N. SUITA
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ON A METRIC INDUCED BY ANALYTIC CAPACITY II

Dedicated to Professor Yûsaku Komatu on his 60th birthday

BY NOBUYUKI SUITA

1. Introduction. Let Ω be a plane region having nonconstant bounded analytic functions. Let $c_B(\zeta)$ be the least upper bound of $|f'(\zeta)|$, $\zeta\in\Omega$ in the class of bounded analytic functions satisfying $|f|\leqq 1$ in Ω. In our earlier paper [5] of these reports we have proved that the curvature in the metric $ds_B=|c_B(\zeta)||d\zeta|$ is not greater than -4, by making use of a supporting metric due to Ahlfors [1].

In the present paper we will show that a method of Bergman [3] provides the same result and further a more precise estimation $\kappa(\zeta)<-4$ for a finitely connected region bounded by more than one curves.

2. Extremal problems. Let Ω be a plane region bounded by a finite number of analytic Jordan curves. The class of analytic functions f such that $|f(z)|^2$ has a harmonic majorant in Ω is called the *Hardy class* of index two, denoted by $H_2(\Omega)$. Every function f in $H_2(\Omega)$ has a non-tangential boundary value almost everywhere on the boundary $\partial\Omega$ of Ω which will be denoted by the same notation $f(z)$, $z\in\partial\Omega$. The function $f(z)$ is measurable and square integrable on $\partial\Omega$ [4]. We define the inner product (f, g) of f and $g\in H_2(\Omega)$ by

$$(f, g)=\int_{\partial\Omega} f(z)\overline{g(z)}|dz|\,.$$

Then $H_2(\Omega)$ becomes a Hilbert space. There exists the Szegö kernel function $k(z, \bar{\zeta})$ in $H_2(\Omega)$ which is characerized by the reproducing property:

$$f(\zeta)=\int_{\partial\Omega} f(z)\overline{k(z, \bar{\zeta})}|dz|\,, \tag{1}$$

for $f\in H_2(\Omega)$ [3]. The following problems were dealt with by Bergman [3].

Consider two extremal problems:

I) Minimize $\|f\|^2=(f, f)$ in the subclass of $H_2(\Omega)$ each member of which satisfies $f(\zeta)=1$ for $\zeta\in\Omega$.

II) Minimize $\|f\|^2$ in the subclass of $H_2(\Omega)$ each member of which satisfies $f(\zeta)=0$ and $f'(\zeta)=1$.

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It is easy to prove that there exists a unique solution $F_0(z)=k(z, \bar{\zeta})/k(\zeta, \bar{\zeta})$ and that the minimum value $\lambda_0(\zeta)$ is equal to $1/k(\zeta, \bar{\zeta})$. Problem II has also a unique solution given by

$$F_1(z)=-\begin{vmatrix} 0 & k(z, \bar{\zeta}) & k_{\bar{\zeta}}(z, \bar{\zeta}) \\ 0 & k_{00} & k_{01} \\ 1 & k_{10} & k_{11} \end{vmatrix} \Bigg/ \begin{vmatrix} k_{00} & k_{01} \\ k_{10} & k_{11} \end{vmatrix}$$

and the minimum value $\lambda_1(\zeta)$ is equal to $k_{00}/(k_{00}k_{11}-|k_{01}|^2)$. Here $k_{\alpha\beta}$ denotes $\partial^{\alpha+\beta}/\partial\zeta^\alpha\partial\bar{\zeta}^\beta k(\zeta, \bar{\zeta})$. Note that $k_{01}=\bar{k}_{10}$.

In order to verify the extremality of F_1 we state a lemma which will be useful in the next section:

LEMMA. *F_1 is extremal if and only if it is orthogonal to every function $g\in H_2(\Omega)$ satisfying $g(\zeta)=0$ and $g'(\zeta)=0$.*

Proof. Let F be a competing function in Problem II. Set $g=F-F_1$. Then we have

$$\|F_1+\varepsilon g\|^2=\|F_1\|^2+\mathrm{Re}\{\bar{\varepsilon}(F_1, g)\}+|\varepsilon|^2\|g\|^2 .$$

Since ε is arbitrary we have $(F_1, g)=0$. Conversely we find from $(F_1, g)=0$

$$\|F\|^2=\|F_1\|^2+\|F-F_1\|^2 , \tag{2}$$

which implies the extremality of F_1.

It is easy to see the orthogonal property of F_1 by making use of the reproducing property (1). Moreover the identity (2) shows the unicity of the extremal function.

