OZAWA, M.
KŌDAI MATH. SEM REP.
24 (1972), 502—512

RADIAL DISTRIBUTION OF ZEROS AND DEFICIENCY OF A CANONICAL PRODUCT OF FINITE GENUS

BY MITSURU OZAWA

1. Introduction. Edrei and Fuchs [1] proved the following

THEOREM A. *Let $f(z)$ be an entire function of finite order ρ, having only negative zeros. If $\rho>1$, then $\delta(0, f)>0$.*

This reveals a quite interesting fact that a simple geometrical restriction is enough to make zero a deficient value. Edrei, Fuchs and Hellerstein [2] made the above result better. They gave a numerical bound

$$\delta(0, f)\geqq\frac{A}{1+A}$$

with an absolute constant $A>0$. By a rough estimation their constant A satisfies $A<0.0017$. This is, of course, far from the best. There is still no reasonable conjecture for the best possible A.

They [2] gave the following result. (We state it here only in the case of genus one.)

THEOREM B. *Let $g(z)$ be a canonical product of genus one and having zeros $\{a_\mu\}$ in the sector*

$$|\pi-\arg a_\mu|\leqq\frac{\pi}{60}.$$

If the order of g is greater than one, then

$$\delta(0, g)\geqq\frac{A}{1+A},$$

where A is the constant already mentioned.

Again $\pi/60$ is far from the best together with A. In this paper we shall prove the following

THEOREM 1. *Let $g(z)$ be a canonical product of genus q, having only negative*

Received January 18, 1973.

zeros. If $q \geqq 2$, *then*

$$\delta(0, g) \geqq \frac{A(q)}{1+A(q)}$$

where

$$A(q) = \frac{\cos \pi/2q}{\pi \sin \pi/2q} \int_1^\infty \frac{ds}{s^q(s+1)^2} \geqq \frac{1}{12\pi}.$$

If q tends to infinity, then $A(q)$ *tends to* $1/2\pi^2$. *If* $q=1$, *then*

$$\delta(0, g) \geqq \frac{A(1)}{1+A(1)},$$

where

$$A(1) = \left(1 - \frac{\sqrt{3}}{9}\pi\right)\frac{\sqrt{3}}{2\pi}.$$

Our method of proof depends heavily upon the extremely precise analysis about the behavior of $\log|g(re^{i\theta})|$ on $|z|=r$ due to Hellerstein and Williamson [3] and the representation of $m(r, g)$ due to Shea [4]. In principle we can imagin how to get the best possible numerical bound of A by our method, although it is very hard to give any explicit form.

THEOREM 2. *Let* $g(z)$ *be a canonical product of genus* q, *having only zeros* $\{-a_k\}$ *which satisfy*

$$|\arg a_k| \leqq \beta < \frac{\pi}{2(q+1)} \quad \text{if } q \text{ is odd,}$$

$$0 \leqq \arg a_k \leqq \beta < \frac{(q-1)\pi}{2q(q+1)} \quad \text{if } q \text{ is even} \geqq 2.$$

Then with a positive constant $A = A(q, \beta)$

$$\delta(0, g) \geqq \frac{A}{1+A}.$$

COROLLARY. *Let* $g(z)$ *be a canonical product of genus* q, *whose zeros* a_μ *satisfy*

$$\sum_{\mu=1}^\infty \frac{1}{|a_\mu|^q} = \infty, \ \sum_{\mu=1}^\infty \frac{1}{|a_\mu|^{q+1}} < \infty, \ q \geqq 1$$

and lie in

$$\left|\arg a_\mu - \frac{2\pi k}{q}\right| \leqq \frac{\beta}{q}, \ \beta < \frac{\pi}{4}$$

$$(k = 0, 1, \cdots, q-1).$$

Then

$$\delta(0, g) \geqq \frac{A}{1+A},$$

where

$$A = A(1, \beta)$$
$$= \frac{1}{\pi}\int_1^\infty \frac{1}{s^2}\left(\frac{s+\sqrt{2}/2}{s^2+\sqrt{2}s+1} - \frac{s\sin 2\beta + \sin\beta}{s^2+2s\cos\beta+1}\right)ds,$$

defined in Theorem 2.

This corollary gives a better result than that of the case of even genus in Theorem 2 in the opening of one sector and the value of A.

