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# DIFFERENTIABLE SOLUTIONS OF ALGEBRAIC EQUATIONS ON MANIFOLDS

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**1. Introduction.** Yano [7] introduced the notion of an *f-structure*, which is a non-null (1, 1) tensor field $f$ of constant rank $r$ on a $C^\infty$ manifold of dimension $r+m$, satisfying $f^3+f=0$. An almost complex and an almost contact structure are particular cases of an $f$-structure the existence of an $f$-structure being equivalent to a reduction of the structural group of the tangent bundle to $U(r/2)\times O(m)$. They were studied by various authors ([1], [2], [6], etc.) with particular focus on the case of globally framed structures [2]. Extending the concept of an $f$-structure, Goldberg and Yano [3] introduced the notion of a polynomial structure on a manifold.

An $f$-structure is a particular case of an almost product structure [7], [8]. The purpose of this paper is to point out the close relation of the polynomial structures on manifolds and the almost product structures as defined by Walker [8]. In § 2 it is shown that any polynomial structure generates an almost product structure. From this follow necessary and sufficient conditions for a distribution to be globally framed and for a manifold to be parallelizable. In § 3 reductions of the structural group of the tangent bundle of a polynomial structure are obtained, similar to that for $f$-structures (see [7]). It is shown that for any polynomial structure with structure polynomial decomposable into distinct irreducible quadratic factors over the reals $R$ that there is an underlying almost complex structure. In § 5 an analogue of the normal $f$-structures [2] is examined which is more general in the sense that the tensor field $f$ is not required to satisfy an algebraic equation.

**2. Almost product structure.** Let $M$ be a differentiable manifold. A $C^\infty$ tensor field $f$ of type (1, 1) on $M$ is said to define a *polynomial structure* if $f$ satisfies the algebraic equation

$$P(x)=x^m+a_m x^{m-1}+\cdots+a_2 x+a_1 I=0, \tag{2. 1}$$

where $I$ is the identity mapping and $f^{m-1}(p), f^{m-2}(p), \cdots, f(p), I$ are linearly independent for every $p\in M$. Clearly, $f$ is non-singular if and only if $a_1\neq 0$. The polynomial $P(x)$ is called the *structure polynomial*. If $P(x)=x^2+I$ we have an almost complex structure.

An *almost product structure* on a differentiable manifold $M$ is a system of differentiable distributions $T_1, T_2, \cdots, T_k$ such that

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(i) $$T(p)=T_1(p)+\cdots+T_k(p),$$

(ii) $$T_a(p)\cap T_b(p)=0, \qquad a\neq b,$$

for every $p\in M$, where $T(p)$ is the tangent space of $M$ at $p$. It is defined by a system of projectors $\pi_i(p)$: $T(p)\to T_i(p)$, $i=1, 2, \cdots, k$, which are $C^\infty$ tensor fields of type (1, 1) on $M$, satisfying $\sum_{i=1}^k \pi_i=I$, $\pi_i\pi_j=\delta_{ij}\pi_i$, where $\delta_{ij}$ is the Kronecker delta. The distributions $T_i$, $i=1, 2, \cdots, k$, are the basic distributions of the structure. A distribution of the form $T'=\sum\delta^j T_j$, $\delta^j=0$ or 1, will be called a *distribution of the structure*; the distribution $T''=\sum(1-\delta^j)T_j$ is called the *complementary distribution* to $T'$.

THEOREM 1. *A polynomial structure on a differentiable manifold $M$, defined by a $C^\infty$ tensor field $f$ of type* (1, 1), *induces an almost product structure on $M$. The number of distributions of the structure is equal to the number of distinct irreducible factors over $R$ of the structure polynomial, and the projectors are expressed as polynomials in $f$.*

*Proof.* Let $p(x)$ be the structure polynomial and $p(x)=p_1(x)^{e_1}p_2(x)^{e_2}\cdots p_k(x)^{e_k}$, where the $p_i(x)$ are distinct monic irreducible polynomials over $R$. Since $p_1(x)^{e_1}$ $\cdots$, $p_k(x)^{e_k}$ are relatively prime in the ring $F[x]$ of polynomials over $R$, applying the Euclidean algorithm, we obtain polynomials $h_1(x)$, $h_2(x)$, $\cdots$, $h_k(x)$ such that

$$_1p(x)h_1(x)+{}_2p(x)h_2(x)+\cdots+{}_kp(x)h_k(x)=1,$$

where $_ip(x)$ is the polynomial obtained from $p(x)$ by deleting the factor $p_i(x)^{e_i}$. If we put $l_i={}_ip(f)h_i(f)$, $i=1, 2, \cdots, k$, then

$$l_1+l_2+\cdots+l_k=I, \qquad l_il_j=\delta_{ij}l_i.$$

The $l_i$, $i=1, 2, \cdots, k$, are thus $C^\infty$ projectors defining an almost product structure with distributions $T_i$, $i=1, 2, \cdots, k$, where $T_i(p)=l_iT(p)$.

Let $M$ be a differentiable manifold with two complementary $C^\infty$ distributions $T_1$, $T_2$ of constant dimensions and projectors $\pi_1$, $\pi_2$, respectively. If there are $m$ vector fields $E_a$, $a=1, 2, \cdots, m$, globally defined on $M$, spanning the distribution $T_1$, and $m$ pfaffian forms $\eta^a$ satisfying

$$\eta^a(E_b)=\delta_b^a, \qquad a, b=1, 2, \cdots, m$$

where $\delta_b^a$ is the Kronecker delta, and if $\pi_1=E_a\otimes\eta^a$, then the distribution $T_1$ is said to be *globally framed*. (The summation convention is used here and in the sequel.)

An $m$-dimensional $C^\infty$ manifold is called *parallelizable* if there are $C^\infty$ vector fields $X_1$, $X_2$, $\cdots$, $X_m$, globally defined on $M$ such that, for every point $p\in M$, $X_1(p)$, $\cdots$, $X_m(p)$ span $T(p)$, the tangent space of $M$ at $p$. Such distributions can be trivially defined on parallelizable manifolds.

COROLLARY 1. *Let $M$ be a simply connected* (*paracompact*) *manifold. A* *necessary and sufficient condition for $M$ to have a globally framed distribution is that there exist a polynomial structure on $M$, defined by a tensor field $f$ of constant rank $k$, with structure polynomial of the form*

$$xP(x),$$

*where $P(x)$ is of degree $k$ having $k$ distinct non-zero real roots.*

*Proof.* Let $c_1, c_2, \cdots, c_k$ be the roots of $P(x)$ where $c_i \neq 0$, $c_i \neq c_j$, $i \neq j$, $i, j = 1, 2, \cdots, k$. Then,

$$P(x) = (x-c_1)(x-c_2) \cdots (x-c_k).$$

Applying Theorem 1, we obtain an almost product structure on $M$, with projectors given by

$$l_1 = f \cdot {}_1p(f)h_1(f), \qquad l_2 = f \cdot {}_2p(f)h_2(f) \cdots,$$

$$l_k = f \cdot {}_kp(f)h_k(f), \qquad l_{k+1} = P(f)h_{k+1}(f)$$

where ${}_ip(x)$ is $P(x)$ with the factor $x-c_i$ deleted. If $T_1, T_2, \cdots, T_{k+1}$ are the corresponding distributions, then $T_{k+1}$ is the null space of $f$. Since $f$ is of rank $k$, $T_{k+1}$ must be of dimension $n-k$ and each of the $T_i$'s, $i=1, 2, \cdots, k$, must be of dimension one.

We define on $M$ a metric $g$ and a connection $L$ such that the distributions $T_1, T_2, \cdots, T_{k+1}$ are orthogonal with respect to $g$, parallel with respect to $L$, and $g$ is invariant by parallel translation (see Appendix). If $\psi(p)$ is the holonomy group of $L$ at $p \in M$ then the distributions $T_i(p)$, $i=1, 2, \cdots, k+1$ are invariant by $\psi(p)$, and since $M$ is simply connected

$$\psi(p) = \psi_1(p) \times \psi_2(p) \times \cdots \times \psi_k(p) \times \psi_{k+1}(p)$$

where the $\psi_i(p)$, $i=1, 2, \cdots, k+1$, are normal subgroups of $\psi(p)$. Since $\psi_i(p)$ is irreducible on $T_i(p)$, $i=1, 2, \cdots, k+1$, and acts trivially on $T_j(p)$, $i \neq j$, and since the $T_i(p)$, $i=1, 2, \cdots, k$, are of dimension one, it follows that the $\psi_i(p)$ coincide with the identity subgroup.

Let $E_1(p), E_2(p), \cdots, E_k(p)$ be unit length vectors spanning $T_1(p), T_2(p), \cdots, T_k(p)$, respectively. By parallel translation we define the vector fields $E_1, E_2, \cdots, E_k$, which span the distributions $T_1, T_2, \cdots, T_k$, respectively. Defining the pfaffian forms $\eta^a$ by

$$\eta^a = g(E_a, \cdot), \qquad a=1, 2, \cdots, k,$$

it follows that

$$\eta^a(E_b) = \delta^a_b.$$

The orthogonal complement of $T_{k+1}$ is therefore a globally framed distribution. Suppose now that $T_1$, $T_2$ are two complementary distributions and that $T_1$ is framed by the globally defined vector fields $E_a$, $a=1, 2, \cdots, k$. Let $\eta^a$ be the dual forms.

We define the tensor field $f$ of type (1, 1) and rank $k$ by

$$f=\sum c_a E_a \otimes \eta^a, \qquad f\pi_2=0$$

where the $c_a$'s are real, $c_a \neq c_b$ for $a \neq b$, and $\pi_2$ is the projector corresponding to the distribution $T_2$. Observe that $\eta^a((f-c_a I)X)=0$, (since $\eta^a(E_b)=0$) for every vector field $X$ of $M$. It follows that $f$ satisfies the equation $x(x-c_1)(x-c_2)\cdots(x-c_k)=0$.

COROLLARY 2. *A necessary and sufficient comdition for a simply connected $C^\infty$ manifold $M$ of dimension $m$ to be parallelizable is that there exist a polynomial structure on $M$ with structure polynomial of degree $m$ having $m$ distinct non-zero real roots.*

COROLLARY 3. *There exists a polynomial structure of degree* 3 (7) *on* $S^3$ ($S^7$) *with* 3 (7) *distinct non-zero real roots.*

**3. Reduction of the structural group.** Let $M$ be a differentiable manifold with a polynomial structure defined by $f$ and structure polynomial

$$P(x)=x^2+a_2x+a_1I \tag{3.1}$$

which is irreducible over $R$.

If $\alpha \pm \beta i$ are the roots of (3. 1), then $f$ satisfies the equation

$$(x-\alpha)^2+\beta^2=0. \tag{3.2}$$

If in (3. 2) we put $f=\beta J+\alpha I$, then $J$ satisfies the equation $x^2+I=0$ and, consequently, it defines an almost complex structure on $M$. We shall call $J$ the almost complex structure induced by $f$.

THEOREM 2. *Let $M$ be a $C^\infty$ manifold with a polynomial structure. If the structure polynomial has only distinct complex roots, then there is an underlying almost complex structure on $M$.*

*Proof.* Let $f$ denote the tensor field which defines the polynomial structure on $M$. According to Theorem 1, it defines an almost product structure on $M$. Let $T_1, T_2, \cdots, T_k$ be the basic distributions of this structure with corresponding projectors $l_1, l_2, \cdots, l_k$. Let $P(x)=q_1(x)q_2(x)\cdots q_k(x)$ be the factorization of the structure polynomial, where $q_i(x)$, $i=1, 2, \cdots, k$, are monic irreducible over $R$ quadratic polynomials.

The restriction of $f$ to each $T_i(p)$, $i=1, 2, \cdots, k$, has minimal polynomial $q_i(x)$, for each $p \in M$. Hence, that the restriction of $f$ to each $T_i(p)$ induces a complex structure $J_i$ on $T_i(p)$ for each $p \in M$, that is

$$J_i^2(l_iX)=-l_iX, \qquad J_i(l_jX)=0, \qquad i \neq j \tag{3.3}$$

for any vector field $X$. From (3. 3) it follows that the tensor field $F=J_1l_1+J_2l_2$

$+\cdots+J_k l_k$ defines an almost complex structure on $M$.