3. Curvature. Let $c_B(\zeta)$ be the analytic capacity of $\partial\Omega$. There exists a unique function $f_0(z)$, called the *Ahlfors function*, which satisfies $f_0'(\zeta)=c_B(\zeta)$ and $|f_0(z)|\leqq 1$ in Ω. Especially, if Ω is a plane region bounded by n analytic Jordan curves, f_0 maps Ω onto an n-sheeted unit disc [2].

The curvature $\kappa(\zeta)$ in the metric $ds_B=c_B(\zeta)|d\zeta|$ is given by

$$-4\frac{\dfrac{\partial^2 \log c_B(\zeta)}{\partial\zeta\,\partial\bar{\zeta}}}{c_B(\zeta)^2} .$$

The differentiability of $c_B(\zeta)$ is guaranteed by the identity $c_B(\zeta)=2\pi k(\zeta, \bar{\zeta})$ [5]. A direct calculation gives

$$\kappa(\zeta)=\frac{|k_{01}|^2-k_{00}k_{11}}{\pi^2 k_{00}{}^4}=-\frac{\lambda_0(\zeta)^2}{\pi^2\lambda_1(\zeta)} . \tag{3}$$

We now state

THEOREM. *If there exists nonconstant bounded analytic functions on Ω, the* *curvature $\kappa(\zeta)$ is dominated by -4. Furthermore, if Ω is bounded by a finite number of Jordan curves, the equality $\kappa(\zeta)=-4$ at one point $\zeta\in\Omega$ implies that Ω is conformally equivalent to the unit disc.*

Proof. We first prove the second statement. Since $\kappa(\zeta)$ is conformally invariant we may suppose that Ω is a bounded region whose boundary consists of a finite number of analytic Jordan curves. By (3) we have

$$\kappa(\zeta)=-\frac{1}{\pi^2\|F_1\|^2k_{00}^3},$$

F_1 being the extremal function in Problem II. We take a competing function

$$\varphi(z)=\frac{f_0(z)k(z,\bar{\zeta})}{2\pi k_{00}^2}$$

in the same problem. Then we have

$$\|F_1\|^2\leqq\|\varphi\|^2=\frac{1}{4\pi^2k_{00}^3}$$

and hence $\kappa(\zeta)\leqq-4$. Suppose the number of boundary components of Ω is greater than one. Then the Ahlfors function $f_0(z)$ has at least one zero point ζ_1 other than ζ, say. We use

$$g(z)=\frac{(z-\zeta)f_0(z)}{z-\zeta_1}$$

as a test function in Lemma. We shall show that the integral

$$\int_{\partial\Omega}g(z)\overline{\varphi(z)}|dz|=\frac{1}{2\pi k_{00}^2}\int_{\partial\Omega}g(z)\overline{f_0(z)k(z,\bar{\zeta})}|dz| \tag{4}$$

does not vanish which implies that φ is not extremal. The Szegö kernel function $k(z,\bar{\zeta})$ of Ω has its adjoint kernel $l(z,\zeta)$ which is regular in Ω except for a simple pole at ζ with residue $(2\pi)^{-1}$ and never vanishes [3]. $k(z,\bar{\zeta})$ and $l(z,\zeta)$ satisfies a fundamental relation

$$l(z,\zeta)dz=i\overline{k(z,\bar{\zeta})|dz|},$$

along $\partial\Omega$ [3]. Then we find that the integral (4) is equal to

$$\frac{1}{2\pi ik_{00}^2}\int_{\partial\Omega}\frac{z-\zeta}{z-\zeta_1}l(z,\zeta)dz=\frac{(\zeta_1-\zeta)l(\zeta_1,\zeta)}{k_{00}^2}\neq0.$$

This means that $\kappa(\zeta)<-4$.

To prove the first statement, we take a canonical exhaustion $\{\Omega_n\}$ of Ω, each member of which is bounded by a finite number of analytic Jordan curves. The sequence of Szegö kernel functions $k_n(z,\bar{\zeta})$ of Ω_n converges to a function $k(z,\bar{\zeta})$. The convergence is uniform on every compact subset of $\Omega\times\Omega$ as a sequence of functions analytic in z and $\bar{\zeta}$ [5]. Therefore the sequence of curvatures $\kappa_n(\zeta)$ in Ω_n converges to $\kappa(\zeta)$ in Ω. Thus we have $\kappa(\zeta)\leqq-4$.

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DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY