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A NOTE ON SEMIGROUPS OF MARKOV OPERATORS ON $C(X)$

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1. Introduction.

Let X be a compact Hausdorff space, and let $C(X)$ be the commutative C^*-algebra of all continuous complex functions on X. A bounded linear operator T of $C(X)$ into itself is called a Markov operator it $T \geqq 0$, $\|T\|=1$, and $T1=1$.

Let Σ be a semigroup of Markov operators. For each $f \in C(X)$, $\overline{\text{co}}\{Tf: T\in\Sigma\}$ denotes the closed convex hull of $\{Tf: T\in\Sigma\}$. $g\in C(X)$ is called a Σ-invariant function if $Tg=g$ for all $T\in\Sigma$.

In ergodic theory the following conditions on Σ are interesting: (I) Each $\overline{\text{co}}\{Tf: T\in\Sigma\}$ contains exactly one Σ-invariant function. (II) Each $\overline{\text{co}}\{Tf: T\in\Sigma\}$ contains at least one Σ-invariant function. In Theorem 1, we shall give some necessary and sufficient conditions that (I) holds.

Let $C(X)^*$ be the dual Banach space of $C(X)$. $\mu\in C(X)^*$ is called a state if $\mu\geqq 0$ and $\|\mu\|=\mu(1)=1$. If T is a Markov operator and if μ is a state, then $T^*\mu$ is also a state where T^* denotes the adjoint operator of T. A state μ is called a Σ-invariant state if $T^*\mu=\mu$ for all $T\in\Sigma$.

Let K_Σ be the set of all Σ-invariant states. Then K_Σ is a weak*-compact convex subset of $C(X)^*$. $\mu\in K_\Sigma$ is called an extremal Σ-invariant state if μ is an extreme point of K_Σ.

A proper closed ideal I of $C(X)$ is called a Σ-invariant ideal if $T(I)\subset I$ for all $T\in\Sigma$. There exists at least one maximal Σ-invariant ideal, and each Σ-invariant ideal is contained in some maximal Σ-invariant ideal. If μ is a Σ-invariant state, then $I_\mu=\{f\in C(X): \mu(|f|)=0\}$ is a Σ-invariant ideal.

In Theorem 2, we shall show that if (I) holds, then $\mu\to I_\mu$ is a bijection of the set of all extremal Σ-invariant states onto the family of all maximal Σ-invariant ideals.

Our discussion is much due to Deleeuw and Glicksberg [1], Schaefer [2], Sine [3], and Takahashi [4].

2. Theorems.

co Σ denotes the set of all finite convex linear combinations of operators in Σ. co Σ is also a semigroup of Markov operators. We note that $\overline{\text{co}}$ $\{Tf:$

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$T\in\Sigma\}=\overline{\{Af: A\in\mathrm{co}\,\Sigma\}}$. $\tilde{f}$ denotes the unique Σ-invariant function in $\overline{\mathrm{co}}\,\{Tf: T\in\Sigma\}$ whenever (I) holds.

LEMMA 1. *If* (I) *holds, then for any* $\varepsilon>0$ *and* $f_i\in C(X)$ $(i=1, 2, \cdots, n)$, *there exists an* $A\in\mathrm{co}\,\Sigma$ *such that* $\|\tilde{f}_i-Af_i\|\leqq\varepsilon$ $(i=1, 2, \cdots, n)$.

Proof. It is easy to see that $\widetilde{Af}=\tilde{f}$ for all $f\in C(X)$ and $A\in\mathrm{co}\,\Sigma$. First we choose an $A_1\in\mathrm{co}\,\Sigma$ such that $\|\tilde{f}_1-A_1f_1\|\leqq\varepsilon$. Next we choose an $A_2\in\mathrm{co}\,\Sigma$ such that $\|\widetilde{A_1f_2}-A_2(A_1f_2)\|\leqq\varepsilon$. Let $A=A_2A_1$. Then $A\in\mathrm{co}\,\Sigma$ and $\|\tilde{f}_i-Af_i\|\leqq\varepsilon$ $(i=1, 2)$. An induction argument completes the proof.