COROLLARY 3. *Let $M$ be a $C^\infty$ manifold of even dimension $m$ with a polynomial structure defined by the tensor field $f$ of constant rank $r$ and structure polynomial of the form $xP(x)$, where $P(x)$ has constant term* 1, *and its factors are distinct irreducible quadratic polynomials. If the distribution defined by the projector $\pi_2=P(f)$ is globally framed, then there is an underlying almost complex structure on $M$.*

*Proof.* First note that $\pi_1=-P(f)+I$, and $\pi_2=P(f)$ are projectors defining two complementary distributions $T_1$, $T_2$, respectively. It follows from Theorem 2 that $f$ restricted to $T_1$ induces on it a complex structure $J_1$, that is $(J_1^2+I)\pi_1=0$. Thus, the dimension $r$ of $T_1$ is even, and so also the dimension $m-r$ of $T_2$. Let $T_2$ be spanned by the globally defined vector fields $E_a$ with dual forms $\eta^a$, $a=1, 2, \cdots, m-r$.

We define the tensor field

$$J_2=E_{2i}\otimes\eta^{2i-1}-E_{2i-1}\otimes\eta^{2i}, \qquad i=1, 2, \cdots, \frac{m-r}{2}.$$

Clearly, $(J_2^2+I)\pi_2=0$, so $J=J_1\pi_1+J_2\pi_2$ is an almost complex structure on $M$.

It follows immediately, under the assumptions of Theorem 2, that the structural group of the tangent bundle $L(M)$ is reducible to $GL(m/2, C)$, the complex linear group of complex dimension $m/2$. Similarly, under the assumptions of Corollary 3, the structural group of $L(M)$ is reducible to $GL(r/2, C)\times GL((m-r)/2, C)$.

More refined reductions can be obtained if we assume that the ranks of the projectors $l_i$, $i=1, 2, \cdots, k$, are constants $r_i$, $i=1, 2, \cdots, k$, respectively. In this case, under the assumptions of Theorem 2, the group of $L(M)$ is reducible to

$$GL(r_1/2, C)\times GL(r_2/2, C)\times\cdots\times GL(r_k/2, C).$$

**4. Integrability.** The torsion of an almost product structure is defined by

$$H=\sum_1^k l_i[l_i, l_i] \tag{4.1}$$

(see [8]), where $l_i$, $i=1, 2, \cdots, k$, are the projectors of the structure and $[l_i, l_i]$ is the Nijenhuis tensor (see §5). It is known [8] that the almost product structure is integrable if and only if $H=0$. A polynomial structure defined by $f$ is said to be *integrable* if $[f, f]=0$.

Considering the almost product structure generated by $f$, the $l_i$ in (4.1) are expressed as polynomials of $f$. Making use of this fact and the identity

$$[f_1, f_2f_3]+[f_1f_3, f_2]=f_1[f_2, f_3]+f_2[f_1, f_3]+[f_1, f_2]f_3+[f_1, f_2]\cdot f_3,$$

where

$$[f_1, f_2]f_3(X, Y)=[f_1, f_2](f_3X, Y)$$

$$[f_1, f_2]\cdot f_3(X, Y)=[f_1, f_2](X, f_3Y),$$

(see [8]), the torsion $H$ is expressed by

$$H=a_{hst}f^h[f,f]f^s\cdot f^t.$$

Hence, *If the polynomial structure is integrable, then the almost product structure generated by the polynomial structure is also integrable.*

**5. Normal *f*-product structures.** For any $C^\infty$ tensor field $F$ on $M$ of type (1, 1), the Nijenhuis tensor field $[F, F]$ is given by

$$[F,F](X,Y)=[FX,FY]-F[FX,Y]-F[X,FY]+F^2[X,Y].$$

An almost complex structure $F$ on $M$ is integrable, if and only if, $[F, F]=0$.

Let $M$ be a differentiable manifold with an almost product structure defined by the distributions $T_1$, $T_2$ of ranks $r_1$, $r_2$ and projectors $\pi_1$, $\pi_2$, respectively. In addition, we assume that $T_2$ is globally framed with (globally defined) vector fields $E_a$ spanning $T_2$ and dual forms $\eta^a$ such that $\pi_2=E_a\otimes\eta^a$.

If there exists a $C^\infty$ linear transformation field $f$ on $M$ inducing an endomorphism on $T_1(p)$, for each $p\in M$, and $f\equiv 0$ on $T_2$, i.e., rank $f=r_1$, $f=f\pi_1=\pi_1 f$, $f\pi_2=0$, we shall say that we have an *f-product structure* on $M$. The globally framed $f$-manifolds are examples of such structures. An $f$-product structure is said to be *normal* if the $d\eta^a$ are of bidegree (1, 1) with respect to $f$, i.e., if

$$d\eta^a(fX,Y)+d\eta^a(X,fY)=0,$$

and if

$$[f,f]+E_a\otimes d\eta^a=0. \tag{5.1}$$

These relations are clearly satisfied on a normal globally framed $f$-manifold [4]. A *normal f-product structure* will be denoted by $M(f, E_a, \eta^a)$.

Let $R^{r_2}$ be the $r_2$-dimensional affine space. For an arbitrary point $(p, x)\in M\times R^{r_2}$, the tangent space $T(p, x)$ of $M\times R^{r_2}$ at $(p, x)$ will be identified with the direct sum $T(p, x)=T(p)+T_3(x)$, where $T(p)$ is the tangent space of $M$ at $p$ and $T_3(x)$ is the tangent space of $R^{r_2}$ at $x$. From the almost product structure of $M$ we have $T(p)=T_1(p)+T_2(p)$. Hence, the distributions $T_1$, $T_2$, $T_3$ define an almost product structure on $M\times R^{r_2}$. Let $\pi_1$, $\pi_2$, $\pi_3$ be the corresponding projectors.

The frame field $\{E_1, E_2, \cdots, E_{r_2}\}$ defines a non-singular linear mapping $e$: $R^{r_2}\to T_2(p)$ for each $p\in M$. Let $X_a=e^{-1}(E_a)$, $a=1, 2, \cdots, r_2$. We define a $C^\infty$ tensor field $F$ of type (1, 1) on $M\times R^{r_2}$ by

$$F=f+e^{-1}\pi_2-e\pi_3.$$

We then have the following basic theorem.