THEOREM 3. *Let $g(z)$ be a canonical product of genus q with zeros $\{-a_\mu\}$ such that*

$$\sum \frac{1}{|a_\mu|^q} = \infty$$

and

$$|\arg a_\mu| \leqq \beta < \frac{\pi}{2(q+1)}.$$

Then

$$q \leqq \mu \leqq \rho \leqq q+1,$$

where ρ and μ indicate the order and the lower order of $g(z)$, respectively.

In Theorem 1 we have given a numerical bound of $A(q)$. By a minor modification of our method we can give a slightly improved bound of $A(q)$. In the lower genus cases we can easily improve it.

2. Proof of Theorem 1.

$$1-\delta(0, g) = \varlimsup_{r\to\infty} \frac{N(r, g)}{m(r, g)} = \varlimsup_{r\to\infty} \frac{N(r, 0)}{N(r, 0)+m(r, 0, g)}.$$

Hence it is sufficient to estimate $m(r, 0, g)$ from below by $A(q)N(r, 0)$. Assume $q=2p+1, p\geqq 1$ in the first place. Hellerstein and Williamson's analysis gives

$$m(r, 0, g) = \frac{1}{\pi}\sum_{j=1}^{p}\int_{\alpha_{2j}}^{\alpha_{2j+1}} \log\frac{1}{|g(re^{i\theta})|}\,d\theta + \frac{1}{\pi}\int_{\alpha_{q+1}}^{\pi} \log\frac{1}{|g(re^{i\theta})|}\,d\theta.$$

Here $\{\alpha_j\}$ is given in [3], Main Lemma. Since by their lemma

$$\log|g(re^{i\theta})| < 0$$

in and only in $(\alpha_{2j}, \alpha_{2j+1})$ and (α_{q+1}, π), which may be empty, we have the above representation of $m(r, 0, g)$. Here α_j and α_{q+1} satisfy

$$\frac{2j-1}{2(q+1)}\pi < \alpha_j < \frac{2j-1}{2q}\pi, \qquad j=1, \cdots, q,$$

$$\frac{2q+1}{2(q+1)}\pi < \alpha_{q+1} \leqq \pi, \qquad \alpha_0 = 0.$$

If we select $\{\alpha_j^*\}$ such that $\alpha_{j-1}^* < \alpha_{j-1} < \alpha_j^* < \alpha_j$ for $j \leqq (q+2)/2$ and $\alpha_j^* \in ((2j-1)\pi/2(q+1), (2j-1)\pi/2q)$ for $j > (q+2)/2$, then

$$m(r, 0, g) \geqq \frac{1}{\pi} \sum_{j=0}^{p} \int_{\alpha_{2j}^*}^{\alpha_{2j+1}^*} \log \frac{1}{|g(re^{i\theta})|} d\theta.$$

We may take $\alpha_0^* = \alpha_1^* = 0$. There is such a selection. Let α_j^* be $(j-1)\pi/q$. Then if $j > (q+2)/2$

$$\frac{2j-1}{2q}\pi > \alpha_j^* > \frac{2j-1}{2(q+1)}\pi$$

and if $j \leqq (q+2)/2, j \geqq 2$

$$\alpha_{j-1} < \frac{2j-3}{2q}\pi < \alpha_j^* < \frac{2j-1}{2(q+1)}\pi < \alpha_j$$

and $\alpha_0^* = \alpha_1^* = 0$. On the other hand

$$= \frac{1}{\pi} \int_{\alpha_{2j}}^{\alpha_{2j+1}^*} \log |g(re^{i\theta})| d\theta$$

$$= -\frac{1}{\pi} \left(\int_0^{\alpha_{2j+1}^*} - \int_0^{\alpha_{2j}^*} \right) \log |g(re^{i\theta})| d\theta$$

$$= -\frac{1}{\pi} \int_0^\infty \frac{N(sr, 0)}{s^{q+1}} \left[\frac{s \sin (q+1)\alpha_{2j+1}^* + \sin q\alpha_{2j+1}^*}{s^2 + 2s \cos \alpha_{2j+1}^* + 1} - \frac{s \sin (q+1)\alpha_{2j}^* + \sin q\alpha_{2j}^*}{s^2 + 2s \cos \alpha_{2j}^* + 1} \right] ds$$

