this case

$$\Psi^2(\theta)=c_2^2\cos^4\theta+\Psi_D^2(\theta), \qquad c_2\neq 0,$$

we see that

$$\begin{cases} \text{(i) if } d_{12}\neq 0, \text{ then } \theta(t) \text{ is non-singular, and that} \\ \text{(ii) if } d_{12}=0, \text{ then it has singular points at } \theta=\frac{1}{2}\pi \text{ and } \frac{3}{2}\pi. \end{cases}$$

Investigating the behaviors of the canonical measure $m(d\theta)$ and the canonical scale $s(\theta)$, associated with $\theta(t)$, we can see the natures of the singular points, which are shown in Figures 1, 2 and 3 of Appendix.

2°) If $d_{12}\neq 0$, then $\Psi^2(\theta)>0$. Thus, we have 2°) of Summary in Section 5, by the same argument as in 1°).

3°) If $d_{12}=0$ and $b_{12}\neq 0$ (see Figures 1 and 2), then we can show easily that $\theta(t)$ is recurrent, making use of Lemma 2.1. Therefore, we have 3°) of Summary.

4°) If $d_{12}=0$ and $b_{12}=0$, then the singular points are the natural bundaries (see Figure 1). According to the behaviors of $s(\theta)$ in the neighbourhood of the singular points, we have that

(i) in case $d_{11}\neq d_{22}$

$$\begin{cases} \text{if } \kappa_4^{(1)}>-1, \text{ then } \frac{1}{2}\pi+0 \text{ and } \frac{3}{2}\pi+0 \text{ are attracting, and} \\ \text{if } \kappa_4^{(1)}\leqq -1, \text{ then they are repelling,} \end{cases}$$

and that

(ii) in case $d_{11}=d_{22}$ and $\kappa_4^{(2)}\equiv -b_{11}+b_{22}\neq 0$,

$$\begin{cases} \text{if } \kappa_4^{(2)}>0, \text{ then } \frac{1}{2}\pi+0 \text{ and } \frac{3}{2}\pi+0 \text{ are repelling, and} \\ \text{if } \kappa_4^{(2)}<0, \text{ then they are attracting,} \end{cases}$$

and that

(iii) in case $d_{11}=d_{22}$, $\kappa_4^{(2)}=0$, and $\kappa_4^{(3)}\equiv b_{21}-c_1c_2-d_{21}d_{11}\neq 0$,

$$\begin{cases} \text{if } \kappa_4^{(3)}>0, \text{ then } \frac{1}{2}\pi+0 \text{ and } \frac{3}{2}\pi+0 \text{ are repelling and} \\ \frac{1}{2}\pi-0 \text{ and } \frac{3}{2}\pi-0 \text{ are attracting, and} \\ \text{if } \kappa_4^{(3)}<0, \text{ then the former are attracting and latter are repelling,} \end{cases}$$

(iv) in case $d_{11}=d_{22}$, $\kappa_4^{(2)}=0$, and $\kappa_4^{(3)}=0$,

$$\left\{ \frac{1}{2}\pi+0 \text{ and } \frac{3}{2}\pi+0 \text{ are always attracting} \right.$$

where

$$\kappa_4^{(1)}\equiv\frac{2\{(-b_{11}+b_{22})+d_{22}(d_{11}-d_{22})\}}{(-d_{11}+d_{22})^2}\,.$$

Note that the singular points $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ are "trap"s in any case. It follows from the above, Lemmas 2.2, 2.3 and 2.4, that, for $\theta_0\neq\frac{1}{2}\pi$ and $\frac{3}{2}\pi$,

(i) in case $d_{11}\neq d_{22}$,

$$\begin{cases} \text{if } \kappa_4^{(1)}>-1, \text{ then} \\ \qquad P_{\theta_0}\left\{\lim_{t\to\infty}\theta(t)=\frac{1}{2}\pi \text{ or } \frac{3}{2}\pi\right\}=1 \\ \text{if } \kappa_4^{(1)}\leqq -1, \text{ then } \theta^{\theta_0}(t) \text{ is recurrent on } \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right) \text{ and } \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right) \end{cases}$$

and that

(ii) in case $d_{11}=d_{22}$ and $\kappa_4^{(2)}\neq 0$,

$$\begin{cases} \text{if } \kappa_4^{(2)}>0, \text{ then } \theta^{\theta_0}(t) \text{ is recurrent on } \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right) \text{ and } \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right), \text{ and} \\ \text{if } \kappa_4^{(1)}<0, \text{ then} \\ \qquad P_{\theta_0}\left\{\lim_{t\to\infty}\theta(t)=\frac{1}{2}\pi \text{ or } \frac{3}{2}\pi\right\}=1\,, \end{cases}$$

and that

(iii) in case $d_{11}=d_{22}$, $\kappa_4^{(2)}=0$, and $\kappa_4^{(3)}\neq 0$,

if $\kappa_4^{(3)}>0$, then

$$P_{\theta_0}\left\{\lim_{t\to\infty}\theta(t)=\frac{1}{2}\pi\right\}=1 \qquad \text{for } \theta_0\in\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$$

$$P_{\theta_0}\left\{\lim_{t\to\infty}\theta(t)=\frac{3}{2}\pi\right\}=1 \qquad \text{for } \theta_0\in\left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$$

if $\kappa_4^{(3)}<0$, then

$$P_{\theta_0}\left\{\lim_{t\to\infty}\theta(t)=-\frac{1}{2}\pi\right\}=1 \qquad \text{for } \theta_0\in\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$$

$$P_{\theta_0}\left\{\lim_{t\to\infty}\theta(t)=\frac{1}{2}\pi\right\}=1 \qquad \text{for } \theta_0\in\left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$$

and that

(iv) in case $d_{11}=d_{22}$, $\kappa_4^{(2)}=0$, and $\kappa_4^{(3)}=0$,

$$P_{\theta_0}\left\{\lim_{t\to\infty}\theta(t)=\frac{1}{2}\pi \quad \text{or} \quad \frac{3}{2}\pi\right\}=1.$$

Thus, taking (3.4) into account, we have 4°) of Summary.