THEOREM 3. *For a normal f-product structure*

$$[F,F]=0.$$

COROLLARY 4. *For a normal f-product structure* $M(f, E_a, \eta^a)$, *the vector fields* $E_a$ *are infinitesimal automorphisms, i.e.,*

$$[E_a, E_b]=0, \tag{5.2}$$

$$L_{E_a}\eta^b=0, \tag{5.3}$$

$$L_{E_a}f=0, \tag{5.4}$$

*where* $L_X$ *denotes the Lie derivative in the direction of* $X$ *and* $[E_a, E_b]=L_{E_a}E_b$, $a, b=1, 2. \cdots, r_2$.

This generalizes Lemma 2 in [4].

To prove Theorem 3, we establish a number of lemmas.

LEMMA 1.

$$F\pi_1=f\pi_1=\pi_1 f=f, \qquad FE_a=X_a, \qquad FX_a=-E_a, \tag{5.5}$$

$$\pi_3 F\pi_3=\pi F\pi_2=0, \tag{5.6}$$

$$\pi_3 F=\pi_3 F\pi_2=F\pi_2=\pi_3 F\pi, \tag{5.7}$$

$$F^2\pi_2=-\pi_2, \qquad F^2\pi_3=-\pi_3, \qquad \pi_3 F^2\pi=0, \tag{5.8}$$

$$\pi_3 F^3=-F\pi_2=-\pi_3 F,$$

$$\pi_3[\pi X, \pi Y]=0, \qquad X_a(\pi X)=(\pi X)X_a=0, \qquad [X_a, X_b]=0, \tag{5.9}$$

where $\pi=\pi_1+\pi_2$.

*Proof.* (5. 5) and (5. 6) follow directly from the definitions, and (5. 7) follows from (5. 5) and (5. 6). The first two relations of (5. 8) follow directly from the definition of $F$ and the third follows from (5. 5) and the first of (5. 8). The last relation of (5. 8) follows from the first two of (5. 8), from (5. 6) and the first relation first formula of (5. 9) follows from the fact that the distribution defined by $\pi$ is of (5. 5). The integrable, and the last two relations of (5. 9) follow from the fact that $X_a=\partial/\partial x^a$, $a=1, 2, \cdots, r_2$, where $(x^1, \cdots, x^{r_2})$ are the natural coordinates of $R^{r_2}$, and $\pi X$ is a vector field over $M$.

LEMMA 2.

$$[f, f]+E_a\otimes d\eta^a=0 \tag{5.10}$$

*implies*

$$\pi[F, F](\pi X, \pi Y)=0. \tag{5.11}$$

*Proof.* Let $U$ be a coordinate neighborhood of $M$ such that the distribution $T_1$ is spanned in $U$ by the differentiable tensor fields $X_\alpha, \alpha=1, 2, \cdots, r_1$. Then, $T_1+T_2$ is spanned in $U$ by

$$X_1, X_2, \cdots, X_{r_1}, \qquad E_1, E_2, \cdots, E_{r_2}. \tag{5.12}$$

Since $[F, F]$ is bilinear over the module of vector fields $\mathfrak{X}(M)$ of $M$, it is enough to prove (5.11) for the vectors of a local frame (5.12). We have

$$[F. F](X_\alpha, E_b)=[FX_\alpha, FE_b]+F^2[X_\alpha, E_b]-F[FX_\alpha, E_b]-F[X_\alpha, FE_b]. \tag{5.13}$$

By Lemma 1

$$F^2[X_\alpha, E_b]=f^2[X_\alpha, E_b]-\pi_2[X_\alpha, E_b],$$

$$F[FX_\alpha, E_b]=f[fX_\alpha, E_b]+F\pi_2[fX_\alpha, E_b],$$

$$[FX_\alpha, FE_b]=0,$$

$$\pi F\pi_2[fX_\alpha, E_b]=0,$$

so, from (5.13)

$$\begin{aligned}\pi[F, F](X_\alpha, E_b)&=f^2[X_\alpha, E_b]-f[fX_\alpha, E_b]-\pi_2[X_\alpha, E_b]\\ &=S_f(X_\alpha, E_b),\end{aligned} \tag{5.14}$$

where we have put $S_f=[f, f]+E_a\otimes d\eta^a$.

Applying the last two relations of (5.9), the first two relations of (5.8) and the first formula of (5.5), we obtain

$$\begin{aligned}[F, F](E_a, E_b)&=f^2[E_a, E_b]-\pi_2[E_a, E_b]\\ &=S_f(E_a, E_b)\\ &=\pi[F, F](E_a, E_b).\end{aligned} \tag{5.15}$$

Finally,

$$[F, F](X_\alpha, X_\beta)=[fX_\alpha, fX_b]+F^2[X_\alpha, X_\beta]-F[fX_\alpha, X_\beta]-F[X_\alpha, fX_\beta]. \tag{5.15a}$$

Applying the first relations of (5.8) and (5.6) yields

$$\begin{aligned}\pi[F, F](X_\alpha, X_\beta)=[fX_\alpha, fX_\beta]+f^2[X_\alpha, X_\beta]-\pi_2[X_\alpha, X_\beta]\\ -f[fX_\alpha, X_\beta]-f[X_\alpha, fX_\beta]=S_f(X_\alpha, X_\beta).\end{aligned} \tag{5.16}$$

The lemma now follows from (5.14), (5.15) and (5.16).

LEMMA 3. *For a normal f-product structure*

(5. 17) $$\pi_3F[F, F](F\pi X, \pi Y)=\pi_3[F, F](\pi X, \pi Y).$$

*Proof.* We show

(5. 18) $$\pi_3F[F, F](FX_\alpha, X_\beta)=\pi_3[F, F](X_\alpha, X_\beta).$$

From (5. 15a)

$$F[F, F](FX_\alpha, X_\beta)=F[f^2X_\alpha, fX_\beta]+F^3[fX_\alpha, X_\beta]-F^2[f^2X_\alpha, X_\beta]-F^2[fX_\alpha, fX_\beta].$$

Applying the third and fourth formulas of (5. 8), we obtain

(5. 19) $$\pi_3F[F, F](FX_\alpha, X_\beta)=\pi_3F[f^2X_\alpha, fX_\beta]-\pi_3F\pi_2[fX_\alpha, X_\beta].$$

From the normality condition $S_f(fX_\alpha, X_\beta)=0$, we get

$$\pi_2[f^2X_\alpha, fX_\beta]=\pi_2[fX_\alpha, X_\beta],$$

so (5. 19) becomes

$$\pi_3F[F, F](FX_\alpha, X_\beta)=0.$$

On the other hand,

$$\pi_3[F, F](X_\alpha, X_\beta)=-\pi_3F\pi_2\{[fX_\alpha, X_\beta]+[X_\alpha, fX_\beta]\}.$$

Since the $d\eta^a$ are of bidegree (1. 1) with respect to $f$

$$\pi_3[F, F](X_\alpha, X_\beta)=0$$

which completes the proof of (5. 18)

We now show that

(5. 20) $$\pi_3F[F, F](FX_\alpha, E_b)=\pi_3[F, F](X_\alpha, E_b).$$

Using the second formula of (5. 9) and the third and fourth of (5. 8)

$$\pi_3F[F, F](FX_\alpha, E_b)=-F\pi_2[fX_\alpha, E_b],$$

$$\pi_3[F, F](X_\alpha, E_b)=-F\pi_2[fX_\alpha, E_b],$$

from which (5. 20) follows.

We also show that

(5. 21) $$\pi_3F[F, F](FE_a, E_b)=\pi_3[F, F](E_a, E_b).$$

Using the last two relations of (5. 9) and the first of (5. 8)

$$[F, F](FE_a, E_b)=F[E_a, E_b],$$

$$[F, F](E_a, E_b)=F^2[E_a, E_b],$$

from which (5. 21) follows.

Finally, we show

$$\pi_3 F[F, F](FE_a, X_\alpha) = \pi_3[F, F](E_a, X_\alpha). \tag{5.22}$$

Using the first formula of (5. 8), the second of (5. 7) and of (5. 9),

$$\pi_3 F[F, F](FE_a, X_\alpha) = -\pi_3 F[E_a, FX_\alpha] = \pi_3[F, F](E_a, X_\alpha).$$

The relations (5. 18), (5. 20), (5. 21) and (5. 22) complete the proof of the lemma.

LEMMA 4.

$$\pi_3 F[F, F](\pi_3 X, \pi Y) = -\pi_3[F, F](\pi_3 X, F\pi Y). \tag{5.23}$$

*Proof*. Applying the second formula of (5. 7) and the last two of (5. 9)

$$\pi_3 F[F, F](X_a, X_\alpha) = \pi_3 F[FX_a, fX_\alpha] = -\pi_3[F, F](X_a, FX_\alpha). \tag{5.24}$$

Moreover,

$$\pi_3 F[F, F](X_a, E_b) = \pi_3[F, F](X_a, FE_b) = 0. \tag{5.25}$$

Now, (5. 24) and (5. 25) yield (5. 23).

LEMMA 5.

$$\pi_3 F[F, F](F\pi_3 X, \pi Y) = \pi_3[F, F](\pi_3 X, \pi Y). \tag{5.26}$$

*Proof*. Applying the second formula of (5. 9) and the second, third and fourth of (5. 8) gives

$$\pi_3 F[F, F](FX_a, X_\alpha) = -\pi_3 F[FX_a, X_\alpha] = \pi_3[F, F](X_a, X_\alpha). \tag{5.27}$$

Also

$$\pi_3 F[F, F](FX_a, E_b) = \pi_3[F, F](X_a, E_b) = -\pi_3 F[FX_a, E_b]. \tag{5.28}$$

Now, (5. 27) and (5. 28) yield (5. 26).

LEMMA 6.

$$\pi_3[F, F](\pi_3 X, \pi_3 Y) = 0. \tag{5.29}$$

*Proof*. Applying the last two relations of (5. 9), we obtain

$$\pi_3[F, F](X_a, X_b) = \pi_3[E_a, E_b] = 0,$$

thereby giving (5. 29).

LEMMA 7. *For a normal f-product structure*

$$[F, F](\pi X, \pi Y) = 0. \tag{5.30}$$

*Proof.* By Lemma 2, it is enough to show

$$\pi_3[F, F](\pi X, \pi Y)=0. \tag{5.31}$$

From Lemma 3

$$\pi_3[F, F](\pi X, \pi Y)=\pi_3 F[F, F](F\pi X, \pi Y). \tag{5.32}$$

(a) Suppose $F\pi X=\pi Z$. Then, (5.32) becomes

$$\pi_3[F, F](\pi X, \pi Y)=\pi_3 F[F, F](\pi Z, \pi Y).$$

Applying the first relation of (5.7) and Lemma 2, we have

$$\pi_3 F[F, F](\pi Z, \pi Y)=\pi_3 F\pi[F, F](\pi Z, \pi Y)=0$$

which establishes (5.32) in this case.

(b) Suppose $F\pi X=\pi_3 Z$. Then applying Lemma 4

$$\pi_3 F[F, F](F\pi X, \pi Y)=\pi_3 F[F, F](\pi_3 Z, \pi Y)=-\pi_3[F, F](\pi_3 Z, F\pi Y).$$

If $F\pi Y=\pi_3 W$, then by Lemma 6

$$\pi_3[F, F](\pi_3 Z, F\pi Y)=\pi_3[F, F](\pi_3 Z, \pi_3 W)=0.$$

If $F\pi Y=\pi W$ then, by Lemma 5

$$\pi_3[F, F](\pi_3 Z, F\pi Y)=\pi_3[F, F](\pi_3 Z, \pi W)$$

$$=\pi_3 F[F, F](F\pi_3 Z, \pi W)=\pi_3 F\pi[F, F](\pi F\pi_3 Z, \pi W).$$

But, from Lemma 2,

$$\pi[F, F](\pi F\pi_3 Z, \pi W)=0.$$

The lemma follows from (a) and (b).

LEMMA 8.

$$[F, F](\pi X, \pi Y)=0$$

*implies*

$$[F, F]=0.$$

*Proof.* We must show that

(a) $[F, F](\pi X, \pi_3 Y)=0$ and (b) $[F, F](\pi_3 X, \pi_3 Y)=0$.