by Shea's representation. Hence we have

$$m(r, 0, g) \geqq \frac{1}{\pi} \sum_{j=2}^{q} \int_0^\infty \frac{N(sr, 0)}{s^q} \frac{\sin ((j-1)\pi/q)}{s^2 + 2s \cos ((j-1)\pi/q) + 1} ds$$

$$\geqq \frac{1}{\pi} \sum_{j=1}^{q-1} \sin \frac{j}{q}\pi \int_0^\infty \frac{N(sr, 0)}{s^q} \frac{ds}{(s+1)^2}$$

$$\geqq \frac{N(r, 0)}{\pi} \int_1^\infty \frac{ds}{s^q (s+1)^2} \frac{\sin ((q-1)\pi/2q)}{\sin (\pi/2q)}.$$

Hence we have

$$\delta(0, g) \geqq \frac{A(q)}{1+A(q)}$$

with

$$A(q)=\frac{1}{\pi}\frac{\cos(\pi/2q)}{\sin(\pi/2q)}\int_1^\infty \frac{ds}{s^q(s+1)^2}.$$

Since

$$\int_1^\infty \frac{ds}{s^q(s+1)^2} > \frac{1}{4}\int_1^\infty \frac{ds}{s^{q+2}} = \frac{1}{4(q+1)},$$

$$\int_1^\infty \frac{ds}{s^q(s+1)^2} < \frac{1}{4}\int_1^\infty \frac{ds}{s^q} = \frac{1}{4(q-1)},$$

we have

$$\lim_{q\to\infty} A(q) = \frac{1}{2\pi^2}.$$

Further

$$A(q) > \frac{1}{4\pi}\frac{\cos(\pi/2q)}{\sin(\pi/2q)}\frac{1}{q+1}$$

$$\geqq \frac{1}{12\pi}$$

for $q \geqq 2$.

Assume $q=2p, p\geqq 1$. Then

$$m(r, g)=N(r, 0)+m(r, 0, g),$$

$$m(r, 0, g)=\frac{1}{\pi}\sum_{j=1}^{q}\int_{\alpha_{2j-1}}^{\alpha_{2j}} \log\frac{1}{|g(re^{i\theta})|}\,d\theta+\frac{1}{\pi}\int_{q+1}^{\pi}\log\frac{1}{|g(re^{i\theta})|}\,d\theta$$

$$\geqq -\frac{1}{\pi}\sum_{j=1}^{p}\int_{\alpha_{2j-1}^*}^{\alpha_{2j}^*}\log|g(re^{i\theta})|\,d\theta$$

by the same $\alpha_j^* = (j-1)\pi/q$. Then the same process leads the same expression for $A(q)$. Hence we have the desired result.

If $q=1$, we have

$$m(r, 0, g) \geqq \frac{1}{\pi}\int_0^{\alpha_1}\log\frac{1}{|g(re^{i\theta})|}\,d\theta$$

$$\geqq \frac{1}{\pi}\int_0^{\pi/3}\log\frac{1}{|g(re^{i\theta})|}\,d\theta.$$

The last integral is by Shea's representation

$$\frac{\sqrt{3}}{2}\frac{1}{\pi}\int_0^\infty \frac{N(sr,0)}{s^2}\frac{s+1}{s^2+s+1}ds$$

$$\geqq \frac{\sqrt{3}}{2\pi}N(r,0)\int_0^\infty \frac{s+1}{s^2(s^2+s+1)}ds.$$

By an easy calculation

$$m(r,0,g)\geqq \frac{\sqrt{3}}{2\pi}N(r,0)\left(1-\frac{\sqrt{3}}{9}\pi\right).$$

Hence

$$\delta(0,g)\geqq \frac{A(1)}{1+A(1)},$$

$$A(1)=\frac{\sqrt{3}}{2\pi}\left(1-\frac{\sqrt{3}}{9}\pi\right).$$

3. Proof of Theorem 2.

In the first place assume $q=2p+1$. Let

$$\psi(x,y)=\frac{1}{2}\log(1+2y\cos x+y^2)+\sum_{j=1}^{q}(-1)^j\frac{y^j}{j}\cos jx.$$