Let $B(\Sigma)$ be the commutative C^*-algebra of all bounded complex functions on Σ. For each $f\in C(X)$ and $\nu\in C(X)^*$, we define $f\otimes\nu\in B(\Sigma)$ by $(f\otimes\nu)(T)=\nu(Tf)$. Let $L(\Sigma)$ be the linear span of $\{f\otimes\nu: f\in C(X), \nu\in C(X)^*\}$ in $B(\Sigma)$. We note that $1\in L(\Sigma)$ and $\varphi^*\in L(\Sigma)$ if $\varphi\in L(\Sigma)$ where φ^* denotes the complex conjugate function of φ, and that φ_s (or ${}_s\varphi)\in L(\Sigma)$ if $S\in\Sigma$ and $\varphi\in L(\Sigma)$ where φ_s (or ${}_s\varphi$) denotes the right (or left) translation of φ by S. $m\in L(\Sigma)^*$ is called a right (or left) invariant mean on $L(\Sigma)$ if $m(\varphi)\geqq 0$ whenever $\varphi\geqq 0$, $\|m\|=m(1)=1$, and $m(\varphi_s)$ (or $m({}_s\varphi))=m(\varphi)$ for all $S\in\Sigma$ and $\varphi\in L(\Sigma)$. A right and left invariant mean m on $L(\Sigma)$ is called a two-sided invariant mean on $L(\Sigma)$. If m is a right invariant mean on $L(\Sigma)$, then for each state μ we can define $\tilde{\mu}\in K_\Sigma$ by $\tilde{\mu}(f)=m(f\otimes\mu)$. In the following theorem, M_Σ denotes the set of all Σ-invariant functions in $C(X)$.

THEOREM 1. *The following conditions are equivalent:*
(1) (I) *holds.*
(2) *There exists a two-sided invariant mean on* $L(\Sigma)$, *and* M_Σ *separates* K_Σ.
(3) *There exists a right invariant mean on* $L(\Sigma)$, *and* M_Σ *separates* K_Σ.
(4) *There exists a right invariant mean on* $L(\Sigma)$, *and* (II) *holds.*

Proof. (1) implies (2): If μ_1 and μ_2 are distinct Σ-invariant states, then $\mu_1(f)\neq\mu_2(f)$ for some $f\in C(X)$. This implies that $\mu_1(\tilde{f})=\mu_1(f)\neq\mu_2(f)=\mu_2(\tilde{f})$. Thus M_Σ separates K_Σ. For each $\varphi=\sum_{i=1}^n f_i\otimes\nu_i\in L(\Sigma)$, we define $m(\varphi)=\sum_{i=1}^n \nu_i(\tilde{f}_i)$. We shall show that $m(\varphi)$ is independent of the particular representation of φ and that m is a two-sided invariant mean on $L(\Sigma)$. Suppose $\sum_{i=1}^n f_i\otimes\nu_i$ is identically zero. By Lemma 1, for any $\varepsilon>0$ there exists an $A\in\mathrm{co}\,\Sigma$ such that $\|\tilde{f}_i-Af_i\|\leqq\varepsilon$ $(i=1, 2, \cdots, n)$. Then we have

$$|\sum_{i=1}^n \nu_i(\tilde{f}_i)|\leqq|\sum_{i=1}^n \nu_i(\tilde{f}_i-Af_i)|+|\sum_{i=1}^n \nu_i(Af_i)|\leqq(\sum_{i=1}^n \|\nu_i\|)\varepsilon\,.$$