4. The determination of $J(\theta_0)$ (continuation).

In this section, we shall determine $J(\theta_0)$ in case that the matrix C has the forms (III) and (IV)

(III) Since for this case

$$\Psi^2(\theta)=(-c_1+c_2)^2\sin^2\theta\cos^2\theta+\Psi_D^2(\theta),$$

we see that

(i) if $d_{12}\neq 0$ and $d_{21}\neq 0$, then $\theta(t)$ is non-singular,

(ii) if $d_{12}=0$ and $d_{12}\neq 0$, then it has singular points at $\theta=\frac{1}{2}\pi$ and $\frac{3}{2}\pi$,

(iii) if $d_{12}\neq 0$ and $d_{21}=0$, then it has singular points at $\theta=0$ and π,

(iv) if $d_{12}=0$ and $d_{21}=0$, then it has singular points at $\theta=0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi$.

According to the behaviors of $m(d\theta)$ and $s(\theta)$, we can classify the singular points, as it is shown in Figure 4 through Figure 18 of Appendix.

5°) If $d_{12}\neq 0$ and $d_{21}\neq 0$, then $\theta(t)$ is recurrent on the circumference.

6°) If $d_{12}=0$, $d_{21}\neq 0$, and $b_{12}\neq 0$ (see Figures 4 and 5), then it is snown that $\theta(t)$ is recurrent on the circumference, by applying Lemma 2.1 to $\theta(t)$.

7°) If $d_{12}=0$, $d_{21}\neq 0$, and $b_{12}=0$, then there exists the natural boundary points, as it is shown in Figure 6. Investigating the behaviors of $s(\theta)$, we have that

$$\begin{cases} \text{if } \kappa_7 > -1, \text{ then } \frac{1}{2}\pi \pm 0 \text{ and } \frac{3}{2}\pi \pm 0 \text{ are attracting, and} \\ \text{if } \kappa_7 \leqq -1, \text{ then they are repelling,} \end{cases}$$

where

$$\kappa_7 \equiv \frac{2\{(-b_{11}+b_{22})+c_2(c_1-c_2)+d_{22}(d_{11}-d_{22})\}}{(-c_1+c_2)^2+(-d_{11}+d_{22})^2}.$$

By virtue of Lemmas 2.2 and 2.4, it follows from the above that, for $\theta_0 \neq \frac{1}{2}\pi$ and $\frac{3}{2}\pi$,

$$\begin{cases} \text{if } \kappa_7 > -1, \text{ then} \\ \qquad P_{\theta_0}\left\{\lim_{t\to\infty} \theta(t) = \frac{1}{2}\pi \text{ or } \frac{3}{2}\pi\right\} = 1 \\ \text{if } \kappa_7 \leqq -1, \text{ then } \theta^{\theta_0}(t) \text{ is recurrent on } \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right) \end{cases}$$

$$\left(\left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)\right) \text{ for } \theta_0 \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right) \left(\text{respectively, } \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)\right).$$

Thus, taking (3.2) into account, we have 7°) of Summary.

8°) $d_{12} \neq 0$, $d_{21}=0$, and $b_{21} \neq 0$ (see Figures 7 and 8).
9°) $d_{12} \neq 0$, $d_{21}=0$, and $b_{21}=0$ (see Figure 9).
10°) $d_{12}=d_{21}=0$ and $b_{12}b_{21}<0$ (see Figures 10 and 11).

In the case of 8°) through 10°), we have 8°) through 10°) of Summary, by a slight change in the preceding argument.

11°) $d_{12}=d_{21}=0$ and $b_{12}b_{21}>0$ (see Figures 12 and 13).

By virtue of Lemma 2.1, it is shown that, for $\theta_0 \in \left[-\frac{1}{2}\pi, 0\right]$ or $\left[\frac{1}{2}\pi, \pi\right]$, $\theta^{\theta_0}(t)$ goes into either $\left(0, \frac{1}{2}\pi\right)$ or $\left(\pi, \frac{3}{2}\pi\right)$ after a finite time with probability 1. Therefore, we may assume that $\theta(t)$ starts only from a point θ_0 in $\left(0, \frac{1}{2}\pi\right)$ or in $\left(\pi, \frac{3}{2}\pi\right)$. Noting that the both boundaries of $\left(0, \frac{1}{2}\pi\right)$ and of $\left(\pi, \frac{3}{2}\pi\right)$ are repelling, we see that $\theta(t)$ has an invariant measure on $\left(0, \frac{1}{2}\pi\right)$ and on $\left(\pi, \frac{3}{2}\pi\right)$. If $b_{12}<0$, then we can show that $\theta(t)$ has also an invariant measure on $\left(\frac{1}{2}\pi, \pi\right)$ and on $\left(\frac{3}{2}\pi, 2\pi\right)$, by virtue of a similar argument. Thus, we have 11°) of Summary, noting (3.4) and the fact that an invariant measure coincides with the canonical measure $m(d\theta)$ in this case (see Maruyama-Tanaka [9]).

12°) If $d_{12}=d_{21}=0$, $b_{12}\neq 0$, and $b_{21}=0$, then there exists the natural boundary points, as it is shown in Figures 14 and 15. According to the behaviors of $s(\theta)$, we have that

$$\begin{cases} \text{if } \kappa_{12}\geqq 1, \text{ then } 0\pm 0 \text{ and } \pi\pm 0 \text{ are repelling, and} \\ \text{if } \kappa_{12}<1, \text{ then they are attracting,} \end{cases}$$

where

$$\kappa_{12}\equiv\frac{2\{(-b_{11}+b_{22})+c_1(c_1-c_2)+d_{11}(d_{11}-d_{22})\}}{(-c_1+c_2)^2+(-d_{11}+d_{22})^2}.$$

Suppose that $b_{12}>0$. By virtue of Lemmas 2.2, 2.3, and 2.4, it follows from the above that,

$$\begin{cases} \text{if } \kappa_{12}\geqq 1, \text{ then} \\ \qquad \theta^{\theta_0}(t)=\begin{cases} 0 & \theta_0=0\,, \\ \text{recurrent on } \left(0, \frac{1}{2}\pi\right) & \theta_0\in(0, \pi)\,, \\ \pi & \theta_0=\pi\,, \\ \text{recurrent on } \left(\pi, \frac{3}{2}\pi\right) & \theta_0\in(\pi, 2\pi)\,, \end{cases} \\ \text{and if } \kappa_{12}<1, \text{ then, with probability 1,} \\ \qquad \lim_{t\to\infty}\theta^{\theta_0}(t)=\begin{cases} 0 & \theta_0\in\left[0, \frac{1}{2}\pi\right], \\ 0 \text{ or } \pi & \theta_0\in\left(\frac{1}{2}\pi, \pi\right), \\ \pi & \theta_0\in\left[\pi, \frac{3}{2}\pi\right], \\ 0 \text{ or } \pi & \theta_0\in\left(\frac{3}{2}\pi, 2\pi\right), \end{cases} \end{cases}$$

(see Figure 14), because $\frac{1}{2}\pi-0$ and $\frac{3}{2}\pi-0$ are repelling and $\frac{1}{2}\pi+0$ and $\frac{3}{2}\pi+0$ are attracting. If $b_{12}<0$ (see Figure 15), we can repeat the same argument, and we have 12°) of Summary.