Then

$$\frac{\partial\psi(x,y)}{\partial x}=\frac{(-1)^{q+1}y^{q+1}}{1+2y\cos x+y^2}(\sin(q+1)x+y\sin qx),$$

$$\frac{\partial\psi(x,y)}{\partial y}=\frac{(-1)^q y^q}{1+2y\cos x+y^2}(\cos(q+1)x+y\cos qx).$$

Hence $\partial\psi/\partial x\geqq 0$ for $0\leqq x\leqq\pi/(q+1)$, $y\geqq 0$, which shows that $\psi(x,y)$ is monotone increasing for x there. $\partial\psi/\partial y\leqq 0$ for $y\leqq 0$, $0\leqq x\leqq\pi/2(q+1)$ and hence $\psi(x,y)$ is monotone decreasing for y there. Since $\psi(x,0)=0$, $\psi(x,y)<0$ for $y>0$ and $0\leqq x\leqq\pi/2(q+1)$. Let $z=re^{i\theta}$, $a_\mu=|a_\mu|e^{i\phi_\mu}$, $y_\mu=r/|a_\mu|$. Look at values of $\psi(\theta-\phi_\mu,y)$ in $0\leqq\theta\leqq\pi/2(q+1)-\beta$. By the assumption $|\phi_\mu|\leqq\beta$. Then for $\phi_\mu\geqq 0$

$$\psi(\theta-\phi_\mu,y_\mu)=\psi(-\theta+\phi_\mu,y_\mu)$$
$$\leqq\psi(\theta+\phi_\mu,y_\mu)\leqq\psi(\theta+\beta,y_\mu)<0.$$

Let $\hat{g}(w)$ be

$$\prod_{k=0}^{\infty}\left(1+\frac{w}{|a_k|}\right)\exp\left(\sum_{j=1}^{q}(-1)^j\frac{1}{j}\left(\frac{w}{|a_k|}\right)^j\right).$$

Then for $0\leqq\theta\leqq\pi/2(q+1)-\beta$, $z=re^{i\theta}$

$$\log|g(z)|\leqq\log|\hat{g}(ze^{i\beta})|.$$

Hence for Θ in $\beta\leqq\Theta\leqq\pi/2(q+1)$, $w=|z|e^{i\theta}$, $\Theta-\beta=\theta$

$$\log|g(z)|\leqq\log|\hat{g}(w)|.$$

Therefore

$$1-\delta(0,g)=\overline{\lim_{r\to\infty}}\frac{N(r,0,g)}{m(r,g)}$$

$$=\overline{\lim_{r\to\infty}}\frac{N(r,0,g)}{N(r,0,g)+m(r,0,g)},$$

$$m(r,0,g)\geqq\frac{1}{\pi}\int_0^{\pi/2(q+1)-\beta}\log\frac{1}{|g(re^{i\theta})|}\,d\theta$$

$$\geqq\frac{1}{\pi}\int_\beta^{\pi/2(q+1)}\log\frac{1}{|\hat{g}(re^{i\theta})|}\,d\Theta$$

$$=\frac{1}{\pi}\int_1^\infty\frac{N(sr,0)}{s^{q+1}}\left\{\frac{s+\sin(q\pi/2(q+1))}{s^2+2s\cos(\pi/2(q+1))+1}\right.$$

$$\left.-\frac{s\sin(q+1)\beta+\sin q\beta}{s^2+2s\cos\beta+1}\right\}ds.$$

Further

$$m(r,0,g)\geqq\frac{N(r,0)}{\pi}\int_1^\infty\frac{1}{s^{q+1}}\left\{\frac{s+\cos(\pi/2(q+1))}{s^2+2s\cos(\pi/2(q+1))+1}\right.$$

$$\left.-\frac{s\sin(q+1)\beta+\sin q\beta}{s^2+2s\cos\beta+1}\right\}ds,$$

since the integrand of the above integral is positive for $s>0$ by $\beta<\pi/2(q+1)$. Let us put the right hand side term by $N(r,0)A(q,\beta)$. Then $A(q,\beta)>0$ and