For (a), we have

$$[F, F](X_a, X_\alpha)=-[E_a, FX_\alpha]+F[E_a, X_\alpha],$$

$$[F, F](E_a, X_\alpha)=F^2[E_a, X_\alpha]-F[E_a, FX_\alpha],$$

that is

$$F[F, F](X_a, X_\alpha)=[F, F](E_a, X_\alpha)=0$$

by assumption. Since $F$ is non-singular, this implies $[F, F](X_a, X_\alpha)=0$. On the other hand,

$$[F, F](X_a, E_b)=F[E_a, E_b],$$

$$[F, F](E_a, E_b)=F^2[E_a, E_b],$$

that is

$$F[F, F](X_a, E_b)=[F, F](E_a, E_b)=0$$

which implies

$$[F, F](X_a, E_b)=0.$$

For (b)

$$[F, F](X_a, X_b)=[E_a, E_b]$$

$$[F, F](E_a, E_b)=F^2[E_a, E_b],$$

that is

$$F^2[F, F](X_a, X_b)=[F, F](E_a, E_b)=0.$$

With this lemma, the proof of Theorem 3 is complete.

*Proof of Corollary* 4. Let $X$ be a vector field on $M$, then $\pi X=X$. From Theorem 3 we have

$$[F, F](X, X_a)=0, \quad \text{that is,} \quad [FX, FX_a]-F[X, FX_a]=0,$$

or

$$[fX+F\pi_2X, E_a]-f[X, E_a]-F\pi_2[X, E_a]=0,$$

that is,

$$[fX, E_a]+[F\pi_2X, E_a]-f[X, E_a]-F\pi_2[X, E_a]=0. \tag{5.33}$$

From the second relation of (5.9) and the first of (5.7)

$$[F\pi_2X, E_a]\in T_3, \qquad F\pi_2[X, E_a]\in T_3.$$

Thus, from (5.33)

$$[fX, E_a]-f[X, E_a]=0, \tag{5.34}$$

$$(5.35) \qquad [F\pi_2X, E_a]-F\pi_2[X, E_a]=0.$$

From (5.34), we have $L_{E_a}f=0$. Since $\pi_2=E_a\otimes\eta^a$, we have $\pi_2X=\eta^a(X)E_a$ and $F\pi_2X=\eta^a(X)F(E_a)=\eta^a(X)X_a$ and $F\pi_2[X, E_b]=\eta^a([X, E_b])X_a$. Since $[X_a, E_b]=0$, (5.35) gives $E_b(\eta^a(X))X_a-\eta^a([E_b, X])=0$, that is, $E_b(\eta^a(X))-\eta^a([E_b, X])=0$, or $L_{E_b}\eta^a=0$. Finally,

$$[F, F](X_a, X_b)=[E_a, E_b].$$

Therefore, $[F, F]=0$ implies $[E_a, E_b]=0$ which completes the proof of the corollary.

Consider a polynomial structure $f$ on $M$ with structure polynomial

$$P(x)=a_{m+1}x^{m+1}+a_mx^m+\cdots+a_2x^2+x.$$

It defines two complementary distributions $T_1$, $T_2$ with projectors

$$\pi_1=-a_{m+1}f^m-a_mf^{m-1}-\cdots-a_2f$$

and

$$\pi_2=a_{m+1}f^m+a_mf^{m-1}+\cdots+a_2f+I,$$

respectively. This polynomial structure is said to be *globally framed* if the distribution $T_2$ is globally framed.

If, in addition, the $d\eta^a$ are of bidegree (1, 1) with respect to $f$ and

$$[f, f]+E_a\otimes d\eta^a=0,$$

the polynomial structure will be called a *normal polynomial structure.*

COROLLARY 5. *For a normal polynomial structure $f$ of constant rank the relations* (5.2), (5.3), (5.4) are satisfied.

*Proof.* Clearly, the almost product structure defined by the complementary distributions discussed above is a normal $f$-product structure.

REMARK 1. If the polynomial structure is an $f$-structure, and if it is globally framed, then $[f, f]+E_a\otimes d\eta^a=0$ implies the bidegree property [2].

REMARK 2. The bidegree property of a normal $f$-product structure is trivially satisfied if $d\eta^a=0$, $a=1, \cdots, k$, which in the case of metric $f$-structures characterizes the $\mathcal{C}$-structures [1].

**6. Appendix.** The following theorem is basically due to A. G. Walker.

THEOREM 4. *Let $M$ be a (paracompact) $C^\infty$ manifold with an almost product structure defined by the distributions $T_1, T_2, \cdots, T_k$ with corresponding projectors $\pi_1, \pi_2, \cdots, \pi_k$. Then, there exists a metric connection on $M$ with respect to which the distributions $T_i$, $i=1, 2, \cdots, k$, are parallel and orthogonal.*

The proof will be split into the following two lemmas:

LEMMA 9. *Let $M$ be a (paracompact) $C^\infty$ manifold with an almost product structure defined by the distributions $T_1, T_2, \cdots, T_k$ with corresponding projectors $\pi_1, \pi_2, \cdots, \pi_k$. Then, there exists a metric connection on $M$ with respect to which the distribution $T_1$ is parallel and the distributions $T_i$, $i=1, 2, \cdots, k$, are orthogonal.*

LEMMA 10. *Let $M$ be a (paracompact) $C^\infty$ manifold with an almost product structure defined by the distributions $T_1, T_2, \cdots, T_k$ with corresponding projectors $\pi_1, \pi_2, \cdots, \pi_k$. If there is a metric connection on $M$ with respect to which the distributions $T_i$, $i=1, 2, \cdots, k$, are orthogonal, and the distributions $T_j$, $j=1, 2, \cdots, m$, $m<k$, are parallel, then there is a metric connection with respect to which the distributions $T_i$, $i=1, 2, \cdots, k$ are orthogonal and the distributions $T_j$, $j=1, 2, \cdots, m+1$ are parallel.*

*Proof of Lemma* 9. Let $\Gamma$ be a connection on $M$. Then the distribution $T_1$ is parallel with respect to $\Gamma$ if

$$\nabla_Z \pi_1 = 0 \tag{6.1}$$

for every vector field $Z$ of $M$, where $\nabla$ denotes covariant differentiation with respect to $\Gamma$.

Defining the tensor field $\alpha_{1\Gamma}$ of type (1, 2) by

$$\alpha_{1\Gamma}(Y, Z) = (\pi_1 - \pi_2 - \cdots - \pi_k)(\nabla_Z \pi_1) Y \tag{6.2}$$

we have, by (6. 1),

$$\alpha_{1\Gamma} = 0. \tag{6.3}$$

Let $h$ be a positive definite metric on $M$ and define the tensor field $g$ by

$$g(X, Y) = h(\pi_1 X, \pi_1 Y) + \cdots + h(\pi_k X, \pi_k Y). \tag{6.4}$$

Clearly, $g$ is a positive definite metric with respect to which the $T_i$, $i=1, 2, \cdots, k$, are orthogonal. Let $C$ be the Levi-Civita connection defined by $g$, then

$$\dot{\nabla}_Z g = 0 \tag{6.5}$$

for every vector field $Z$ of $M$, where $\dot{\nabla}$ denotes covariant differentiation with respect to $C$.

If $S$ is a tensor field of type (1, 2), then $\Gamma = C + S$ is a connection and

$$\nabla_Z Y = \dot{\nabla}_Z Y + S(Y, Z). \tag{6.6}$$

The metric $g$ is preserved by $\Gamma$ if and only if $\nabla_Z g = 0$ which, because of (6. 5) and (6. 6), becomes

$$g(S(X, Z), Y) + g(X, S(Y, Z)) = 0. \tag{6.7}$$

Now, (6. 2) may be written as

$$\alpha_{1\Gamma}(Y, Z)=\alpha_{1c}(Y, Z)+(\pi_1-\pi_2-\cdots-\pi_k)[S(\pi_1 Y, Z)-\pi_1 S(Y, Z)] \tag{6. 8}$$

so that (6. 3) becomes:

$$\alpha_{1c}(Y, Z)=(\pi_1-\pi_2-\cdots-\pi_k)[\pi_1 S(Y, Z)-S(\pi_1 Y, Z)]. \tag{6. 9}$$

We wish to determine $S$ so that (6. 7) and (6. 9) are satisfied.

Using (6. 5) and the fact that the $T_i$, $i=1, 2, \cdots, k$, are orthogonal,

$$\begin{aligned}
g(\alpha_{1c}(X, Z),\ Y)&=g((\pi_1-\pi_2-\cdots-\pi_k)(\dot{\nabla}_Z\pi_1)X,\ Y)\\
&=g((\pi_1-\pi_2-\cdots-\pi_k)\dot{\nabla}_Z(\pi_1 X),\ Y)-g(\pi_1\dot{\nabla}_Z X,\ Y)\\
&=g(\pi_1\dot{\nabla}_Z(\pi_1 X),\ Y)-g(\pi_2\dot{\nabla}_Z(\pi_1 X),\ Y)-\cdots-g(\pi_k\dot{\nabla}_Z(\pi_1 X),\ Y)-g(\pi_1\dot{\nabla}_Z X,\ Y)\\
&=g(\dot{\nabla}_Z(\pi_1 X),\ \pi_1 Y)-g(\dot{\nabla}_Z(\pi_1 X),\ \pi_2 Y)-\cdots-g(\dot{\nabla}_Z(\pi_1 X),\ \pi_k Y)-g(\dot{\nabla}_Z X,\ \pi_1 Y)\\
&=Zg(\pi_1 X,\ \pi_1 Y)-g(\pi_1 X,\ \dot{\nabla}_Z(\pi_1 Y)+g(\pi_1 X,\ \dot{\nabla}_Z(\pi_2 Y))\\
&\quad+\cdots+g(\pi_1 X,\ \dot{\nabla}_Z(\pi_k Y))-Zg(X,\ \pi_1 Y)+g(X,\ \dot{\nabla}_Z(\pi_1 Y))\\
&=-g(\pi_1 X,\ \dot{\nabla}_Z(\pi_1 Y))+g(\pi_1 X,\ \dot{\nabla}_Z(\pi_2 Y))+\cdots+g(\pi_1 X,\ \dot{\nabla}_Z(\pi_k Y))+g(X,\ \dot{\nabla}_Z(\pi_1 Y))\\
&=g(X,\ \pi_1\dot{\nabla}_Z(\pi_2 Y))+\cdots+g(X,\ \pi_1\dot{\nabla}_Z(\pi_k Y))\\
&\quad+g(X,\ \pi_2\dot{\nabla}_Z(\pi_1 Y))+\cdots+g(X,\ \pi_k\dot{\nabla}_Z(\pi_1 Y))\\
&=g(X,\ \pi_1\dot{\nabla}_Z Y)-g(X,\ \pi_1\dot{\nabla}_Z(\pi_1 Y))+g(X,\ (\pi_2+\cdots+\pi_k)\dot{\nabla}_Z(\pi_1 Y))\\
&=g(X,\ (\pi_2+\cdots+\pi_k-\pi_1)(\dot{\nabla}_Z(\pi_1 Y)-\pi_1\dot{\nabla}_Z Y))\\
&=-g(X,\ \alpha_{1c}(Y, Z)).
\end{aligned}$$

Thus,

$$g(\alpha_{1c}(X, Z),\ Y)+g(X,\ \alpha_{1c}(Y, Z))=0. \tag{6. 10}$$

If in the second member of (6. 9) we put $S=\alpha_{1c}$

$$\begin{aligned}
&(\pi_1-\pi_2-\cdots-\pi_k)[\pi_1\alpha_{1c}(Y, Z)-\alpha_{1c}(\pi_1 Y, Z)]\\
&=(\pi_1-\pi_2-\cdots-\pi_k)[\pi_1\dot{\nabla}_Z(\pi_1 Y)-\pi_1\dot{\nabla}_Z Y-\pi_1\dot{\nabla}_Z(\pi_1 Y)+\pi_1\dot{\nabla}_Z(\pi_1 Y)+(\pi_2+\cdots+\pi_k)\dot{\nabla}_Z(\pi_1 Y)]\\
&=(\pi_1-\pi_2-\cdots-\pi_k)[\dot{\nabla}_Z(\pi_1 Y)-\pi_1\dot{\nabla}_Z Y)],
\end{aligned}$$

from which

$$\alpha_{1c}(Y, Z)=(\pi_1-\pi_2-\cdots-\pi_k)[\pi_1\alpha_{1c}(Y, Z)-\alpha_{1c}(\pi_1 Y, Z)]. \tag{6. 11}$$

From (6. 10) and (6. 11), we see that if

$$\alpha_{1c}=S, \tag{6.12}$$

then (6.7) and (6.9) are satisfied.