13°) If $d_{12}=d_{21}=0$, $b_{12}=0$, and $b_{21}\neq 0$ (see Figures 16 and 17), 13°) of Summary is obtained, by a slight change in the argument in 12°).

14°) If $d_{12}=d_{21}=0$ and $b_{12}=b_{21}=0$ (see Figures 18), then 14°) of Summary follows from Example 1 in Section 6.

(IV) Let U_2 be a real constant regular matrix. Since the matrix C, for this case, is commutable for any matrix, if $X'\equiv U_2\cdot X$, the system (1.1) is transformed into the following system:

$$\begin{aligned} (4.1) \qquad dX'(t)=&(U_2\cdot B\cdot U_2^{-1})\cdot X'(t)dt+C\cdot X'(t)dB_1(t) \\ &+(U_2\cdot D\cdot U_2^{-1})\cdot X'(t)dB_2(t)\,, \end{aligned}$$

where the transformed matrix $(U_2\cdot D\cdot U_2^{-1})$ is one of the canonical forms (I) through (IV). We may replace (1.1) by (4.1), in order to discuss the stability of the system (1.1). Denote by B' the transformed matrix $(U_2 \cdot B \cdot U_2^{-1})$, etc.

15°) D' has the form (I).
16°) D' has the form (II).
17°) D' has the form (III).

15°) through 17°) come to the special case of (I) through (III), replacing B by B', C by D', and D by cI, where I is the identity matrix.

18°) If D' has the form (IV), then D' is commutable for any matrix. Then, there exists a real constant regular matrix, such that, if $X'' \equiv U_3 \cdot X'$, the system (4.1) is transformed into the following system:

$$dX''(t)=(U_3 \cdot B' U_3^{-1}) \cdot X''(t)dt + C \cdot X''(t)dB_1(t) + D' \cdot X''(t)dB_2(t)\,, \tag{4.2}$$

where the transformed matrix $(U_3 \cdot B' U_3^{-1})$ is one of the canonical forms. Hence, we may replace (4.1) by (4.2), in order to discuss the stability of the system (1.1).

Denote by B'' the transformed matrix $(U_3 \cdot B' \cdot U_3^{-1})$, etc. The angular component $\theta''(t)$ of $X''(t)$ is given by

$$d\theta''(t)=\Phi''(\theta''(t))dt+\Psi''(\theta''(t))d\tilde{B}(t) \tag{4.3}$$

where $\Phi''(\theta)$ and $\Psi'(\theta)$ are defined by (3.2) and (3.3), in which B, C, and D are respectively replaced by B'', cI, and $d'I$. For this case, $\Psi^2(\theta)=0$, and the equation (4.3) comes into the deterministic differencial equation:

$$\frac{d\theta''(t)}{dt}=\Phi''(\theta''(t))\,. \tag{4.4}$$

By substituting the solutions of the equation (4.4) into $J(\theta_0)$, we have 13°) of Summary.

5. Summary of $J(\theta_0)$ and the extension of Khas'minskii's result.

The following table is the summary of $J(\theta_0)$, which are obtained in Sections 3 and 4.

For simplicity, we use the following notations, in the definitions of the invariant measures $\mu_i(\theta)$:

$$F(\alpha, \beta)\equiv\frac{1}{\Psi^2(\beta)W(\alpha, \beta)}$$

$$F^*(\alpha, \beta)\equiv\frac{1}{\Psi^2(\alpha)W(\alpha, \beta)}$$

$$H(\alpha, \beta)\equiv\frac{\int_\alpha^\beta W(\alpha, \phi)d\phi}{\Psi^2(\beta)W(\alpha, \beta)}$$

$$H^*(\alpha, \beta) \equiv \frac{\int_\alpha^\beta W(\psi, \beta) d\psi}{\Psi^2(\alpha) W(\alpha, \beta)}$$

where

$$W(\alpha, \beta) \equiv \exp\left\{-\int_\alpha^\beta \frac{2\Phi(\theta)}{\Psi^2(\theta)} d\theta\right\}.$$

Denote by N a constant which is defined by the normal condition:

$$\int_0^{2\pi} \mu(\theta) d\theta = 1.$$

1°) (I), $\Rightarrow J(\theta_0) = \int_0^{2\pi} Q(\theta) \mu_1(\theta) d\theta.$

2°) (II), $d_{12} \neq 0 \Rightarrow J(\theta_0) = \int_0^{2\pi} Q(\theta) \mu_2(\theta) d\theta.$

3°) (II), $d_{12} = 0,\ b_{12} \neq 0 \Rightarrow J(\theta_0) = \int_0^{2\pi} Q(\theta) \mu_3(\theta) d\theta.$

4°) (II), $d_{12} = 0,\ b_{12} = 0.$

$$\begin{cases} \text{(a)}\quad d_{11} \neq d_{22},\ \kappa_4^{(1)} > -1 \Rightarrow J(\theta_0) = Q\left(\frac{1}{2}\pi\right) \\ \text{(b)}\quad d_{11} \neq d_{22},\ \kappa_4^{(1)} \leqq -1 \Rightarrow J(\theta_0) = \begin{cases} \int_0^{2\pi} Q(\theta) \mu^{(1)}(\theta) d\theta, & \theta_0 \neq \frac{1}{2}\pi, \frac{3}{2}\pi \\ Q\left(\frac{1}{2}\pi\right) & \theta_0 = \frac{1}{2}\pi, \frac{3}{2}\pi \end{cases} \\ \text{(c)}\quad d_{11} = d_{22},\ \kappa_4^{(2)} \geqq 0 \quad \Rightarrow J(\theta_0) = Q\left(\frac{1}{2}\pi\right) \\ \text{(d)}\quad d_{11} = d_{22},\ \kappa_4^{(2)} < 0 \quad \Rightarrow J(\theta_0) = \begin{cases} \int_0^{2\pi} Q(\theta) \mu_4^{(2)}(\theta) d\theta, & \theta_0 \neq \frac{1}{2}\pi, \frac{3}{2}\pi \\ Q\left(\frac{1}{2}\pi\right) & \theta_0 = \frac{1}{2}\pi, \frac{3}{2}\pi \end{cases} \end{cases}$$