$$\delta(0,g)\geqq\frac{A(q,\beta)}{1+A(q,\beta)}.$$

Next consider the case $q=2p$, $p\geqq1$. In this case $\partial\psi(x,y)/\partial x\leqq0$ for $0\leqq x\leqq\pi/(q+1)$ and hence $\psi(x,y)$ is monotone decreasing there. Since $\partial\psi(\pi/2q,y)/\partial y\leqq0$ for $y\geqq0$, $\psi(\pi/2q,y)$ is monotone decreasing for $y\geqq0$. $\psi(\pi/2q,0)=0$ implies that $\psi(x,y)<0$ for $y>0$ in $\pi/2q\leqq x\leqq\pi/(q+1)$. Further

$$\frac{s\sin(q+1)x+\sin qx}{1+2s\cos x+s^2}$$

is monotone decreasing for $\pi/2q\leqq x\leqq\pi/(q+1)-\beta$. In this case we shall consider $\beta+\pi/2q\leqq\theta\leqq\pi(q+1)$, $z=re^{i\theta}$. By the above analysis we have

$$\log|g(re^{i\theta})|\leqq\log|\hat{g}(re^{i\theta})|<0,$$

where $\hat{g}(w)$ is

$$\prod_{k=1}^{\infty}\left(1+\frac{w}{|a_k|}\right)\exp\left(\sum_{j=1}^{q}\frac{(-1)^j}{j}\left(\frac{w}{|a_k|}\right)^j\right)$$

with $w=ze^{-i\beta}$. Hence

$$m(r,0,g)\geqq\frac{1}{\pi}\int_{\beta+\pi/2q}^{\pi/(q+1)}\log\frac{1}{|g(re^{i\theta})|}\,d\theta$$

$$\geqq\frac{1}{\pi}\int_{\beta+\pi/2q}^{\pi/(q+1)}\log\frac{1}{|\hat{g}(re^{i\theta})|}\,d\theta$$

$$=\frac{1}{\pi}\int_{\pi/2q}^{\pi/(q+1)-\beta}\log\frac{1}{|\hat{g}(re^{i\phi})|}\,d\phi,\qquad \phi=\theta-\beta,$$

$$=-\int_0^{\infty}\frac{N(sr,0)}{s^{q+1}}\left\{\frac{s\sin(\pi-(q+1)\beta)+\sin(q\pi/(q+1)-q\beta)}{s^2+2s\cos\pi/(q+1)-\beta)+1}\right.$$

$$\left.-\frac{s\sin((q+1)\pi/2q)+\sin(\pi/2)}{s^2+2s\cos(\pi/2q)+1}\right\}ds$$

$$\equiv\frac{1}{\pi}\int_0^{\infty}\frac{N(sr,0)}{s^{q+1}}H(s,q,\beta)ds.$$

Here $H(s,q,\beta)>0$ for $s>0$, since

$$\frac{s\sin(q+1)x+\sin qx}{s^2+2s\cos x+1}$$

is monotone decreasing for $\pi/2q\leqq x\leqq\pi/(q+1)-\beta$. Thus

$$m(r,0,g)\geqq\frac{N(r,0)}{\pi}\int_1^{\infty}\frac{H(s,q,\beta)}{s^{q+1}}\,ds$$

$$\equiv N(r,0)A(q,\beta),\ \ A(q,\beta)>0.$$

Therefore

$$\delta(0,g)\geqq\frac{A(q,\beta)}{1+A(q,\beta)}.$$

This gives the desired result.

In the above proof we have only consider a single suitable sector. Hence $A(q,\beta)$ is not good enough for $q\to\infty$. If we count all the possible sectors, then we can get a better estimation for $A(q,\beta)$. Our result in the cases of genus one or two is better than Edrei, Fuchs and Hellerstein's in the opening of the sector and the value of $A(q,\beta)$. However our result for any even genus case is not satisfactory. It is conjectured that we can improve it to

$$|\arg a_k|\leqq\beta,\ \ \beta<\frac{\pi}{2(q+1)}$$

as in the odd case.

4. Proof of Corollary.

Let ω be $\exp(2\pi i/q)$. Consider

$$G(z)=g(z)g(\omega z)\cdots g(\omega^{q-1}z)$$
$$=\Pi E\left(\frac{z^q}{a_\mu{}^q}, 1\right),$$

where

$$E(x, p)=(1-x)\exp\left(\sum_{j=1}^{p}\frac{1}{j}\,x^j\right).$$

Let $H(w)$ be

$$\Pi E\left(\frac{w}{a_\mu{}^q}, 1\right),$$

then $G(z)=H(z^q)$. Since $N(r, 0, G(z))=N(r^q, 0, H(z))$ and $m(r, G(z))=m(r^q, H(z))$, $m(r, 0, G(z))=m(r^q, 0, H(z))$.
Since

$$\Sigma\frac{1}{|a_\mu|^q}=\infty,\quad \Sigma\frac{1}{|a_\mu|^{q+1}}<\infty,$$

the genus of $H(w)$ is equal to one. Hence by Theorem 2

$$1-\delta(0, H)\leqq\frac{1}{1+A(1, \beta)}$$

since

$$|\arg a_\mu{}^q-2\pi k|\leqq\beta,\ \beta<\frac{\pi}{4}.$$

Further

$$1-\delta(0, H)=\varlimsup_{r\to\infty}\frac{N(r, 0, H)}{m(r, H)}$$
$$=\varlimsup_{r\to\infty}\frac{N(r^q, 0, H(z))}{m(r^q, H(z))}=\varlimsup_{r\to\infty}\frac{N(r, 0, G(z))}{m(r, G(z))}$$
$$=1-\delta(0, G).$$

Hence, by $N(r, 0, G)=qN(r, 0, g)$ and $m(r, G)\leqq qm(r, g)$, we have

$$1-\delta(0, g)\leqq 1-\delta(0, G)$$
$$\leqq\frac{1}{1+A(1, \beta)}.$$

This gives the desired result.

This proof is the same as in [2], Lemma 6. The result in this Corollary is better than that of the case of even genus in Theorem 2 in the opening of one sector together with the value of A.

5. Proof of Theorem 3.

Assume $q=2p+1$. Then

$$m(r, g)\geqq m(r, 0, g)$$

$$\geqq\frac{1}{\pi}\int_0^\infty \frac{N(sr, 0)}{s^{q+1}}\left\{\frac{s+\sin(q\pi/2(q+1))}{s^2+2s\cos(\pi/2(q+1))+1}-\frac{s\sin(q+1)\beta+\sin q\beta}{s^2+2s\cos\beta+1}\right\}ds$$

$$\geqq\frac{1}{\pi}\int_0^1 \frac{N(sr, 0)}{s^{q+1}}\frac{ds}{s^2+2s\cos\beta+1}\left(\sin\frac{q\pi}{2(q+1)}-\sin q\beta\right)$$

$$\geqq Mr^q\int_0^r \frac{N(t, 0)}{t^{q+1}}dt,$$

$$M=\frac{\sin(q\pi/2(q+1))-\sin q\beta}{2\pi(1+\cos\beta)}>0.$$

Since

$$\int_0^r \frac{N(t, 0)}{t^{q+1}}dt\to\infty$$

as $r\to\infty$ by

$$\sum\frac{1}{|a_\mu|^q}=\infty.$$

we have

$$\lim_{r\to\infty}\frac{m(r, g)}{r^q}=\infty.$$

Assume $q=2p$. Then similarly

$$m(r, g)\geqq\frac{1}{\pi}\int_\beta^{\pi/2(q+1)}\log|\hat{g}(re^{i\theta})|d\theta.$$

Thus we have similarly

$$m(r, g)\geqq Mr^q\int_0^r \frac{N(t, 0)}{t^{q+1}}dt,$$

$$\lim_{r\to\infty}\frac{m(r, g)}{r^q}=\infty.$$

This implies $\mu\geqq q$. Hence we have the desired result.

REFERENCES

[1] EDREI, A., AND W. H. J. FUCHS, On the growth of meromorphic functions with several deficient values. Trans. Amer. Math. Soc. **93** (1959), 292–328.

[2] ———, AND S. HELLESTEIN, Radial distribution and deficiencies of the values of a meromorphic function. Pacific Journ. Math. **11** (1961), 135–151.

[3] HELLERSTEIN, S., AND J. WILLIAMSON, Entire functions with negative zeros and a problem of R. Nevanlinna. Journ. Analyse Math. **22** (1969), 233–267.

[4] SHEA, D.F., On the Valiron deficiencies of meromorphic functions of finite order. Trans. Amer. Math. Soc. **124** (1966), 201–227.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.