Since $\varepsilon>0$ is arbitrary, $\sum_{i=1}^n \nu_i(\tilde{f}_i)=0$. Thus we may unambiguously define $m(\varphi)$. It is easy to see that m is linear and $m(1)=1$. We shall show that $\|m\|=1$. Since $m(1)=1$, it suffices to show that $\|m\|\leqq 1$. Suppose $\varphi=\sum_{i=1}^n f_i\otimes\nu_i$. Again by Lemma 1, for any $\varepsilon>0$ there exists an $A\in\mathrm{co}\,\Sigma$ such that $\|\tilde{f}_i-Af_i\|\leqq\varepsilon$ $(i=1, 2, \cdots, n)$. Then we have

$$|m(\varphi)| \leqq |\sum_{i=1}^{n} \nu_i(\tilde{f}_i - Af_i)| + |\sum_{i=1}^{n} \nu_i(Af_i)| \leqq (\sum_{i=1}^{n} \|\nu_i\|)\varepsilon + \|\varphi\| .$$

Since $\varepsilon > 0$ is arbitrary, $|m(\varphi)| \leqq \|\varphi\|$. It is easy to see that $m(\varphi^*) = \overline{m(\varphi)}$. Suppose $\varphi = \sum_{i=1}^{n} f_i \otimes \nu_i \geqq 0$. Then $m(\varphi)$ is real. We assume that $\alpha = m(\varphi) < 0$. Let ε be a number such that $0 < \varepsilon < -\alpha$. By Lemma 1, we can choose an $A \in \mathrm{co}\, \Sigma$ such that $\|\tilde{f}_i - Af_i\| \leqq \varepsilon / \sum_{i=1}^{n} \|\nu_i\|$ $(i = 1, 2, \cdots, n)$. Let $\beta = \sum_{i=1}^{n} \nu_i(Af_i)$. Then $\beta \geqq 0$ and $|\alpha - \beta| < \varepsilon$, so we have $0 \leqq \beta = |\alpha - \beta| + \alpha \leqq \varepsilon + \alpha < 0$. This is a contradiction. Thus $m(\varphi) > 0$. If $\varphi = \sum_{i=1}^{n} f_i \otimes \nu_i$, then $\varphi_s = \sum_{i=1}^{n} Sf_i \otimes \nu_i$ and ${}_s\varphi = \sum_{i=1}^{n} f_i \otimes S^* \nu_i$. It is easy to see that $m(\varphi_s) = m({}_s\varphi) = m(\varphi)$.

(2) implies (3): Evident.

(3) implies (4): Let m be a right invariant mean on $L(\Sigma)$, and let δ_x be the point measure at $x \in X$. For each $f \in C(X)$, we can define $Pf \in C(X)$ by $(Pf)(x) = m(f \otimes \delta_x)$ (see [3] and [4]). Then P is a Markov operator such that $PT = P$ for all $T \in \Sigma$ and $Pg = g$ for all $g \in M_\Sigma$. We shall show that Pf is a Σ-invariant function in $\overline{\mathrm{co}}\, \{Tf : T \in \Sigma\}$. Let μ be a state. Then $P^* T^* \mu$, $P^* \mu$ and $\tilde{\mu}$ are Σ-invariant states. If g is a Σ-invariant function, then $(P^* T^* \mu)(g) = \mu(TPg) = \mu(Tg) = \mu(g)$, $(P^* \mu)(g) = \mu(Pg) = \mu(g)$, and $\tilde{\mu}(g) = m(g \otimes \mu) = \mu(g)$. Since M_Σ separates K_Σ, we have $P^* T^* \mu = P^* \mu = \tilde{\mu}$, which implies that $TP = P$ for all $T \in \Sigma$ and $\nu(Pf) = m(f \otimes \nu)$ for all $f \in C(X)$ and $\nu \in C(X)^*$. Thus Pf is a Σ-invariant function. If Pf is not contained in $\overline{\mathrm{co}}\, \{Tf : T \in \Sigma\}$, there exists a $\nu \in C(X)^*$ such that $\sup \{\Re e\, \nu(Tf) : T \in \Sigma\} < \Re e\, \nu(Pf)$, but $\Re e\, \nu(Pf) = \Re e\, m(f \otimes \nu) = m(\Re e\, (f \otimes \nu)) \leqq \sup \{\Re e\, \nu(Tf) : T \in \Sigma\}$. This is a contradiction.