5°) (III), $d_{12} \neq 0,\ d_{21} \neq 0 \quad \Rightarrow J(\theta_0) = \int_0^{2\pi} Q(\theta) \mu_5(\theta) d\theta.$

6°) (III), $d_{12} = 0,\ d_{21} \neq 0,\ b_{12} \neq 0 \Rightarrow J(\theta_0) = \int_0^{2\pi} Q(\theta) \mu_6(\theta) d\theta.$

7°) (III), $d_{12} = 0,\ d_{21} \neq 0,\ b_{12} = 0.$

$$\begin{cases} \text{(a)}\quad \kappa_7 > -1 \quad \Rightarrow \quad J(\theta_0) = Q\left(\frac{1}{2}\pi\right) \\ \text{(b)}\quad \kappa_7 \leqq -1 \quad \Rightarrow \quad J(\theta_0) = \begin{cases} \int_0^{2\pi} Q(\theta) \mu_7(\theta) d\theta, & \theta_0 \neq \frac{1}{2}\pi, \frac{3}{2}\pi \\ Q\left(\frac{1}{2}\pi\right) & \theta_0 = \frac{1}{2}\pi, \frac{3}{2}\pi \end{cases} \end{cases}$$

8°) (III), $d_{12} \neq 0,\ d_{21} = 0,\ b_{21} \neq 0 \Rightarrow J(\theta_0) = \int_0^{2\pi} Q(\theta) \mu_8(\theta) d\theta.$

9°) (III), $d_{12}\neq 0,\ d_{21}=0,\ b_{21}=0.$

$$\begin{cases} \text{(a)}\quad \kappa_9<1 \ \Rightarrow\ J(\theta_0)=Q(0) \\ \text{(b)}\quad \kappa_0\geqq 1 \ \Rightarrow\ J(\theta_0)=\begin{cases} \int_0^{2\pi} Q(\theta)\mu_9(\theta)d\theta, & \theta_0\neq 0,\ \pi \\ Q(0) & \theta_0=0,\ \pi \end{cases} \end{cases}$$

10°) (III), $d_{12}=d_{21}=0,\ b_{12}b_{21}<0 \Rightarrow J(\theta_0)=\int_0^{2\pi} Q(\theta)\mu_{10}(\theta)d\theta.$

11°) (III), $d_{11}=d_{21}=0,\ b_{12}b_{21}>0.$

$$\begin{cases} \text{(a)}\quad b_{12}>0 \ \Rightarrow\ J(\theta_0)=\left(\int_\pi^{\frac{1}{2}\pi}+\int_\pi^{\frac{3}{2}\pi}\right)(Q(\theta)\mu_{11}^{(1)}(\theta))d\theta \\ \text{(b)}\quad b_{12}<0 \ \Rightarrow\ J(\theta_0)=\left(\int_{\frac{1}{2}\pi}^{\pi}+\int_{\frac{3}{2}\pi}^{2\pi}\right)(Q(\theta)\mu_{11}^{(2)}(\theta))d\theta \end{cases}$$

12°) (III), $d_{12}=d_{21}=0,\ b_{12}\neq 0,\ b_{21}=0.$

$$\begin{cases} \text{(a)}\quad \kappa_{12}\geqq 1,\ b_{12}>0 \Rightarrow J(\theta_0)=\begin{cases} \left(\int_0^{\frac{1}{2}\pi}+\int_\pi^{\frac{3}{2}\pi}\right)(Q(\theta)\mu_{12}^{(1)}(\theta))d\theta, & \theta_0\neq 0,\ \pi \\ Q(0) & \theta_0=0,\ \pi \end{cases} \\ \text{(b)}\quad \kappa_{12}\geqq 1,\ b_{12}<0 \Rightarrow J(\theta_0)=\begin{cases} \left(\int_{\frac{1}{2}\pi}^{\pi}+\int_{\frac{3}{2}\pi}^{2\pi}\right)(Q(\theta)\mu_{12}^{(2)}(\theta))d\theta, & \theta_0\neq 0,\ \pi \\ Q(0) & \theta_0=0,\ \pi \end{cases} \\ \text{(c)}\quad \kappa_{12}<1 \qquad \Rightarrow J(\theta_0)=Q(0) \end{cases}$$

13°) (III), $d_{12}=d_{21}=0,\ b_{12}=0,\ b_{21}\neq 0.$

$$\begin{cases} \text{(a)}\quad \kappa_{13}>-1 \qquad \Rightarrow \qquad J(\theta_0)=Q\left(\frac{1}{2}\pi\right) \\ \text{(b)}\quad \kappa_{13}\leqq -1,\ b_{21}>0 \Rightarrow J(\theta_0)=\begin{cases} \left(\int_0^{\frac{1}{2}\pi}+\int_\pi^{\frac{3}{2}\pi}\right)(Q(\theta)\mu_{13}^{(1)}(\theta))d\theta, & \theta_0\neq\frac{1}{2}\pi,\frac{3}{2}\pi \\ Q\left(\frac{1}{2}\pi\right) & \theta_0=\frac{1}{2}\pi,\frac{3}{2}\pi \end{cases} \\ \text{(c)}\quad \kappa_{13}\leqq -1,\ b_{21}<0 \Rightarrow J(\theta_0)=\begin{cases} \left(\int_{\frac{1}{2}\pi}^{\pi}+\int_{\frac{3}{2}\pi}^{2\pi}\right)(Q(\theta)\mu_{13}^{(2)}(\theta))d\theta, & \theta_0\neq\frac{1}{2}\pi,\frac{3}{2}\pi \\ Q\left(\frac{1}{2}\pi\right) & \theta_0=\frac{1}{2}\pi,\frac{3}{2}\pi \end{cases} \end{cases}$$

14°) (III), $d_{12}=d_{21}=0,\ b_{12}=b_{21}=0.$

$$J(\theta_0)=\begin{cases} \max\left\{b_{11}-\frac{1}{2}(c_1^2+d_{11}^2),\ b_{22}-\frac{1}{2}(c_2^2+d_{22}^2)\right\}, & \theta_0\neq 0,\ \frac{1}{2}\pi,\pi,\ \frac{3}{2}\pi \\ Q(0) & \theta_0=0,\ \pi \\ Q\left(\frac{1}{2}\pi\right) & \theta_0=\frac{1}{2}\pi,\frac{3}{2}\pi \end{cases}$$

15°) (IV) D' is the form (I)
16°) (IV) D' is the form (II) } The special cases of 1°) through 14°).
17°) (IV) D' is the form (III)
18°) (IV) D' is the form (IV).

$$\begin{cases} \text{(a) } B'' \text{ is the form (I)} & J(\theta_0)=\int_0^{2\pi} Q''(\theta)d\theta=b_1''-\frac{1}{2}(c^2+d'^2) \\ \text{(b) } B'' \text{ is the form (II)} & J(\theta_0)=Q''\left(\frac{1}{2}\pi\right) \\ \text{(c) } B'' \text{ is the form (III)} & J(\theta_0)=\max\{b_1'', b_2''\}-\frac{1}{2}(c^2+d'^2) \\ \text{(d) } B'' \text{ is the form (IV)} & J(\theta_0)=b''-\frac{1}{2}(c^2+d'^2) \end{cases}$$

In this table,

1°) $$\mu_1(\theta)=N\left\{F(0,\theta)+\frac{W(0,2\pi)-1}{\int_0^{\pi} W(0,\phi)d\phi}H(0,\theta)\right\}$$

2°) $$\mu_2(\theta)=N\left\{F(0,\theta)+\frac{W(0,2\pi)-1}{\int_0^{2\pi} W(0,\phi)d\phi}H(0,\theta)\right\}$$

3°) $$b_{12}>0 \Rightarrow \mu_3(\theta)=\begin{cases} NH\left(-\frac{1}{2}\pi,\theta\right) & -\frac{1}{2}\pi\leqq\theta\leqq\frac{1}{2}\pi \\ \mu_3(\theta-\pi) & \frac{1}{2}\pi\leqq\theta<\frac{3}{2}\pi \end{cases}$$

$$b_{12}<0 \Rightarrow \mu_3(\theta)=\begin{cases} NH^*\left(\theta,\frac{1}{2}\pi\right) & -\frac{1}{2}\pi\leqq\theta<\frac{1}{2}\pi \\ \mu_3(\theta-\pi) & \frac{1}{2}\pi\leqq\theta<\frac{3}{2}\pi \end{cases}$$

4°) $$\kappa_4^{(1)}=\frac{2\{(-b_{11}+b_{22})+d_{22}(d_{11}-d_{22})\}}{(d_{11}-d_{22})^2}$$

$$\kappa_4^{(2)}=-b_{11}+b_{22}$$

$$\mu_4^{(1)}(\theta)=\begin{cases} NF(0,\theta) & -\frac{1}{2}\pi\leqq\theta<\frac{1}{2}\pi \\ \mu_4^{(1)}(\theta-\pi) & \frac{1}{2}\pi\leqq\theta<\frac{3}{2}\pi \end{cases}$$

$$\mu_4^{(2)}(\theta)=\begin{cases} NF(0,\theta) & -\frac{1}{2}\pi\leqq\theta<\frac{1}{2}\pi \\ \mu_4^{(2)}(\theta-\pi) & \frac{1}{2}\pi\leqq\theta<\frac{3}{2}\pi \end{cases}$$

5°) $\mu_5(\theta)$ is the same form as $\mu_1(\theta)$.

6°) $$b_{12}>0 \Rightarrow \mu_6(\theta)=\begin{cases} NH\left(-\frac{1}{2}\pi,\theta\right) & -\frac{1}{2}\pi\leqq\theta<\frac{1}{2}\pi \\ \mu_6(\theta-\pi) & \frac{1}{2}\pi\leqq\theta<\frac{3}{2}\pi \end{cases}$$

$$b_{12}<0 \quad \Rightarrow \quad \mu_6(\theta)=\begin{cases} NH^*\left(\theta, \frac{1}{2}\pi\right) & -\frac{1}{2}\pi \leqq \theta < \frac{1}{2}\pi \\ \mu_6(\theta-\pi) & \frac{1}{2}\pi \leqq \theta < \frac{3}{2}\pi \end{cases}$$

7°) $$\kappa_7=\frac{2\{(-b_{11}+b_{22})+c_2(c_1-c_2)+d_{22}(d_{11}-d_{22})\}}{(c_1-c_2)^2+(d_{11}-d_{22})^2}$$

$$\mu_7(\theta)=\begin{cases} NF(0, \theta) & -\frac{1}{2}\pi \leqq \theta < \frac{1}{2}\pi \\ \mu_7(\theta-\pi) & \frac{1}{2}\pi \leqq \theta < \frac{3}{2}\pi \end{cases}$$

3°) $$b_{21}>0 \quad \Rightarrow \quad \mu_8(\theta)=\begin{cases} NH^*(\theta, \pi) & 0 \leqq \theta < \pi \\ \mu_8(\theta-\pi) & \pi \leqq \theta < 2\pi \end{cases}$$

$$b_{21}<0 \quad \Rightarrow \quad \mu_8(\theta)=\begin{cases} NH(0, \theta) & 0 \leqq \theta < \pi \\ \mu_8(\theta-\pi) & \pi \leqq \theta < 2\pi \end{cases}$$

9°) $$\kappa_9=\frac{2\{(-b_{11}+b_{22})+c_1(c_1-c_2)+d_{11}(d_{11}-d_{22})\}}{(c_1-c_2)^2+(d_{11}-d_{22})^2}$$

$$\mu_9(\theta)=\begin{cases} NF\left(\frac{1}{2}\pi, \theta\right) & 0<\theta<\pi \\ \mu_9(\theta-\pi) & \pi<\theta<2\pi \end{cases}$$

10°) $$b_{12}>0 \quad \Rightarrow \quad \mu_{10}(\theta)=\begin{cases} NH(0, \theta) & 0 \leqq \theta < \frac{1}{2}\pi \\ NH\left(\frac{1}{2}\pi, \theta\right) & \frac{1}{2}\pi \leqq \theta < \pi \\ \mu_{10}(\theta-\pi) & \pi \leqq \theta < 2\pi \end{cases}$$

$$b_{12}<0 \quad \Rightarrow \quad \mu_{10}(\theta)=\begin{cases} NH^*\left(\theta, \frac{1}{2}\pi\right) & 0 \leqq \theta < \frac{1}{2}\pi \\ NH^*(\theta, \pi) & \frac{1}{2}\pi \leqq \theta < \pi \\ \mu_{10}(\theta-\pi) & \pi \leqq \theta < 2\pi. \end{cases}$$

11°) $$\mu_{11}^{(1)}(\theta)=\begin{cases} NF\left(\frac{1}{4}\pi, \theta\right) & 0<\theta<\frac{1}{2}\pi \\ \mu_{11}^{(1)}(\theta-\pi) & \pi<\theta<\frac{3}{2}\pi \end{cases}$$

$$\mu_{11}^{(2)}(\theta)=\begin{cases} NF\left(\frac{5}{4}\pi, \theta\right) & \frac{1}{2}\pi<\theta<\pi \\ \mu_{11}^{(2)}(\theta-\pi) & \frac{3}{2}\pi<\theta<2\pi \end{cases}$$

12°) $$\kappa_{12}=\frac{2\{(-b_{11}+b_{22})+c_1(c_1-c_2)+d_{11}(d_{11}-d_{22})\}}{(c_1-c_2)^2+(d_{11}-d_{22})^2}$$

$$\mu_{12}^{(1)}(\theta)=\begin{cases} NF\left(\frac{1}{4}\pi,\theta\right) & 0<\theta<\frac{1}{2}\pi \\ \mu_{12}^{(1)}(\theta-\pi) & \pi<\theta<\frac{3}{2}\pi \end{cases}$$

$$\mu_{12}^{(2)}(\theta)=\begin{cases} NF\left(\frac{3}{4}\pi,\theta\right) & \frac{1}{2}\pi<\theta<\pi \\ \mu_{12}^{(2)}(\theta-\pi) & \frac{3}{2}\pi<\theta<2\pi \end{cases}$$

13°) $$\kappa_{13}=\frac{2\{(-b_{11}+b_{22})+c_2(c_1-c_2)+d_{22}(d_{11}-d_{22})\}}{(c_1-c_2)^2+(d_{11}-d_{22})^2}$$

$$\mu_{13}^{(1)}(\theta)=\begin{cases} NF\left(\frac{1}{4}\pi,\theta\right) & 0<\theta<\frac{1}{2}\pi \\ \mu_{12}^{(1)}(\theta-\pi) & \pi<\theta<\frac{3}{2}\pi \end{cases}$$

$$\mu_{13}^{(2)}(\theta)=\begin{cases} NF\left(\frac{3}{4}\pi,\theta\right) & \frac{1}{2}\pi<\theta<\pi \\ \mu_{13}^{(2)}(\theta-\pi) & \frac{3}{2}\pi<\theta<2\pi\,. \end{cases}$$

Therefore, we have determined $J(\theta_0)$ for every system with the form (1.1), and the following proposition follows, automatically, from Khas'minskii's result. Denote an angular component of a point $x\in R^2$ by $\theta(x)$.

PROPOSITION 4.1. *$J(\theta(x))$ is determined, with probability* 1, *for an arbitrary point $x\in R^2$. Thus, for $x_0\neq 0$,*

$$\begin{cases} \text{if} & J(\theta(x_0))<0, \text{ then } X^{x_0}(t) \text{ is stable,} \\ \text{if} & J(\theta(x_0))>0, \text{ then it is divergent, and} \\ \text{if} & J(\theta(x_0))=0, \text{ then it is neither stable nor divergent.} \end{cases}$$

6. Examples.

Example 1. Consider the following system:

(6.1) $$dX(t)=\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}X(t)dt+\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}X(t)dB_1(t) + \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}X(t)dB_2(t)\,.$$

Since the system (6.1) are written by two independent linear equations, we can solve them explicitly:

(6.2) $$x_i^{x_0,i}(t)=x_{0,i}\exp\left\{\left(b_i-\frac{1}{2}(c_i^2+d_i^2)t + c_i(B_1(t)-B_1(0))+d_i(B_2(t)-B_2(0))\right)\right\},$$

where $x_0=(x_{0,1}, x_{0,2})$ is a point from which $X(t)$ starts. The following result is obtained by (6.2) and the law of iterated logarithm:

$$\begin{cases} x_i^{x_0,i}(t) \text{ is table if } x_{0,i}\neq 0 \text{ and if } J_i<0, \\ \text{it is divergent if } x_{0,i}\neq 0 \text{ and if } J_i>0, \\ \text{it is neither stable nor divergent if } x_{0,i}\neq 0 \text{ and if } J_i=0, \\ \text{it vanishes if } x_{0,i}=0, \end{cases}$$

where

$$J_i\equiv b_i-\frac{1}{2}(c_i^2+d_i^2).$$

Example 2. Consider the following second order deterministic system:

$$\ddot{x}(t)=b_2\dot{x}(t)+b_1x(t). \tag{6.3}$$

Now, we are concerned with the system which has an addition of the excitation, by the Gaussian white noise $\dot{B}(t)$, to the right hand side of the system (6.3):

$$\ddot{x}(t)=b_2x(t)+(b_1+\sigma\dot{B}(t))\dot{x}(t). \tag{6.4}$$

If $x_1(t)\equiv x(t)$ and $x_2(t)\equiv\dot{x}(t)$, we have

$$dX(t)=\begin{pmatrix}0 & 1\\ b_1 & b_2\end{pmatrix}X(t)dt+\begin{pmatrix}0 & 0\\ \sigma & 0\end{pmatrix}X(t)dB(t). \tag{6.4'}$$

For the system (6.4′), $\theta=\pm\frac{1}{2}\pi$ are the singular points of $\theta(t)$ and their nature are just the same as in Figure 1 of Appendix. Thus, in 3°) of Summary, we have,

$$Q(\theta)=(1+b_1)\sin\theta\cos\theta+b_2\sin^2\theta+\frac{1}{2}\sigma^2\cos^2\theta(1-2\sin^2\theta),$$

$$H\left(-\frac{1}{2}\pi, \theta\right)=\frac{\int_{-\frac{\pi}{2}}^{\theta} W\left(-\frac{1}{2}\pi, \phi\right)d\phi}{\sigma^2\cos^4\theta W\left(-\frac{1}{2}\pi, \theta\right)}$$

in which

$$W\left(-\frac{1}{2}\pi, \theta\right)=\frac{1}{\cos^2\theta}\exp\left\{\frac{\tan\theta}{3\sigma^2}(2\tan^2\theta-3b_2\tan\theta-6b_1)\right\}.$$

Unfortunately, we cannot have the functional relation between b_1, b_2, and σ that determines the algebraic sign of $J(\theta(x_0))$, but the numerical integration of (6.5) have been given by Kozin-Prodromou [8], for the case $b_1=-1$.

Example 3. We shall study the system which has an addition of the dumping term, by the Gaussian white noise $\dot{B}(t)$, to the right hand side of the system (6.3):

(6.5) $$\ddot{x}(t)=(b_2+\sigma\dot{B}(t))\dot{x}(t)+b_1x(t).$$

Making use of the same substitution as Example 2, we have

(6.5′) $$dX(t)=\begin{pmatrix}0 & 1\\ b_1 & b_2\end{pmatrix}X(t)dt+\begin{pmatrix}0 & 0\\ 0 & \sigma\end{pmatrix}X(t)dB(t).$$

The singular points of $\theta(t)$ are $\theta=0$, $\frac{1}{2}\pi$, π, and $\frac{3}{2}\pi$ which have

(i) the same natures as in Figure 12 if $b_1>0$,
(ii) the same natures as in Figure 14 if $b_1=0$, and
(iii) the same natures as in Figure 11 if $b_1<0$.

(i) If $b_1>0$, then we have, in 11°)-(a) of Summary,

(6.6) $$Q(\theta)=(1+b_1)\sin\theta\cos\theta+b_2\sin^2\theta+\frac{1}{2}\sigma^2\sin^2\theta-\sigma^2\sin^4\theta,$$

(6.7) $$F\Big(\frac{1}{4}\pi,\theta\Big)=\frac{2b_2}{\sigma^2}|\cos\theta|^{-\frac{2b_2}{\sigma^2}}|\sin\theta|^{\frac{2b_2}{\sigma^2}-2}\times\exp\Big\{-\frac{2b_1}{\sigma^2}\frac{\cos\theta}{\sin\theta}-\frac{2}{\sigma^2}\frac{\sin\theta}{\cos\theta}\Big\}.$$

(ii) If $b_1=0$, then we have, in 12°) of Summary, that

if $\kappa_{12}=\frac{2b_2}{\sigma^2}<1$, then $J(\theta(x_0))=Q(0)=0$,
if $\kappa_{12}\geqq 1$, then

(6.8) $$J(\theta(x_0))=\begin{cases}\Big(\int_0^{\frac{1}{2}\pi}+\int_\pi^{\frac{3}{2}\pi}\Big)(Q(\theta)\mu_{11}(\theta))d\theta & \theta(x_0)\neq 0,\ \pi\\ Q(0)=0 & \theta(x_0)=0,\ \pi\end{cases}$$

where $Q(\theta)$ and $\mu_{11}(\theta)$ are given by (6.6) and (6.7), with $b_1=0$.

We cannot calculate (6.8), but we can know the stability of $X(t)$ at this time. We can solve (6.5′), i.e.,

$$\begin{cases}x_1(t)=\int_0^t x_2(u)du+x_{0,1},\\ x_2(t)=x_{0,2}\exp\Big\{\Big(b_2-\frac{1}{2}\sigma^2\Big)t+\sigma(B(t)-B(0))\Big\},\end{cases}$$

and we have, by virtue of the law of iterated logarithm, that

if $b_2-\frac{1}{2}\sigma^2>0$, then $x_2(t)$ is divergent, and
if $b_2-\frac{1}{2}\sigma^2=0$, then $x_2(t)$ is neither divergent nor stable but $x_1(t)$ is divergent.

Thus, we have that

$$\begin{cases} X^{x_0}(t) \text{ is neither stable nor divergent if } b_2-\frac{1}{2}\sigma^2<0 \text{ and if } \theta(x_0)\neq 0, \pi, \text{and that} \\ \text{it is divergent if } b_2-\frac{1}{2}\sigma^2\geqq 0 \text{ and if } \theta(x_0)\neq 0, \pi, \\ \text{and that } X^{x_0}(t)=(x_{0,1}, 0) \text{ if } \theta(x_0)=0. \end{cases}$$

(iii) If $b_1<0$, then we have, in 10°) of Summary, $Q(\theta)$ is given by (6.6) and

$$H(\alpha, \theta)=\frac{\int_\alpha^\theta W(\phi)d\phi}{\sigma^2 \cos^4\theta} \qquad \left(\alpha=0, \frac{1}{2}\pi\right),$$

in which

$$W(\phi)=|\cos\phi|^{\frac{2b_2}{\sigma^2}-2}|\sin\phi|^{-\frac{2b_2}{\sigma^2}}\exp\left\{\frac{2b_1}{\sigma^2}\frac{\cos\theta}{\cos\theta}+\frac{2}{\sigma^2}\frac{\sin\theta}{\cos\theta}\right\}.$$

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Appendix.

In this appendix, we show natures of all singular points, on the circumference of the unit circle, associated with the angular component $\theta(t)$, which is give by the equation (3.1).

We use the following notations in the figures:

$\mapsto$ ($\leftarrow\!\dashv$) denotes that the left (right) boundary is an entrance boundary, and

$\vdash\!\leftarrow$ ($\rightarrow\!\dashv$) denotes that the left (right) boundary is an exit boundary, and

⚲ denotes that the left and the right boundary is a natural boundary.

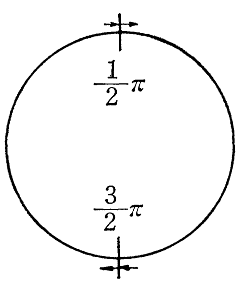

Fig. 1. (II), $b_{12}>0$.

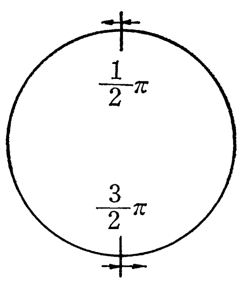

Fig. 2. (II), $b_{12}<0$.

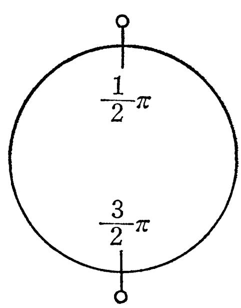

Fig. 3. (II), $b_{12}=0$.

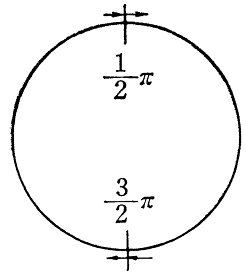

Fig. 4. (III), $d_{12}=0,\ d_{21}\neq 0,\ b_{12}>0.$

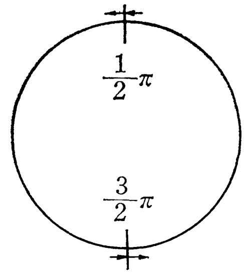

Fig. 5. (III), $d_{12}=0,\ d_{21}\neq 0,\ b_{12}<0.$

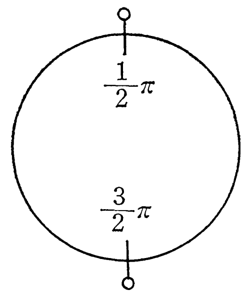

Fig. 6. (III), $d_{12}=0,\ d_{21}\neq 0,\ b_{12}=0$

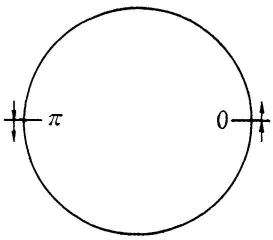

Fig. 7. (III), $d_{12}\neq 0,\ d_{21}=0,\ b_{21}>0.$

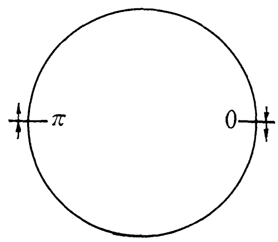

Fig. 8. (III), $d_{12}\neq 0,\ d_{21}=0,\ b_{21}<0.$

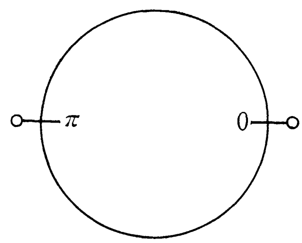

Fig. 9. (III), $d_{12}\neq 0,\ d_{21}=0,\ b_{21}=0.$

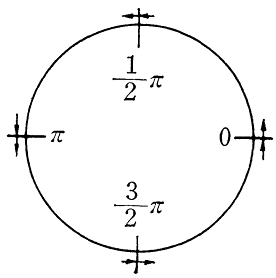

Fig. 10. (III), $d_{12}=d_{21}=0,\ b_{12}<0,\ b_{21}>0.$

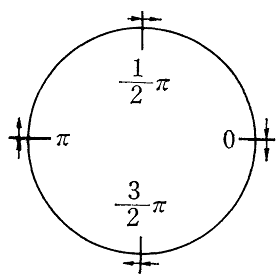

Fig. 11. (III), $d_{12}=d_{21}=0,\ b_{12}>0,\ b_{21}<0.$

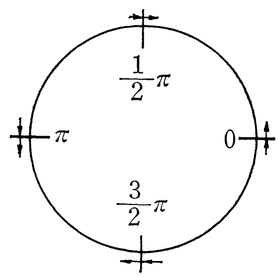

Fig. 12. (III), $d_{12}=d_{21}=0,\ b_{12}>0,\ b_{21}>0.$

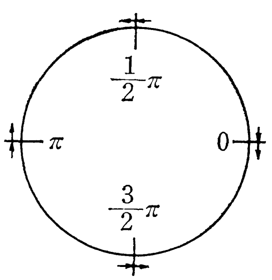

Fig. 13. (III), $d_{12}=d_{21}=0,\ b_{12}<0,\ b_{21}<0.$

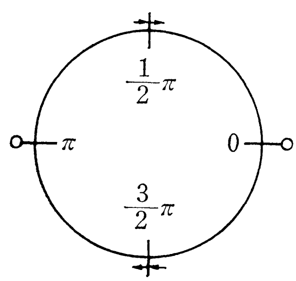

Fig. 14. (III), $d_{12}=d_{21}=0,\ b_{12}>0,\ b_{21}=0.$

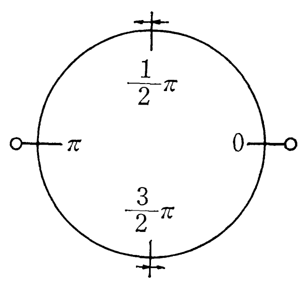

Fig. 15. (III), $d_{12}=d_{21}=0,\ b_{12}<0,\ b_{21}=0.$

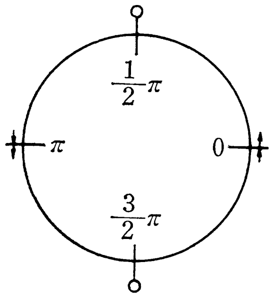

Fig. 16. (III), $d_{12}=d_{21}=0,\ b_{12}=0,\ b_{21}>0$.

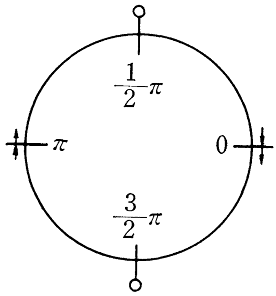

Fig. 17. (III), $d_{12}=d_{21}=0,\ b_{12}=0,\ b_{21}<0$.

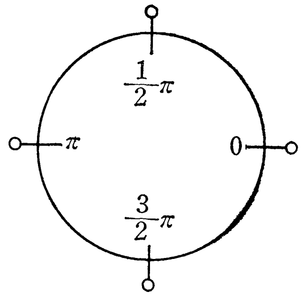

Fig. 18. (III), $d_{12}=d_{21}=0,\ b_{12}=b_{21}=0$.

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