*Proof of Lemma* 10. Let $\Gamma$ be a connection on $M$ with respect to which

$$\dot{\nabla}_Z\pi_1=\dot{\nabla}_Z\pi_2=\cdots=\dot{\nabla}_Z\pi_m=0, \qquad m<k, \tag{6.13}$$

$$\dot{\nabla}_Z g=0 \tag{6.14}$$

for every vector field $Z$ of $M$, where $\dot{\nabla}$ denotes covariant differentiation with respect to $\Gamma$ and $g$ is a metric such that

$$g(\pi_i X, \pi_j Y)=0 \qquad \text{for} \qquad i\neq j. \tag{6.15}$$

Defining the tensor fields $\alpha_{i\Gamma}$, $i=1, 2, \cdots, m+1$, of type (1, 2) by

$$\alpha_{i\Gamma}(Y, Z)=(2\pi_i-\pi)(\dot{\nabla}_Z\pi_i)Y, \qquad \pi=\pi_1+\cdots+\pi_k \tag{6.16}$$

the relations (6.13) become

$$\alpha_{i\Gamma}=0, \qquad i=1, 2, \cdots, m. \tag{6.17}$$

Consider the connection $L=\Gamma+S$, where $S$ is a tensor field of type (1, 2) and denote covariant differentiation with respect to $L$ by $\nabla$. We wish to determine $S$ so that

$$\alpha_{iL}=0, \qquad i=1, 2, \cdots, m+1 \tag{6.18}$$

$$g(S(X, Z), Y)+g(X, S(Y, Z))=0 \tag{6.19}$$

where $\alpha_{iL}$ is given by (6.16) with $\dot{\nabla}$ replaced by $\nabla$.

Applying the identity (6.6) and (6.17) we obtain

$$\alpha_{iL}(Y, Z)=(2\pi_i-\pi)[S(\pi_i Y, Z)-\pi_i S(Y, Z)], \qquad i=1, 2, \cdots, m \tag{6.20}$$

and

$$\alpha_{m+1, L}(Y, Z)=\alpha_{m+1, \Gamma}(Y, Z)+(2\pi_{m+1}-\pi)[S(\pi_{m+1} Y, Z)-\pi_{m+1}S(Y, Z)], \tag{6.21}$$

so (6.18) becomes

$$(2\pi_i-\pi)[S(\pi_i Y, Z)-\pi_i S(Y, Z)]=0, \qquad i=1, 2, \cdots, m \tag{6.22}$$

and

$$\alpha_{m+1, \Gamma}(Y, Z)=(2\pi_{m+1}-\pi)[\pi_{m+1}S(Y, Z)-S(\pi_{m+1} Y, Z)]. \tag{6.23}$$

We observe that

$$(\dot{\nabla}_Z\pi_{m+1})(\pi_i Y)=-\pi_{m+1}\dot{\nabla}_Z(\pi_i Y)=0, \qquad i\neq m+1$$

since $\dot{\nabla}_Z(\pi_i Y)=\pi_i\dot{\nabla}_Z Y$, $i=1, \cdots, m$; hence $\alpha_{m+1,r}(\pi_i Y, Z)=0$, $i=1, \cdots, m$. Similarly, $\pi_i\alpha_{m+1,r}(Y, Z)=0$, $i=1, \cdots, m$.

It follows that

$$(2\pi_i-\pi)[\alpha_{m+1,r}(\pi_i Y, Z)-\pi_i\alpha_{m+1,r}(Y, Z)]=0, \qquad i=1, \cdots, m. \tag{6.24}$$

We also observe that

$$\pi_{m+1}\alpha_{m+1,r}(Y, Z)=\pi_{m+1}\dot{\nabla}_Z(\pi_{m+1}Y)-\pi_{m+1}\dot{\nabla}_Z Y, \tag{6.25}$$

and

$$\alpha_{m+1,r}(\pi_{m+1}Y, Z)=(2\pi_{m+1}-\pi)\dot{\nabla}_Z(\pi_{m+1}Y)-\pi_{m+1}\dot{\nabla}_Z(\pi_{m+1}Y). \tag{6.26}$$

Subtracting (6. 26) from (6. 25) gives

$$\pi_{m+1}\alpha_{m+1,r}(Y, Z)-\alpha_{m+1,r}(\pi_{m+1}Y, Z)=(\dot{\nabla}_Z\pi_{m+1})Y.$$

Hence,

$$\alpha_{m+1,r}(Y, Z)=(2\pi_{m+1}-\pi)[\pi_{m+1}\alpha_{m+1,r}(Y, Z)-\alpha_{m+1,r}(\pi_{m+1}Y, Z)]. \tag{6.27}$$

It follows from (6. 24) and (6. 27) that (6. 22) and (6. 23) are satisfied for $S=\alpha_{m+1,r}$.

In a similar manner to that used to establish (6. 10)

$$g(\alpha_{m+1,r}(X, Z), Y)+g(X, \alpha_{m+1,r}(Y, Z))=0. \tag{6.28}$$

Consequently, if $S=\alpha_{m+1,r}$, (6. 18) and (6. 19) are satisfied.

## References

[1] Blair, D. E., Geometry of manifolds with structural group $U(n)\times O(s)$. J. Differential Geometry **2** (1970), 155–167.

[2] Goldberg, S. I., Framed manifolds. Differential Geometry, in honor of Kentaro Yano, Kinokuniya, Tokyo (1972), 121–132.

[3] Goldberg, S. I. and K. Yano, Polynomial structures on manifolds. Kōdai Math. Sem. Rep. **22** (1970), 199–218.

[4] ———, On normal globally framed $f$-manifolds. Tôhoku Math. J. **22** (1970), 362–370.

[5] ———, Globally framed $f$-manifolds. Illinois J. Math. **15** (1971), 456–474.

[6] Nakagawa, H., $f$-structures induced on submanifolds in spaces almost Hermitian or Kaehlerian. Kōdai Math. Sem. Rep. **18** (1966), 161–183.

[7] Yano, K., On a structure defined by a tensor field $f$ of type (1, 1) satisfying $f^3+f=0$. Tensor **14** (1963), 99–109.

[8] WALKER, A. G., Almost product structures. Proceedings of the third Symposium in Pure Mathematics of the American Mathematical Society, **3** (1961), 94-100.
[9] ———, Connexions for parallel distributions in the large (II). Quart. J. Math. Oxford (2), **9** (1958), 221-231.

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