(4) implies (1): The proof is similar to [4].

THEOREM 2. *If* (I) *holds, then* $\mu \to I_\mu$ *is a bijection of the set of all extremal* Σ*-invariant states onto the family of all maximal* Σ*-invariant ideals.*

Proof. Let I be a maximal Σ-invariant ideal. As well known, there exists an $x_0 \in X$ such that any function in I vanishes at x_0. For each $f \in C(X)$ we define $\mu(f) = \tilde{f}(x_0)$, then μ is a Σ-invariant state which vanishes on I. The Schwarz inequality $\mu(|f|) \leqq \sqrt{\mu(|f|^2)}$ implies that $I \subset I_\mu$ and therefore $I = I_\mu$. Let $K_{\Sigma, I} = \{\mu \in K_\Sigma : I = I_\mu\}$, then $K_{\Sigma, I}$ is a nonempty weak*-compact convex subset of $C(X)^*$. By the Krein-Milman theorem there exists an extreme point μ_0 of $K_{\Sigma, I}$. It is easy to see that μ_0 is also an extreme point of K_Σ.

Let μ be an extremal Σ-invariant state. If I_μ is not maximal, then there exists a maximal Σ-invariant ideal I containing I_μ. We can choose a Σ-invariant function g from $I - I_\mu$ such that $0 \leqq g \leqq 1$ and $0 < \mu(g) < 1$. Let $\mu_1(f) = \mu(\tilde{f}g)/\mu(g)$ and $\mu_2(f) = \mu(\tilde{f}(1-g))/\mu(1-g)$. Then μ_1 and μ_2 are Σ-invariant states, and $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ where $\alpha = \mu(g)$. Since μ is extremal, $\mu_1 = \mu_2$ and therefore $\mu_1(g) = \mu_2(g)$, which implies $\mu(g^2) = (\mu(g))^2$. It follows easily from the Schwarz inequality that $\mu(|g - \mu(g)1|) = 0$. This shows that $g - \mu(g)1 \in I$ and therefore $1 \in I$. This is a contradiction.

Let μ_1 and μ_2 be distinct extremal Σ-invariant states. Then there exists a Σ-invariant function g such that $0 \leqq g \leqq 1$ and $\mu_1(g) \neq \mu_2(g)$. If $I_{\mu_1} = I_{\mu_2}$, then

$0<\mu_1(g)<1$. Let $\mu_3(f)=\mu_1(\tilde{f}g)/\mu_1(g)$ and $\mu_4(f)=\mu_1(\tilde{f}(1-g))/\mu_1(1-g)$. Then μ_3 and μ_4 are Σ-invariant states, and $\mu_1=\alpha\mu_3+(1-\alpha)\mu_4$ where $\alpha=\mu_1(g)$. Since μ_1 is extremal, $\mu_3=\mu_4$. As in the above paragraph, it follows that $g-\mu_1(g)1\in I_{\mu_1}$ and therefore $g-\mu_1(g)1\in I_{\mu_2}$, which implies that $\mu_1(g)=\mu_2(g)$. This is a contradiction. Thus we conclude that I_{μ_1} and I_{μ_2} are distinct.

REFERENCES

[1] DELEEUW, K., AND I. GLICKSBERG, Applications of almost periodic compactifications. Acta Math. **105** (1961), 65-97.

[2] SCHAEFER, H. H., Invariant ideals of positive operators in $C(X)$. Illinois J. Math. **12** (1968), 525-538.

[3] SINE, R., Geometric theory of a single Markov operator. Pacific J. Math. **27** (1968), 155-166.

[4] TAKAHASHI, W., Invariant ideals for amenable semigroups of Markov operators. Kōdai Math. Sem. Rep. **23** (1971), 121-126.

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